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Volatility Smile Expansions in Lévy models

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Betreuer:
Privatdoz. Dipl.-Ing. Dr.techn. Stefan Gerhold
E105 Institut für Wirtschaftsmathematik
E-Mail-Adresse: `stefan.gerhold@tuwien.ac.at`

Autor:
Axel Zrunek, BSc
066 405 Finanz- und Versicherungsmathematik
Anschrift: Sieveringerstraße 113/1/3
1190 Wien
E-Mail-Adresse: `axel.zrunek@aon.at`

Statutory Declaration

I declare in lieu of an oath that I have written this master thesis myself and that I have not used any sources or resources other than stated for its preparation. This master thesis has not been submitted elsewhere for examination purposes.

Vienna, on December 12, 2013 ______________________

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Abstract

This thesis is about investigating tail expansions for the call price and implied volatility at large strikes in exponential Lévy jump-diffusion models. Furthermore, the asymptotics of the density function and the tail probability are studied. To get these expansions, we use the saddle-point method (method of deepest descent) on the Mellin transform of the call price, respectively density function and tail probability. Expansions for the implied volatility skew are derived by using transfer theorems, sharpening previous results from [BF08] and [BF09]. We consider the double exponential Kou and the Merton Jump Diffusion model in this work.

Keywords: *exponential Lévy jump diffusion models, Kou, Merton Jump Diffusion, saddle-point approximation, tail expansions*

Contents

1. Introduction

The intention of this thesis is to get tail expansions for the call price and implied volatility at large strikes as well as for the density function and the tail probability by using the method of saddle-point approximation as in [FG11]. The method basically consists of 3 steps: finding a saddle-point, deriving an asymptotic expansion of the integral around the saddle-point and showing that the remaining tails are negligible. The book [FS09] by P. Flajolet and P. Sedgewick gives a good overview of the method and contains some combinatorical examples, which are similar to those considered here (Examples VIII.6 and VIII.7 in [FS09, p.560 ff.]).
In contrast to the above mentioned paper, where the stochastic volatility model of Heston is considered, is this thesis about exponential Lévy jump diffusion models. The general form of the stock process in this kind of models is

$$S_t = S_0 e^{rt+X_t}, \quad t \in [0, T^*]$$

where $S_0 < 0$ is the initial stock value, $r \geq 0$ is the riskless interest rate and $T^* > 0$ is a finite time horizont.

$$X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$$

is a Lévy jump diffusion process with $\gamma \in \mathbb{R}$, diffusion volatility $\sigma > 0$, $(W_t)_{t\in[0,T^*]}$ being a standard Brownian motion, $(Y_i)_{i\in\mathbb{N}}$ real iid random variables and $(N_t)_{t\in[0,T^*]}$ a Poisson process with jump intensity $\lambda > 0$. Throughout the work, it can be assumed without loss of generality $r = 0$ and $S_0 \equiv 1$ (see [GL, p.4]). Hence, one gets

$$S_t = e^{X_t}. \tag{1.1}$$

The two models considered in this thesis are the Kou model, where the Y_i follow a double exponential distribution, and the Merton Jump Diffusion model, where the Y_i are Gaussian random variables. [CT04] is a good source on Lévy models used in finance and gives more theoretical background information on this topic.
We consider a European call option with maturity $T < T^*$ and log-strike $k := \log K > 0$

$$C(k, T) = \mathbb{E}\left[(S_T - e^k)_+\right],$$

where $(x - y)_+ := \max(x - y, 0)$. Hence, we just need to focus on the random variable S_T and not on the whole stock process $(S_t)_{t\in[0,T^*]}$. Under these assumptions, the price of the call option in the normalized Black Scholes model is

$$c_{BS}(k, \sigma) = \Phi(d_1) - e^k \Phi(d_2)$$

with $\tilde{\sigma} > 0$ being the constant unannualized (dimensionless) volatility in the Black Scholes setting, $d_{1,2} := -\frac{k}{\tilde{\sigma}} \pm \frac{\tilde{\sigma}}{2}$ and $\Phi(x)$ being the cumulative distribution function of a standard

Gaussian distribution.

Remark 1.1 Usually the Black Scholes formula is given with the annualized volatility $\bar{\sigma}$, i.e.

$$\tilde{c}_{BS}(k,\sigma) = \Phi(d_1) - e^k\Phi(d_2)$$

with $d_{1,2} = -\frac{k}{\bar{\sigma}\sqrt{T}} \pm \frac{\bar{\sigma}\sqrt{T}}{2}$.

The (unannualized) implied volatility is defined as the unique value $V(k) > 0$ such that

$$c_{BS}(k, V(k)) = \mathbb{E}\left[(S_T - e^k)_+\right].$$

As it can be seen later in (2.1), the Kou model exhibits moment explosion with critical moment λ_+, i.e.

$$p^* := \sup\left(s \in \mathbb{R}_+ : \ \mathbb{E}\left[(S_T)^s\right] < \infty\right) = \lambda_+ < \infty.$$

From Roger Lee's moment formula and [BF08, Example 5.3], it is known that V has the following asymptotics

$$\lim_{k\to\infty} \frac{V(k)}{k^{1/2}} = \Psi^{1/2}(\lambda_+ - 1), \tag{1.2}$$

where $\Psi(x)$ is defined by

$$\Psi(x) = 2 - 4(\sqrt{x^2 + x} - x). \tag{1.3}$$

For the Merton Jump Diffusion model all moments of S_T exist, so the above mentioned procedure cannot be applied. But as it was shown in [BF09, Example 5.4], the implied volatility $V(k)$ satisfies

$$V(k) \sim \frac{\sqrt{\delta k}}{\sqrt{2\sqrt{2\log k}}},$$

where $\delta > 0$ is the standard deviation of the jumps (see Section 3.1).
These expansions will be refined in this thesis by using the saddle-point approximation. The starting point for this method is always an integral representation of the call price (respectively density and tail probability), which we get by using the Mellin transform. The $\mathcal{O}$-notation and the Mellin transform are used throughout the thesis, therefore some basic results are explained in the appendix.

Finally, some frequently used notation is defined

$$f \ll g \Leftrightarrow f = \mathcal{O}(g),$$
$$f \sim g \Leftrightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

The moment generating and cumulant generating function of a stochastic process $(X_t)_{t\in\mathbb{R}}$ are defined as

$$\begin{aligned} M(s,T) &:= \mathbb{E}\left[\exp(sX_T)\right], \\ m(s,T) &:= \log M(s,T) \end{aligned}$$

for all $s \in \mathbb{R}$, wherever this expectation exists, and $T > 0$ being the maturity of the option.

2. Kou model

2.1. Model definition

The Kou model ([K02]) is an exponential Lévy jump diffusion model as in (1.1) with $(Y_i)_{i\in\mathbb{N}}$ being a sequence of iid double exponential random variables. So for $i \in \mathbb{N}$ the Y_i have the density

$$f(y) = p\lambda_+ e^{-\lambda_+ y}\mathbb{1}_{[0,\infty)}(y) + (1-p)\lambda_- e^{\lambda_- y}\mathbb{1}_{(-\infty,0)}(y)$$

with parameters $\lambda_+ > 1$, $\lambda_- > 0$ and $p \in (0,1)$.
We have

$$S_T = e^{X_T},$$

where the log-price X_T is determined by its moment generating function (see for example [GG13, p.7])

$$M(s,T) = \mathbb{E}\left[\exp(sX_T)\right] = \exp\left[T\left(\frac{\sigma^2 s^2}{2} + bs + \lambda\left(\frac{\lambda_+ p}{\lambda_+ - s} + \frac{\lambda_-(1-p)}{\lambda_- + s} - 1\right)\right)\right], \quad (2.1)$$

where $\lambda > 0$ is the jump intensity and $\sigma > 0$ the diffusion volatility in (1.1). It follows that the cumulant generating function is

$$m(s,T) = \log \mathbb{E}\left[e^{sX_T}\right] = T\left(\frac{\sigma^2 s^2}{2} + bs + \lambda\left(\frac{\lambda_+ p}{\lambda_+ - s} + \frac{\lambda_-(1-p)}{\lambda_- + s} - 1\right)\right).$$

The parameter $b \in \mathbb{R}$ is chosen, such that S_T becomes a martingale. A sufficient condition for this is (see [CT04])

$$\mathbb{E}\left[S_T\right] = M(1,T) = \mathbb{E}\left[S_0\right] = 1. \quad (2.2)$$

Hence the parameter b has to be chosen such that $m(1,T) = 0$, so it must satisfy

$$b = -\left(\frac{\sigma^2}{2} + \lambda\left(\frac{\lambda_+ p}{\lambda_+ - 1} + \frac{\lambda_-(1-p)}{\lambda_- + 1} - 1\right)\right).$$

Remark 2.1 There is a typo for the characteristic function in [CT04, Table 4.3, p.124].

Before starting the saddle-point approximation, let us summarize the two main results of this chapter. Let

$$\alpha_1 = \lambda_+ - 1, \quad \alpha_{1/2} = -2(\lambda\lambda_+ pT)^{1/2}$$

and
$$\alpha_0 = -\log \frac{e^{\frac{T\sigma^2\lambda_+^2}{2} + bT\lambda_+ + \frac{T\lambda\lambda_-(1-p)}{\lambda_- + \lambda_+} - \lambda T}(\lambda\lambda_+ pT)^{1/4}}{2\sqrt{\pi}\lambda_+(\lambda_+ - 1)}. \tag{2.3}$$

Theorem 2.2 *Let the parameters of the models as well as the maturity be fixed. Then the price of a European call option satisfies for every $\epsilon > 0$*

$$C(k,T) = \exp\left(-\alpha_1 k - \alpha_{1/2} k^{-1/2} - \alpha_0\right) k^{-3/4}\left(1 + \mathcal{O}\left(k^{-1/4+\epsilon}\right)\right)$$

as $k \to \infty$.

Theorem 2.3 *Under the assumptions of Theorem 2.2, the implied volatiliy satisfies for every $\epsilon > 0$ the asymptotic formula (as $k \to \infty$)*

$$V(k) = \beta_{1/2} k^{1/2} + \beta_0 + \beta_{\ell-1/2}\frac{\log k}{k^{1/2}} + \beta_{-1/2}\frac{1}{k^{1/2}} + \mathcal{O}\left(\frac{1}{k^{3/4-\epsilon}}\right),$$

where

$$\beta_{1/2} = -2\gamma\sqrt{\alpha_1^2 + \alpha_1} = \Psi^{1/2}(\lambda_+ - 1), \quad \beta_0 = \gamma\alpha_{1/2}, \quad \beta_{\ell-1/2} = \frac{\gamma}{4},$$

$$\beta_{-1/2} = \left(\alpha_0 + \log\frac{1 - (1 + \frac{1}{\alpha_1})^{-1/2}}{\sqrt{4\pi\alpha_1}}\right)\gamma + \left(\frac{1}{2(2\alpha_1)^{3/2}} - \frac{1}{2(2\alpha_1 + 2)^{3/2}}\right)\alpha_{1/2}^2$$

and

$$\gamma = \left(\frac{1}{\sqrt{2\alpha_1 + 2}} - \frac{1}{\sqrt{2\alpha_1}}\right).$$

Theorem 2.3 shows clearly a refinement of the expansion in (1.2)

$$V(k) \sim \Psi(\lambda_+ - 1)k^{1/2}.$$

In Section 2.4 some numerical examples are given and it will be verified that the expansion does give a better approximation of the volatility smile for large k values.

2.2. Call price

First an expansion for the price of a European call option with strike K will be derived (as $k \to \infty$). The starting point is the integral representation of a call price. Using the Mellin transform of $C(k,T)$, the inversion theorem implies (see Lemma B.5 in the appendix)

$$C(k,T) = \frac{e^k}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-ks}\frac{M(s,T)}{s(s-1)}ds, \quad k = \log K, \tag{2.4}$$

whenever $1 < c < \lambda_+$. Here $i = \sqrt{-1}$ is the imaginary unit.

2.2.1. Saddle-point

The idea of the considered method is to choose a saddle-point of the integrand for c in (2.4) and to estimate the central part of the integral with the Laplace method.
It is easy to see that $M(s,T)$ has two singularities, at $-\lambda_-$ and λ_+. As the call-price is an integral on the positive side of the complex plane, we consider the blowup at λ_+. So we get an approximate saddle-point by just considering the dominant term

$$\frac{\partial e^{-ks}\exp\left(T\lambda\frac{\lambda_+ p}{\lambda_+ - s}\right)}{\partial s} = 0.$$

A simple calculation yields the solution of this equation

$$\hat{s} = \lambda_+ - \sqrt{\frac{\lambda\lambda_+ pT}{k}},$$

which is greater than 1 for sufficiently large k. Using the notation $\xi = \lambda\lambda_+ pT$, we get

$$\hat{s} = \lambda_+ - \xi^{1/2}k^{-1/2} = \lambda_+ + \mathcal{O}\left(k^{-1/2}\right). \tag{2.5}$$

Remark 2.4 The approximate saddle-point actually depends on the log-strike k. We write $\hat{s}$ instead of $\hat{s}(k)$ for ease of notation .

Before doing the central expansion, one must determine the asymptotics of the cumulant generating function. This is necessary to determine the interval around the saddlepoint, so we can apply the Laplace method.

Lemma 2.5 *The cumulant generating function of X_T satisfies*

$$\begin{aligned}
m(\hat{s},T) &= \frac{T\sigma^2\lambda_+^2}{2} + b\lambda_+ T + \xi^{1/2}k^{1/2} + \frac{T\lambda\lambda_-(1-p)}{\lambda_- + \lambda_+} - \lambda T + \mathcal{O}\left(k^{-1/2}\right), \\
m'(\hat{s},T) &= k + \mathcal{O}(1), \\
m''(\hat{s},T) &= 2\xi^{-1/2}k^{3/2} + \mathcal{O}(1)
\end{aligned}$$

and

$$m'''(\hat{s}+it,T) = \mathcal{O}\left(k^2\right) \quad \text{for } |t| < k^{-\alpha}, \quad \alpha > 0,$$

where all derivatives are with respect to s.

Proof: We have for $-\lambda_- < s < \lambda_+$

$$\begin{aligned}
m(s,T) &= T\left(\frac{\sigma^2 s^2}{2} + bs + \lambda\left(\frac{\lambda_+ p}{\lambda_+ - s} + \frac{\lambda_-(1-p)}{\lambda_- + s} - 1\right)\right), \\
m'(s,T) &= T\left(\sigma^2 s + b + \lambda\left(\frac{\lambda_+ p}{(\lambda_+ - s)^2} - \frac{\lambda_-(1-p)}{(\lambda_- + s)^2}\right)\right), \\
m''(s,T) &= T\left(\sigma^2 + 2\lambda\left(\frac{\lambda_+ p}{(\lambda_+ - s)^3} + \frac{\lambda_-(1-p)}{(\lambda_- + s)^3}\right)\right)
\end{aligned}$$

and

$$m'''(s,T) = 6\lambda T\left(\frac{\lambda_+ p}{(\lambda_+ - s)^4} - \frac{\lambda_-(1-p)}{(\lambda_- + s)^4}\right).$$

Inserting $\hat{s} = \lambda_+ - \xi^{1/2}k^{-1/2}$ yields the asymptotics

$$\begin{aligned} m(\hat{s},T) &= T\left(\frac{\sigma^2\hat{s}^2}{2} + b\hat{s} + \lambda\left(\frac{\lambda_+ p}{\lambda_+ - \hat{s}} + \frac{\lambda_-(1-p)}{\lambda_- + \hat{s}} - 1\right)\right) \\ &= T\left(\frac{\sigma^2(\lambda_+ - \xi^{1/2}k^{-1/2})^2}{2} + b\lambda_+ + \lambda\left(\frac{\lambda_+ p}{\xi^{1/2}k^{-1/2}} + \frac{\lambda_-(1-p)}{\lambda_- + \lambda_+ - \xi^{1/2}k^{-1/2}} - 1\right)\right) \\ &\quad + \mathcal{O}\left(k^{-1/2}\right). \end{aligned}$$

Here we use $(1+x)^\omega = 1 + \mathcal{O}(x)$ for small x and $\omega \in \mathbb{R}$. Hence we get,

$$\begin{aligned} m(\hat{s},T) &= \frac{T\sigma^2\lambda_+^2}{2} + b\lambda_+ T + \xi^{1/2}k^{1/2} + T\lambda\lambda_-(1-p)\frac{1}{\lambda_- + \lambda_+}\left(1 - \frac{\xi^{1/2}k^{-1/2}}{\lambda_- + \lambda_+}\right)^{-1} - \lambda T \\ &\quad + \mathcal{O}\left(k^{-1/2}\right) \\ &= \frac{T\sigma^2\lambda_+^2}{2} + b\lambda_+ T + \xi^{1/2}k^{1/2} + \frac{T\lambda\lambda_-(1-p)}{\lambda_- + \lambda_+} - \lambda T + \mathcal{O}\left(k^{-1/2}\right). \end{aligned}$$

Similarly, one gets by using again the same argument from Lemma A.3

$$\begin{aligned} m'(\hat{s},T) &= T\left(\frac{\sigma^2\left(\lambda_+ - \xi^{1/2}k^{-1/2}\right)^2}{2} + b + \lambda\left(\frac{\lambda_+ p}{\xi k^{-1}} + \frac{\lambda_-(1-p)}{\left(\lambda_- + \lambda_+ + \xi^{1/2}k^{-1/2}\right)^2}\right)\right) \\ &= \mathcal{O}(1) + k + \frac{T\lambda\lambda_-(1-p)}{(\lambda_- + \lambda_+)^2}\left(1 + \frac{\xi^{1/2}k^{-1/2}}{\lambda_+ + \lambda_-}\right)^{-2} \\ &= k + \mathcal{O}(1), \end{aligned}$$

as well as

$$\begin{aligned} m''(\hat{s},T) &= T\left(\sigma^2 + 2\lambda\left(\frac{\lambda_+ p}{(\xi^{1/2}k^{-1/2})^3} + \frac{\lambda_-(1-p)}{(\lambda_- + \lambda_+ + \xi^{1/2}k^{-1/2})^3}\right)\right) \\ &= T\sigma^2 + 2\xi^{-1/2}k^{3/2} + \frac{T\lambda\lambda_-(1-p)}{(\lambda_- + \lambda_+)^3}\left(1 + \frac{\xi^{1/2}k^{-1/2}}{\lambda_- + \lambda_+}\right)^{-3} \\ &= T\sigma^2 + 2\xi^{-1/2}k^{3/2} + \frac{T\lambda\lambda_-(1-p)}{(\lambda_- + \lambda_+)^3} + \mathcal{O}(k^{-1/2}) = 2\xi^{-1/2}k^{3/2} + \mathcal{O}(1). \end{aligned}$$

For the third derivative we need to consider $s = \hat{s} + it$, in order to estimate the error term of the Taylor approximation in the Laplace method. So we get

$$\begin{aligned} \left|m'''(s,T)\right| &= 6\lambda T\left|\frac{\lambda_+ p}{(\xi^{1/2}k^{-1/2} - it)^4} + \frac{\lambda_-(1-p)}{(\lambda_- + \lambda_+ - \xi^{1/2}k^{-1/2} + it)^4}\right| \\ &\leq \frac{6\xi}{\left|\xi^{1/2}k^{-1/2} - it\right|^4} + \frac{\lambda_-(1-p)}{\left|\lambda_- + \lambda_+ - \xi^{1/2}k^{-1/2} + it\right|^4} \\ &\leq \frac{6\xi}{\left|\xi^{1/2}k^{-1/2}\right|^4} + \frac{\lambda_-(1-p)}{\lambda_- + \lambda_+}\left(1 + \mathcal{O}\left(k^{-1/2}\right)\right) = \mathcal{O}\left(k^2\right). \end{aligned}$$

■

Now we have derived the asymptotics of the cumulant generating function and its derivatives. This is crucial to find an interval for the central approximation and to apply Taylors theorem on $M(\hat{s}+it,T)$.

2.2.2. Central approximation

In the next step, the interval around the saddle-point $\hat{s}$ has to be chosen. Following the heuristic argument in [FS09, p.554], the length of the interval around the saddle-point (denoted by δ) has to be chosen such that the conditions

$$\begin{aligned} m''(\hat{s},T)\delta^2 &\xrightarrow[k\to\infty]{} \infty, \\ m'''(\hat{s},T)\delta^3 &\xrightarrow[k\to\infty]{} 0 \end{aligned}$$

are fulfilled. These conditions ensure that one gets a Gaussian integral for the second derivative and that error term of the Taylor approximation (depending on the third derivative $m'''(\hat{s}+it,T)$ and some $t\in\delta$) becomes arbitrarily small for large log-strikes k.
Considering intervals of the form $(\hat{s}-k^{-\alpha},\hat{s}+k^{-\alpha})$ together with the results of Lemma 2.5 , yields the conditions

$$\begin{aligned} m''(\hat{s},T)4k^{-2\alpha} &= \mathcal{O}\left(k^{3/2-2\alpha}\right) \xrightarrow[k\to\infty]{} \infty \\ m'''(\hat{s},T)8k^{-3\alpha} &= \mathcal{O}\left(k^{2-3\alpha}\right) \xrightarrow[k\to\infty]{} 0. \end{aligned} \tag{2.6}$$

Therefore the parameter α must fulfill $\frac{2}{3}<\alpha<\frac{3}{4}$. After having determined the interval $(\hat{s}-k^{-\alpha},\hat{s}+k^{-\alpha})$, we are ready to show the asymptotics of the central part in (2.4).

Lemma 2.6 *Let $2/3<\alpha<3/4$. Then*

$$\begin{aligned} &\frac{e^k}{2\pi i}\int_{\hat{s}-ik^{-\alpha}}^{\hat{s}+ik^{-\alpha}} e^{-ks}\frac{M(s,T)}{s(s-1)}ds \\ &= \frac{e^{\frac{T\sigma^2\lambda_+^2}{2}+bT\lambda_+ + \frac{T\lambda\lambda_-(1-p)}{\lambda_-+\lambda_+}-\lambda T}\xi^{1/4}}{2\sqrt{\pi}\lambda_+(\lambda_+-1)} e^{k(1-\lambda_+)+2\xi^{1/2}k^{1/2}}k^{-3/4}\left(1+\mathcal{O}\left(k^{-3\alpha+2}\right)\right). \end{aligned}$$

Proof: Applying Taylor's theorem and using the results from Lemma 2.5 yields

$$m(\hat{s}+it,T)=m(\hat{s},T)+im'(\hat{s},T)t-m''(\hat{s},T)t^2/2+\mathcal{O}\left(t^3k^2\right).$$

where $|t|<k^{-\alpha}$. Using $m'(\hat{s},T)=k+\mathcal{O}(1)$ and that $e^x=1+\mathcal{O}(x)$ for small x, we get

$$\begin{aligned} M(\hat{s}+it,T) &= \exp\left(m(\hat{s}+it,T)\right) \\ &= \exp\left(m(\hat{s},T)+im'(\hat{s},T)t-\frac{m''(\hat{s},T)}{2}t^2+\mathcal{O}\left(t^3k^2\right)\right) \\ &= \exp\left(m(\hat{s},T)+itk+\mathcal{O}(t)-\frac{m''(\hat{s},T)}{2}t^2+\mathcal{O}\left(t^3k^2\right)\right) \\ &= \exp\left(m(\hat{s},T)+itk-\frac{m''(\hat{s},T)}{2}t^2\right)\left(1+\mathcal{O}\left(t^3k^2\right)\right), \quad |t|<k^{-\alpha}, \end{aligned}$$

and by using $(1+x)^{-1} = 1 + \mathcal{O}(x)$ for small x, we obtain

$$\begin{aligned}\frac{1}{(\hat{s}+it)(\hat{s}+it-1)} &= \frac{1}{\lambda_+}\left(1-\frac{\xi^{1/2}k^{-1/2}-it}{\lambda_+}\right)^{-1}\frac{1}{\lambda_+-1}\left(1-\frac{\xi^{1/2}k^{-1/2}-it}{\lambda_+-1}\right)^{-1}\\ &= \frac{1}{\lambda_+(\lambda_+-1)}\left(1+\mathcal{O}\left(k^{-1/2}\right)\right).\end{aligned}$$

Thus we get

$$\begin{aligned}\frac{e^k}{2\pi i}\int_{\hat{s}-ik^{-\alpha}}^{\hat{s}+ik^{-\alpha}} e^{-ks}\frac{M(s,T)}{s(s-1)}ds &= \frac{e^{k(1-\hat{s})}}{2\pi}\int_{-k^{-\alpha}}^{k^{-\alpha}} e^{-itk}\frac{M(\hat{s}+it,T)}{(\hat{s}+it)(\hat{s}+it-1)}dt\\ = \frac{e^{k(1-\hat{s})}M(\hat{s},T)}{2\pi\lambda_+(\lambda_+-1)}&\int_{-k^{-\alpha}}^{k^{-\alpha}}\exp\left(-\frac{m''(\hat{s},T)}{2}t^2\right)\left(1+\mathcal{O}\left(k^{-3\alpha+2}\right)\right)dt.\end{aligned}$$

Here we used the second condition in (2.6), which ensures that the error term $\mathcal{O}\left(k^{-3\alpha+2}\right)$ tends to zero. Setting $u := m''(\hat{s},T)^{1/2}$, we get from Lemma 2.5

$$u = \sqrt{\frac{2k^{3/2}}{\xi^{1/2}}\left(1+\mathcal{O}\left(k^{-3/2}\right)\right)} = \frac{\sqrt{2}k^{3/4}}{\xi^{1/4}}\left(1+\mathcal{O}\left(k^{-3/2}\right)\right)$$

and

$$\frac{1}{u} = \frac{\xi^{1/4}}{\sqrt{2}k^{3/4}}\left(1+\mathcal{O}\left(k^{-3/2}\right)\right).$$

By substituting $\omega = ut$ and using the fact that Gaussian integrals have exponentially decaying tails, i.e. $\int_c^\infty e^{-t^2/2}dt = \mathcal{O}\left(e^{-c^2/2}\right)$, $c>0$ (see [FS09, p.557]), we get

$$\begin{aligned}\int_{-k^{-\alpha}}^{k^{-\alpha}}\exp\left(-\frac{m''(\hat{s},T)}{2}t^2\right)dt &= \frac{1}{u}\int_{-uk^{-\alpha}}^{uk^{-\alpha}}\exp\left(-\frac{\omega^2}{2}\right)d\omega\\ &= \frac{1}{u}\left(\int_{-\infty}^{\infty}\exp\left(-\frac{\omega^2}{2}\right)d\omega + \mathcal{O}\left(e^{-u^2/(2k^{2\alpha})}\right)\right)\\ &= \frac{\sqrt{2\pi}}{u}\left(1+\mathcal{O}\left(e^{-u^2/(2k^{2\alpha})}\right)\right)\\ &= \sqrt{\pi}k^{-3/4}\xi^{1/4}\left(1+\mathcal{O}\left(k^{-3/2}\right)\right).\end{aligned}$$

Here we used the first condition in (2.6), which ensures that $uk^{-\alpha}$ tends to infinity, so we get Gaussian integrals with exponentially decaying tails.
This implies

$$\begin{aligned}\frac{e^k}{2\pi i}\int_{\hat{s}-ik^{-\alpha}}^{\hat{s}+ik^{-\alpha}} e^{-ks}\frac{M(s,T)}{s(s-1)}ds &= \frac{e^{k(1-\hat{s})}M(\hat{s},T)\xi^{1/4}}{2\sqrt{\pi}\lambda_+(\lambda_+-1)}k^{-3/4}\left(1+\mathcal{O}\left(k^{-3/2}\right)+\mathcal{O}\left(k^{-3\alpha+2}\right)\right)\\ &= \frac{e^{k(1-\hat{s})}M(\hat{s},T)\xi^{1/4}}{2\sqrt{\pi}\lambda_+(\lambda_+-1)}k^{-3/4}\left(1+\mathcal{O}\left(k^{-3\alpha+2}\right)\right).\end{aligned}$$

Finally we use Lemma 2.5 to get

$$M(\hat{s},T) = \exp\left(\frac{T\sigma^2\lambda_+^2}{2} + bT\lambda_+ + \xi^{1/2}k^{1/2} + \frac{T\lambda\lambda_-(1-p)}{\lambda_-+\lambda_+} - \lambda T\right)\left(1+\mathcal{O}\left(k^{-1/2}\right)\right).$$

Hence, the integral is equal to

$$\frac{e^{\frac{T\sigma^2\lambda_+^2}{2}+bT\lambda_+ +\frac{T\lambda\lambda_-(1-p)}{\lambda_-+\lambda_+}-\lambda T}\xi^{1/4}}{2\sqrt{\pi}\lambda_+(\lambda_+-1)}e^{k(1-\lambda_+)+2\xi^{1/2}k^{1/2}}k^{-3/4}(1+\mathcal{O}\left(k^{-3\alpha+2}\right))$$

and we get the desired result.

■

2.2.3. Estimation of the tails

To complete the saddle-point approximation, one has to show that the tails of the integral are negligible, i.e.

$$\int_{\hat{s}-i\infty}^{\hat{s}-ik^{-\alpha}} e^{-ks}\frac{M(s,T)}{s(s-1)}ds = o\left(\int_{\hat{s}-ik^{-\alpha}}^{\hat{s}+ik^{-\alpha}} e^{-ks}\frac{M(s,T)}{s(s-1)}ds\right)$$

and

$$\int_{\hat{s}+ik^{-\alpha}}^{\hat{s}+i\infty} e^{-ks}\frac{M(s,T)}{s(s-1)}ds = o\left(\int_{\hat{s}-ik^{-\alpha}}^{\hat{s}+ik^{-\alpha}} e^{-ks}\frac{M(s,T)}{s(s-1)}ds\right),$$

(see Definition A.2 for the o-notation).

Lemma 2.7 *Let $2/3 < \alpha < 3/4$. Then we have*

$$\frac{e^k}{2\pi i}\int_{\hat{s}+ik^{\alpha}}^{\hat{s}+i\infty} e^{-ks}\frac{M(s,T)}{s(s-1)}ds \ll e^{k(1-\lambda_+)+2\xi^{1/2}k^{1/2}-\xi^{-1/2}k^{3/2-2\alpha}/2}$$

and

$$\frac{e^k}{2\pi i}\int_{\hat{s}-i\infty}^{\hat{s}-ik^{-\alpha}} e^{-ks}\frac{M(s,T)}{s(s-1)}ds \ll e^{k(1-\lambda_+)+2\xi^{1/2}k^{1/2}-\xi^{-1/2}k^{3/2-2\alpha}/2}.$$

Proof: We just consider the first integral, the proof of the second one goes the same way. Let $s = \hat{s}+it = \lambda_+ - \xi^{1/2}k^{-1/2}+it$, where $t \geq k^{-\alpha}$. Then we have (as $\text{Re}(s) = \lambda_+ - \xi^{1/2}k^{-1/2} > 1$ for sufficiently big k)

$$\begin{aligned} |s(s-1)| &\geq |\text{Im}(s)|\,|\text{Im}(s-1)| = t^2 \\ |s(s-1)| &\geq |\text{Re}(s)|\,|\text{Re}(s-1)| \gg 1 \end{aligned}$$

Hence, we get

$$|s(s-1)| \gg 1+t^2. \tag{2.7}$$

Furthermore, we have

$$\begin{aligned} |M(s,T)| &\ll \exp\left(\text{Re}\left(\frac{\xi}{\xi^{1/2}k^{-1/2}-it}\right)\right) \\ &= \exp\left(\frac{\xi^{3/2}k^{-1/2}}{\xi k^{-1}+t^2}\right) = \exp\left(\frac{\xi^{1/2}k^{1/2}}{1+t^2\xi^{-1}k}\right) \ll \exp\left(\frac{\xi^{1/2}k^{1/2}}{1+k^{1-2\alpha}\xi^{-1}}\right). \end{aligned}$$

Using the fact $1/(1+x) \leq 1 - x/2$ for $x \leq 1$ and that $k^{1-2\alpha}$ is smaller than 1 for sufficently big k, we get

$$|M(s,T)| \ll \exp\left(\xi^{1/2}k^{1/2} - \xi^{-1/2}k^{3/2-2\alpha}/2\right). \tag{2.8}$$

So we get from (2.7) and (2.8)

$$\frac{e^k}{2\pi}\left|e^{-ks}\frac{M(s,T)}{s(s-1)}\right| \ll \frac{e^{k(1-\lambda_+)+2\xi^{1/2}k^{1/2}-\xi^{-1/2}k^{3/2-2\alpha}/2}}{1+t^2},$$

and obtain

$$\begin{aligned}\frac{e^k}{2\pi i}\int_{\hat{s}+ik^{-\alpha}}^{\hat{s}+i\infty} e^{-ks}\frac{M(s,T)}{t(s-1)}ds &\ll e^{k(1-\lambda_+)+2\xi^{1/2}k^{1/2}-\xi^{-1/2}k^{3/2-2\alpha}/2}\int_{k^{-\alpha}}^{\infty}\frac{dt}{1+t^2}\\ &\ll e^{k(1-\lambda_+)+2\xi^{1/2}k^{1/2}-\xi^{-1/2}k^{3/2-2\alpha}/2}.\end{aligned}$$

■

After having derived an approximate saddle-point in (2.5), a central approximation in Lemma 2.6 and estimating the remaining tails in Lemma 2.7, we are ready to prove the first central result.

Proof (*Theorem 2.2*): We have

$$\begin{aligned} C(k,T) &= \frac{e^k}{2\pi i}\int_{\hat{s}-i\infty}^{\hat{s}+i\infty} e^{-ks}\frac{M(s,T)}{s(s-1)}ds \\ &= \frac{e^k}{2\pi i}\int_{\hat{s}-i\infty}^{\hat{s}-ik^{-\alpha}} e^{-ks}\frac{M(s,T)}{s(s-1)}ds + \frac{e^k}{2\pi i}\int_{\hat{s}-ik^{-\alpha}}^{\hat{s}+ik^{-\alpha}} e^{-ks}\frac{M(s,T)}{s(s-1)}ds \\ &\quad + \frac{e^k}{2\pi i}\int_{\hat{s}+ik^{-\alpha}}^{\hat{s}+i\infty} e^{-ks}\frac{M(s,T)}{s(s-1)}ds. \end{aligned}$$

The second integral gives the main term, while the two others are asymptotically of smaller order. Hence,

$$\begin{aligned} C(k,T) &= \frac{e^{\frac{T\sigma^2\lambda_+^2}{2}+bT\lambda_+ + \frac{T\lambda\lambda_-(1-p)}{\lambda_-+\lambda_+}-\lambda T}\xi^{1/4}}{2\sqrt{\pi}\lambda_+(\lambda_+-1)} e^{k(1-\lambda_+)+2\xi^{1/2}k^{1/2}}k^{-3/4}\Bigg(1+\mathcal{O}\left(k^{-3\alpha+2}\right) \\ &\quad + \mathcal{O}\left(k^{3/4}e^{-\xi^{-1/2}k^{3/2-2\alpha}}\right)\Bigg) \\ &= \frac{e^{\frac{T\sigma^2\lambda_+^2}{2}+bT\lambda_+ + \frac{T\lambda\lambda_-(1-p)}{\lambda_-+\lambda_+}}\xi^{1/4}}{2\sqrt{\pi}\lambda_+(\lambda_+-1)} e^{k(1-\lambda_+)+2\xi^{1/2}k^{1/2}}k^{-3/4}\left(1+\mathcal{O}\left(k^{-3\alpha+2}\right)\right), \\ &= \exp\left(-\alpha_1 k - \alpha_{1/2}k^{-1/2} - \alpha_0\right)k^{-3/4}\left(1+\mathcal{O}\left(k^{-3\alpha+2}\right)\right) \end{aligned}$$

where $2/3 < \alpha < 3/4$ and α_0, $\alpha_{1/2}$ and α_1 are the constants defined in (2.3). Let $\epsilon > 0$, then the choice of $\alpha = 3/4 - \epsilon/3$ proves the Theorem.

■

2.3. Implied volatility

To get an expansion for the implied volatility, we need the following result (see [GL, Corollary 7.11]).

Lemma 2.8 *If the absolute logarithm of the call price* $L := -\log C(k,T)$ *satisfies*

$$L = \alpha_1 k + \alpha_{1/2} k^{1/2} + a_\ell \log k + \alpha_0 + \mathcal{O}\left(k^{-r}\right),$$

where the α_i *are constants (with* $\alpha_1 > 0$*) for* $i = 0,\ 1/2,\ 1,\ \ell$ *and* $0 < r < 1/2$*, then the dimensionless implied volatility has the expansion*

$$V(k) = \beta_{1/2} k^{1/2} + \beta_0 + \beta_{\ell-1/2} \frac{\log k}{k^{1/2}} + \frac{\beta_{-1/2}}{k^{1/2}} + \mathcal{O}\left(\frac{1}{k^{r+1/2}}\right),$$

where

$$\begin{aligned}
\beta_{1/2} &:= \sqrt{2\alpha_1 + 2} - \sqrt{2\alpha_1}, \\
\beta_0 &:= \left(\frac{1}{\sqrt{2\alpha_1+2}} - \frac{1}{\sqrt{2\alpha_1}}\right) \alpha_{1/2}, \\
\beta_{\ell-1/2} &:= \left(\frac{1}{\sqrt{2\alpha_1+2}} - \frac{1}{\sqrt{2\alpha_1}}\right)\left(\alpha_\ell - \frac{1}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}
\beta_{-1/2} &:= \left(\frac{1}{\sqrt{2\alpha_1+2}} - \frac{1}{\sqrt{2\alpha_1}}\right)\left(\alpha_0 + \log \frac{1-(1+1/\alpha_1)^{-1/2}}{\sqrt{4\pi\alpha_1}}\right) \\
&\quad + \left(\frac{1}{2(2\alpha_1)^{3/2}} - \frac{1}{2(2\alpha_1+2)^{3/2}}\right)\alpha_{1/2}^2.
\end{aligned}$$

Remark 2.9 The absolute log of the call price is actually a function in k, but for ease of notation we write simply L instead of $L(k)$.

With this result, it is easy to prove Theorem 2.3 which gave an asymptotic expansion for the implied volatility $V(k)$.

Proof (*Theorem 2.3*):
From the results in Theorem 2.2, we see that the requirements for Lemma 2.8 are fulfilled with $\alpha_1 = \lambda_+ - 1 > 0$, $\alpha_\ell = 3/4$ and $r = 1/2 - \epsilon$.

■

Remark 2.10 Finally, it should be noted, that there already exist asymptotic expansions for the Kou model. In [AS09, Example 7.6], an expansion is given for the tail probability of X_T.

2.4. Numerical tests

2.4.1. Call price

Here we will test the accuracy of the constants in the expansion for the call price: $\alpha_0, \alpha_{1/2}$ and α_1. A test can be based on the results of Theorem 2.2. So we have

$$-\frac{\log(C(k,T))}{k} = \alpha_1 + \alpha_{1/2}k^{-1/2} + \alpha_0 k^{-1} + \frac{3\log k}{4k} + \mathcal{O}\left(k^{-5/4+\epsilon}\right) \xrightarrow[k\to\infty]{} \alpha_1, \tag{A1}$$

$$-\frac{\log\left(\exp\left(\alpha_1 k\right) C(k,T)\right)}{k^{1/2}} = \alpha_{1/2} + \alpha_0 k^{-1/2} + \frac{3\log k}{4k^{1/2}} + \mathcal{O}\left(k^{-3/4+\epsilon}\right) \xrightarrow[k\to\infty]{} \alpha_{1/2}, \tag{A2}$$

$$-\log\left(C(k,T)\exp\left(\alpha_1 k + \alpha_{1/2}k^{1/2}\right)k^{3/4}\right) = \alpha_0 + \mathcal{O}\left(k^{-1/4+\epsilon}\right) \xrightarrow[k\to\infty]{} \alpha_0. \tag{A3}$$

To calculate the call price we use its integral representation from (2.4) (with $c = \hat{s}$)

$$\begin{aligned} C(k,T) &= \frac{e^k}{2\pi i}\int_{\hat{s}-i\infty}^{\hat{s}+i\infty} e^{-ks}\frac{M(s,T)}{s(s-1)}ds \\ &= \frac{e^k}{2\pi}\int_{-\infty}^{\infty} e^{-k(\hat{s}+is)}\frac{M(\hat{s}+is,T)}{(\hat{s}+is)(\hat{s}-1+is)}ds. \end{aligned}$$

and integrate it numerically. We use the parameters in Table 2.1, which are taken from [M11, p.19] except for the riskfree interest rate r which is assumed to be zero.

r	T	σ	λ	λ_+	λ_-	p
0.0	1	0.1	5	15	15	0.5

Table 2.1.: Parameters for the Kou model.

First, we have to determine the parameter b, so that (2.2) is fulfilled and the stock process becomes a martingale. We get

$$b = -\left(\sigma^2/2 + \lambda\left(\frac{\lambda_+ p}{\lambda_+ - 1} + \frac{\lambda_-(1-p)}{\lambda_- + 1} - 1\right)\right) = -0.027$$

and so the constants are (rounded to three decimal places)

$$\alpha_0 = -8.741, \quad \alpha_{1/2} = -12.247 \text{ and } \alpha_1 = 14.$$

The plots of the numerical tests explained above are summarized in Figures 2.3 - 2.1. One sees, that there is only a very slow convergence (even for big values $k > 40$). The plot of the test (A3) for the coefficient of the least dominant term, α_0, does not show any convergence at all. The plotting range is from $k = 0$ till $k = 50$, which is an unrealistically high value for a log-strike. Nevertheless these results can be used to do some qualitativve analysis (see Chapter 4).
A reason for the slow convergence is, that we have used the approximated saddle-point $\hat{s}$, in order to get the expansion for the call price. Figure 2.4 shows the difference between the approximated and the numerically calculated exact saddle-point.

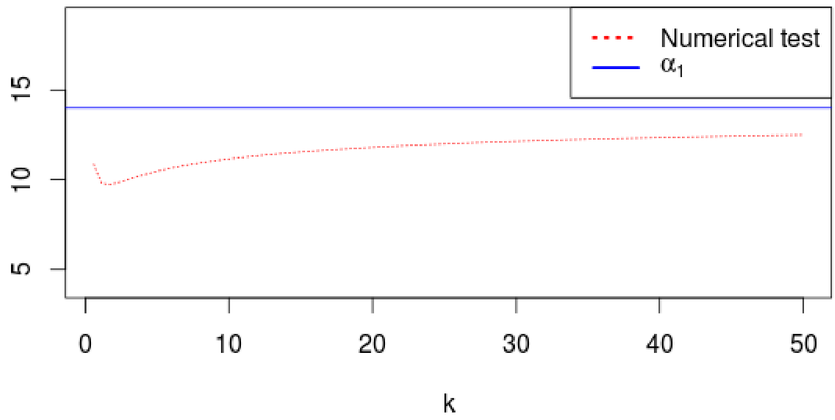


Figure 2.1.: Numerical check (A1) for the constant $\alpha_1 = 14$.

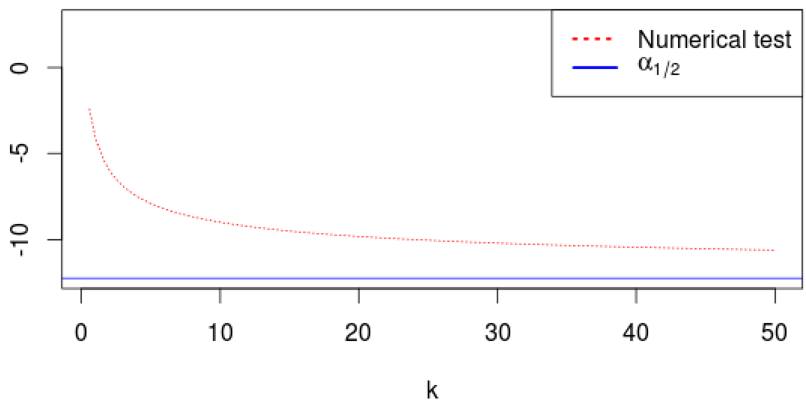


Figure 2.2.: Numerical check (A2) for the constant $\alpha_{1/2} = -12.247$.

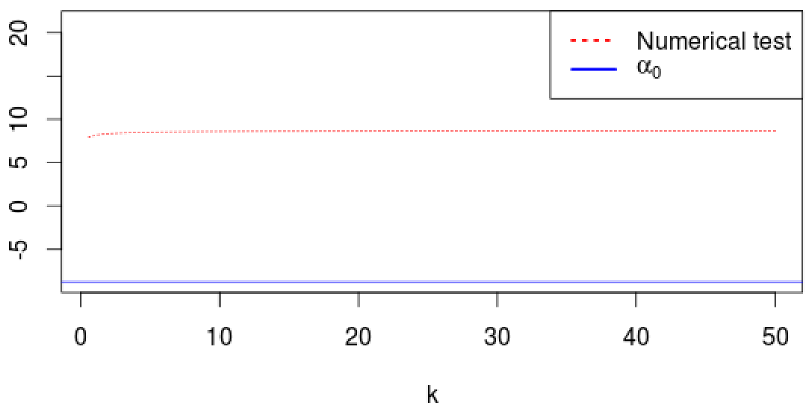


Figure 2.3.: Numerical check (A3) for the constant $\alpha_0 = -8.741$.

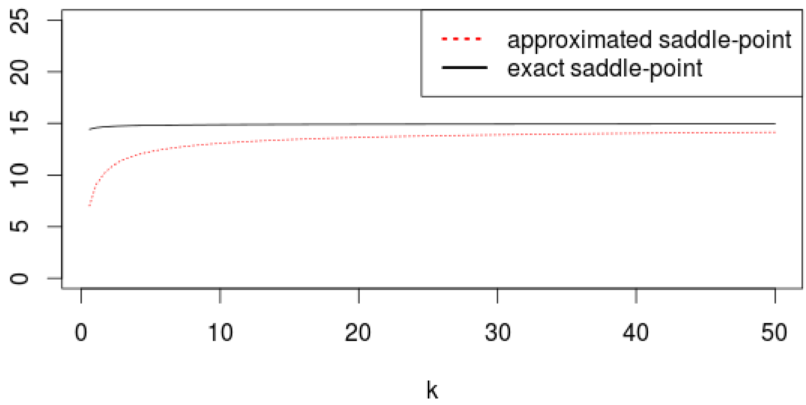


Figure 2.4.: Approximated saddle-point (dotted) and exact saddle-point(solid).

2.4.2. Implied volatility

The procedure to test the constansts of the expansion for $V(k)$ is not much different. From Theorem 2.3 , we get

$$\frac{V(k)}{k^{1/2}} = \beta_{1/2} + \frac{\beta_0}{k} + \frac{\beta_{-1/2}}{k} + \mathcal{O}\left(k^{-5/4+\epsilon}\right) \xrightarrow[k\to\infty]{} \beta_{1/2}, \tag{B1}$$

$$V(K) - \beta_{1/2}k^{1/2} = \beta_0 + \beta_{\ell-1/2}\frac{\log k}{k^{1/2}} + \frac{\beta_{-1/2}}{k^{1/2}} + \mathcal{O}\left(k^{-3/4+\epsilon}\right) \xrightarrow[k\to\infty]{} \beta_0, \tag{B2}$$

$$\frac{\left(V(k) - \beta_{1/2}k^{1/2} - \beta_0\right)k^{1/2}}{\log k} = \beta_{\ell-1/2} + \frac{\beta_{-1/2}}{\log k} + \mathcal{O}\left(\frac{k^{-1/4+\epsilon}}{\log k}\right) \xrightarrow[k\to\infty]{} \beta_{\ell-1/2}, \tag{B3}$$

$$\left(V(k) - \beta_{1/2}k^{1/2} - \beta_0 - \beta_{\ell-1/2}\frac{\log k}{k^{1/2}}\right)k^{1/2} = \beta_{-1/2} + \mathcal{O}\left(k^{-1/4+\epsilon}\right) \xrightarrow[k\to\infty]{} \beta_{-1/2}. \tag{B4}$$

By using the parameters from Table 2.1, we get for the constants (rounded to three decimal places)

$$\beta_0 = 0.078, \quad \beta_{1/2} = 0.186, \quad \beta_{\ell-1/2} = -0.002 \text{ and } \beta_{-1/2} = 0.144.$$

The plots of the numerical checks are summarized in Figures 2.5-2.8.
In contrast to the previous test for the call price, the convergence to the constants of the smile expansion is very good. Just the test for the coefficient of the least dominant term, $\beta_{-1/2}$, does not show much convergence. Here was the plotting range restricted to values of k between 0 and 4, which are still very high values for log-strikes.

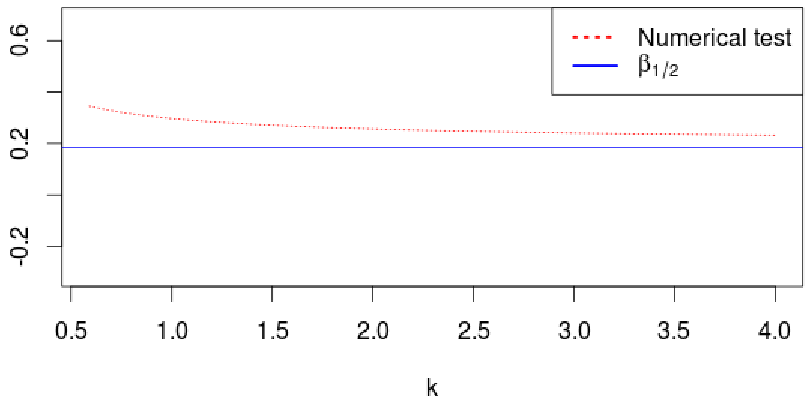


Figure 2.5.: Numerical check (B1) for the constant $\beta_{1/2} = 0.186$.

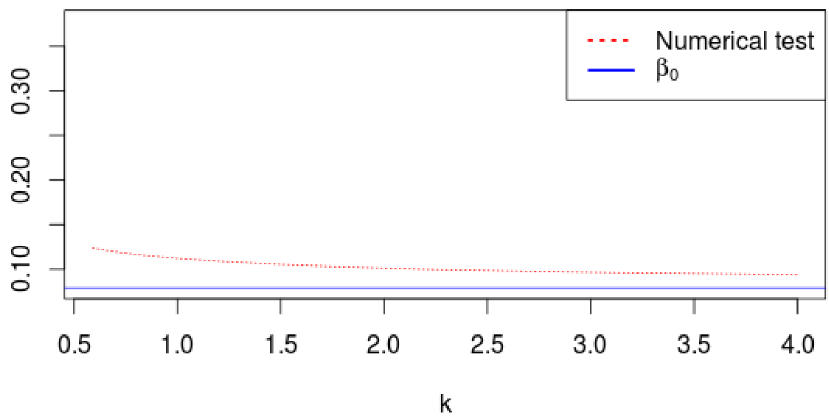


Figure 2.6.: Numerical check (B2) for the constant $\beta_0 = 0.078$.

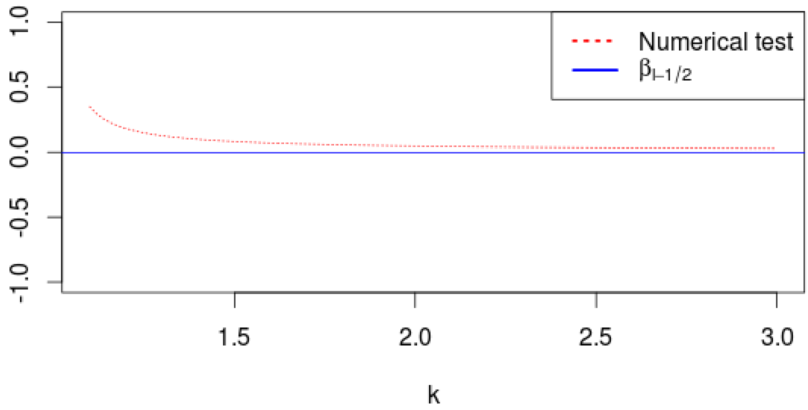


Figure 2.7.: Numerical check (B3) for the constant $\beta_{\ell-1/2} = -0.002$.

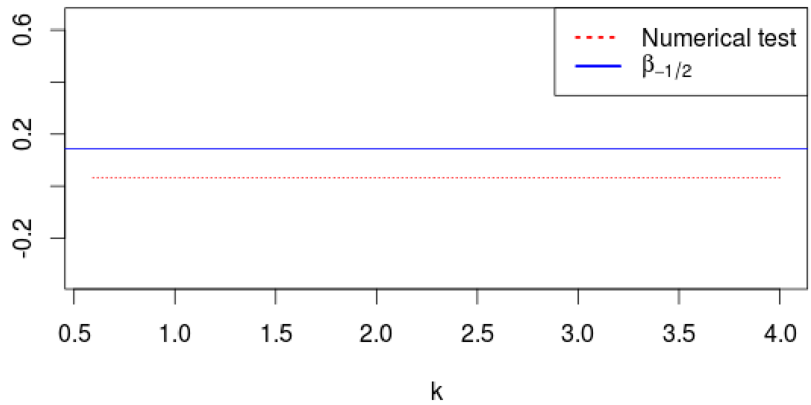


Figure 2.8.: Numerical check (B4) for the constant $\beta_{-1/2} = 0.144$.

Finally, the whole expansion for the implied volatility $V(k)$ from Theorem 2.3 is plotted and is compared with the exact value as well as with the first-order expansion (1.2), which was given in [BF08]. This is done for different parameters and maturities.

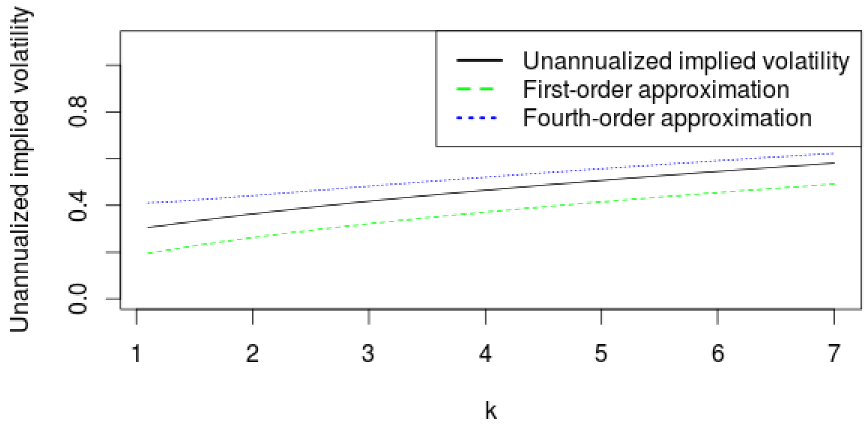


Figure 2.9.: Implied volatility (solid) compared with the fourth-order expansion (dotted) and Lee formula (dashed) in terms of log-strike. The parameters used here are $T = 1$, $\sigma = 0.1$, $\lambda = 5$, $\lambda_+ = 15$, $\lambda_- = 15$, $p = 0.5$.

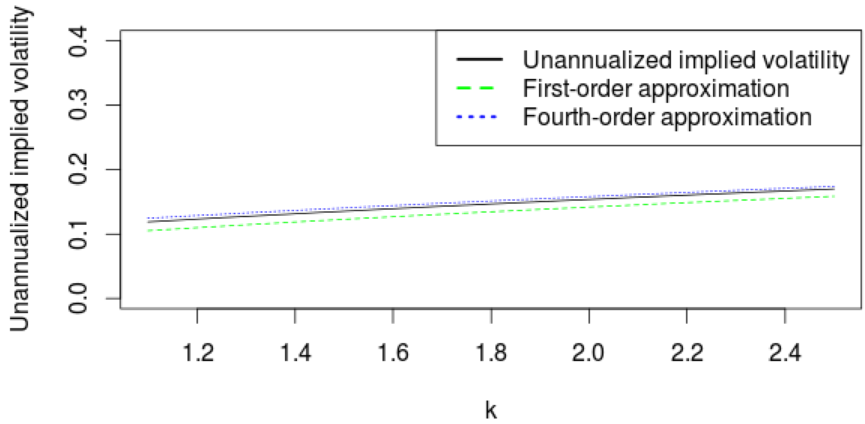


Figure 2.10.: Implied volatility (solid) compared with the fourth-order expansion (dotted) and Lee formula (dashed) in terms of log-strike. The parameters used here are $T = 0.1$, $\sigma = 0.2$, $\lambda = 10$, $p = 0.3$, $\lambda_- = 25$, $\lambda_+ = 50$.

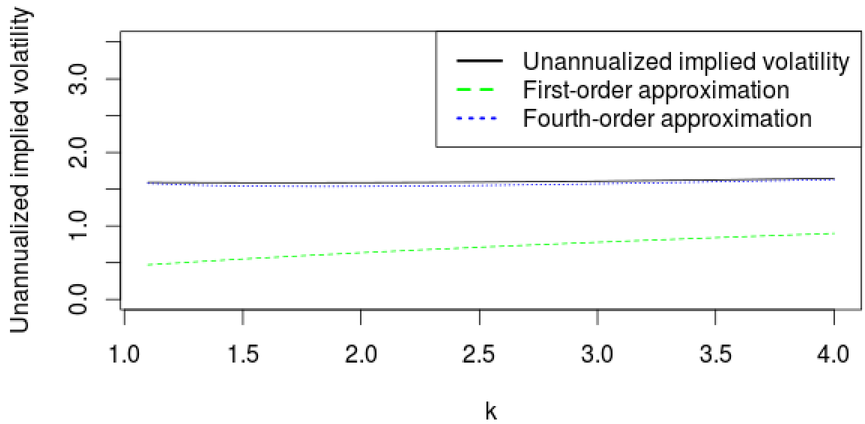


Figure 2.11.: Implied volatility (solid) compared with the fourth-order expansion (dotted) and Lee formula (dashed) in terms of log-strike. The parameters used here are $T = 6$, $\sigma = 0.4$, $\lambda = 1$, $p = 0.2$, $\lambda_- = 2$, $\lambda_+ = 3$.

3. Merton Jump Diffusion model

3.1. Model defintion and general results

The Merton Jump Diffusion model ([M76]) is as the Kou model an exponential Lévy jump diffusion model. The difference lies in the distribution of the jumps Y_i, which follow a Gaussian distribution (mean $\mu \in \mathbb{R}$ and variance $\delta^2 > 0$). So we have

$$S_T = e^{X_T}$$

and X_T is given by its moment generating function (see for example [FG13, p.5])

$$M(s,T) = \mathbb{E}\left[\exp\left(sX_T\right)\right] = \exp\left(T\left(\frac{1}{2}\sigma^2 s^2 + bs + \lambda(e^{\delta^2 s^2/2+\mu s} - 1\right)\right), \quad s \in \mathbb{R},$$

where $\sigma > 0$ is the diffusion volatility and $\lambda > 0$ the jump intensity of the Poisson process in (1.1). The parameter $b \in \mathbb{R}$ is chosen, such that

$$\mathbb{E}\left[S_T\right] = M(1,T) = \mathbb{E}\left[S_0\right] = 1$$

is fulfilled, which ensures that S_T becomes a martingale. Hence, we get

$$b = -\left(\sigma^2/2 + \lambda\left(e^{\delta^2/2+\mu} - 1\right)\right).$$

The cumulant generating function is hence

$$m(s,T) = \log M(s,T) = T\left(\frac{1}{2}\sigma^2 s^2 + bs + \lambda(e^{\delta^2 s^2/2+\mu s} - 1)\right), \quad s \in \mathbb{R}.$$

The moment generating function $M(s,T)$ exists for all $s \in \mathbb{R}$, hence the tail-wing formula of Lee is not applicable, as there is no moment explosion. But in [BF09, Section 5.4] is an asymptotic formula for the (unannualized) implied volatility in the Merton Jump Diffusion model given

$$V(k) \sim \delta \frac{k}{2\sqrt{2T\log k}}, \quad \text{as } k \to \infty.$$

The goal of this chapter is to refine this result.

Before starting with the saddle-point approximation, let us summarize the main results of this chapter.

Theorem 3.1 *Let the parameters as well as the maturity be fixed, then the call price satisfies for* $k \to \infty$

$$\begin{aligned} C(k,T) &= \frac{\delta^2 e^{k(1-\hat{s})} M(\hat{s},T)}{2\log k\sqrt{2\pi m''(\hat{s},T)}} \left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right) \\ &= \frac{\delta^2 e^{k(1-\hat{s})+T\left(\sigma^2\hat{s}^2/2+b\hat{s}+\lambda\left(e^{\delta^2\hat{s}^2/2+\mu\hat{s}}-1\right)\right)}}{2\log k\sqrt{2\pi T\left(\sigma^2+\lambda(\delta^2\hat{s}+\mu)\left((\delta^2\hat{s}+\mu)+\delta^2\right)e^{\delta^2\hat{s}^2/2+\mu\hat{s}}\right)}} \left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right), \end{aligned}$$

where $\hat{s}$ *is defined implicitely by* $m'(\hat{s},T)=k$.

Theorem 3.2 *Under the assumptions of Theorem 3.1, the density of* X_T *satisfies for* $x \to \infty$

$$\begin{aligned} f(x) &= \frac{e^{-x\hat{s}} M(\hat{s},T)}{\sqrt{2\pi m''(\hat{s},T)}} \left(1+\mathcal{O}\left(x^{-1/2+\epsilon}\right)\right) \\ &= \frac{e^{-x\hat{s}+T\left(\sigma^2/2\hat{s}^2+b\hat{s}+\lambda\left(e^{\delta^2/2\hat{s}^2+\mu\hat{s}}-1\right)\right)}}{\sqrt{2\pi T\left(\sigma^2+\lambda(\delta^2\hat{s}+\mu)\left((\delta^2\hat{s}+\mu)+\delta^2\right)e^{\delta^2\hat{s}^2/2+\mu\hat{s}}\right)}} \left(1+\mathcal{O}\left(x^{-1/2+\epsilon}\right)\right), \end{aligned}$$

where $\epsilon > 0$ *and* $\hat{s}$ *is implicitely defined by*

$$m'(\hat{s},T) = x.$$

Theorem 3.3 *Under the assumptions of Theorem 3.1, the tail probability of* X_T *satisfies for* $x \to \infty$

$$\begin{aligned} 1-F(x) &= \frac{\delta e^{-x\hat{s}} M(\hat{s},T)}{2\sqrt{\pi m''(\hat{s},T)}\log x} \left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log x}}\right)\right) \\ &= \frac{\delta e^{-x\hat{s}+T\left(\sigma^2/2\hat{s}^2+b\hat{s}+\lambda\left(e^{\delta^2\hat{s}^2/2+\mu\hat{s}}-1\right)\right)}}{2\sqrt{\log x\pi T\left(\sigma^2+\lambda(\delta^2\hat{s}+\mu)\left((\delta^2\hat{s}+\mu)+\delta^2\right)e^{\delta^2\hat{s}^2/2+\mu\hat{s}}\right)}} \left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log x}}\right)\right), \end{aligned}$$

where $\hat{s}$ *is implicitely defined by*

$$m'(\hat{s},T) = x.$$

Theorem 3.4 *Under the assumptions of Theorem 3.1, the implied volatility satisfies for* $k \to \infty$

$$\begin{aligned} V(k) &= \sqrt{2}\left(\sqrt{k+L-\frac{3}{2}\log L+\log\frac{k}{4\sqrt{\pi}}+\frac{9\log L}{4L}}\right. \\ &\quad \left. - \sqrt{L-\frac{3}{2}\log L+\log\frac{k}{4\sqrt{4}}+\frac{9\log L}{4L}}\right) + \mathcal{O}\left(\frac{k^2}{L^2}\frac{1+|\log k|+k}{L}\right), \end{aligned}$$

where $L = -\log C(k,T)$ *is the absolute log value of the call option.*

3.2. Call price

3.2.1. Saddle-point

As previously in (2.4), the starting point is the Mellin transform of the call price

$$C(k,T) = \frac{e^k}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-ks}\frac{M(s,T)}{s(s-1)}ds, \quad k = \log K > 0,$$

whenever $1 < c < \infty$.
The denominator of the integrand is ignored in the calculation of the saddle-point. So the saddle-point $\hat{s}$ is defined as the point satisfying

$$\left(e^{-k\hat{s}}M(\hat{s},T)\right)' = 0,$$

where $'$ denotes the partial derivative with respect to s. This is equivalent to

$$m'(\hat{s},T) = k \Leftrightarrow T\left(\sigma^2\hat{s} + b + \lambda e^{\frac{\delta^2\hat{s}^2}{2}+\mu\hat{s}}(\delta^2\hat{s}+\mu)\right) = k. \tag{3.1}$$

The equation cannot be solved explicitly, so $\hat{s}$ is defined implicetely as the solution of (3.1).

Remark 3.5 Again we write $\hat{s}$ instead of $\hat{s}(k)$ for ease of notation.

In order to study the behaviour of $m(\hat{s},T)$ and its derivatives, we need an asymptotic representation of $\hat{s}$, which can be derived by using the bootstrapping method (see [B81, p.26]).

Lemma 3.6 *The saddle-point $\hat{s}$ satisfies*

$$\frac{\delta^2}{2}\hat{s}^2 = \log k - \frac{\mu}{\delta}\sqrt{2\log k} - \log\sqrt{\log k} + \frac{\mu^2}{\delta^2} - \log\frac{\sqrt{2}}{\delta} + \mathcal{O}\left(\frac{\log\log k}{\log k}\right)$$

and

$$\hat{s} = \frac{\sqrt{2\log k}}{\delta} - \frac{\mu}{\delta^2} + \mathcal{O}\left(\frac{\log\log k}{\sqrt{\log k}}\right),$$

as $k \to \infty$.

Proof: The idea of bootstrapping is to derive first a simple approximation of $\hat{s}$ and insert it into the original equation and to improve these approximations iteratively. From (3.1) it is easy to see that $\hat{s}$ is increasing in k, if we consider $\hat{s}$ as a function of k.

1. First we concentrate on the dominant term of the left-hand side in (3.1), so we have for sufficiently big $\hat{s}$

   $$e^{\delta^2\hat{s}^2/2} \leq T\left(\sigma^2\hat{s} + b + \lambda e^{\frac{\delta^2\hat{s}^2}{2}+\mu\hat{s}}(\delta^2\hat{s}+\mu)\right) = k$$

   or equivalently

   $$\delta^2\hat{s}^2/2 \leq \log k,$$

   so one gets the first estimate

   $$\hat{s} = \mathcal{O}\left(\sqrt{\log k}\right). \tag{3.2}$$

2. Again we concentrate on the dominant term and see that the following inequalities must hold for sufficiently big $\hat{s}$

$$\begin{aligned} \lambda T\delta^2\hat{s}e^{\delta^2\hat{s}^2/2+\mu\hat{s}} &\le k \\ \Leftrightarrow \log\left(\lambda T\delta^2\right) + \log\hat{s} + \delta^2\hat{s}^2/2 + \mu\hat{s} &\le \log k. \end{aligned} \tag{3.3}$$

By using the asymptotics in (3.2) and that $\log\hat{s}$ grows like $\log\sqrt{\log k}$ and hence slowlier than $\mu\hat{s}$, we get

$$\frac{\delta^2}{2}\hat{s}^2 = \log k + \mathcal{O}\left(\sqrt{\log k}\right).$$

By factoring out the dominant term and using Lemma A.3 with $(1+x)^{1/2} = 1+\mathcal{O}(x)$ for small x, we get the asymptotics for $\hat{s}$

$$\begin{aligned} \hat{s}^2 &= \frac{2\log k}{\delta^2}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right), \\ \hat{s} &= \frac{\sqrt{2\log k}}{\delta}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right). \end{aligned}$$

3. From (3.3) we see

$$\frac{\delta^2\hat{s}^2}{2} + \mu\hat{s} + \log\hat{s} = \log k + \mathcal{O}(1),$$

hence we get

$$\begin{aligned} \frac{\delta^2\hat{s}^2}{2} &= -\frac{\mu}{\delta}\sqrt{2\log k} + \mathcal{O}(1) + \mathcal{O}(\log\log k) + \log k \\ &= \log k - \frac{\mu}{\delta}\sqrt{2\log k} + \mathcal{O}(\log\log k). \end{aligned} \tag{3.4}$$

By factoring out the dominant term

$$\hat{s}^2 = \frac{2}{\delta^2}\log k\left(1 - \frac{\sqrt{2}\mu}{\delta}\frac{1}{\sqrt{\log k}} + \mathcal{O}\left(\frac{\log\log k}{\log k}\right)\right)$$

and using $\sqrt{1+x} = 1+\frac{1}{2}x+\mathcal{O}\left(x^2\right)$ for small x, we get

$$\begin{aligned} \hat{s} &= \frac{\sqrt{2}}{\delta}\sqrt{\log k}\left(1 - \frac{\mu}{\sqrt{2}\delta}\frac{1}{\sqrt{\log k}} + \mathcal{O}\left(\frac{\log\log k}{\log k}\right)\right) \\ &= \frac{\sqrt{2}}{\delta}\sqrt{\log k} - \frac{\mu}{\delta^2} + \mathcal{O}\left(\frac{\log\log k}{\sqrt{\log k}}\right). \end{aligned} \tag{3.5}$$

4. Again, we refine the inequality and get by using $\log(1+x) = x+\mathcal{O}\left(x^2\right)$ for small x

$$\begin{aligned} \lambda T(\delta^2\hat{s}+\mu)e^{\delta^2\hat{s}^2/2+\mu\hat{s}} &\le k \\ \Leftrightarrow \log(\lambda T) + \log\left(\delta^2\hat{s}(1+\frac{\mu}{\delta^2\hat{s}}\right) + \delta^2\hat{s}^2/2 + \mu\hat{s} &\le \log k \\ \Leftrightarrow \log(\lambda T\delta^2) + \log\hat{s} + \frac{\mu}{\delta^2\hat{s}} + \mathcal{O}\left(\frac{1}{\hat{s}^2}\right) + \delta^2\hat{s}^2/2 + \mu\hat{s} &\le \log k. \end{aligned}$$

Putting all terms except $\frac{\delta^2}{2}\hat{s}^2$ on the right side and using the asymptotics of $\hat{s}$ from (3.4), yields

$$\begin{aligned}\frac{\delta^2\hat{s}^2}{2} &= \log k - \frac{\mu}{\delta}\sqrt{2\log k} - \log\sqrt{\log k} + \frac{\mu^2}{\delta^2} - \log\frac{\sqrt{2}}{\delta} - \log\left(\lambda T\delta^2\right) \\ &\quad + \mathcal{O}\left(\frac{\log\log k}{\log k}\right).\end{aligned}$$

After factoring out the dominant term

$$\begin{aligned}\frac{\delta^2\hat{s}^2}{2} &= \log k\Bigg(1 - \frac{\sqrt{2}\mu}{\delta\sqrt{\log k}} - \frac{\log\sqrt{\log k}}{\log k} + \left(\frac{\mu^2}{\delta^2} - \log\frac{\sqrt{2}}{\delta} - \log(\lambda T\delta^2)\right)\frac{1}{\log k} \\ &\quad + \mathcal{O}\left(\frac{\log\log k}{(\log k)^2}\right)\Bigg)\end{aligned}$$

and using again $\sqrt{1+x} = 1+\frac{x}{2}+\mathcal{O}\left(x^2\right)$ for small x, we obtain the desired asymptotics

$$\begin{aligned}\hat{s} &= \frac{\sqrt{2\log k}}{\delta}\left(1 - \frac{\mu}{\delta\sqrt{2\log k}} + \mathcal{O}\left(\frac{\log\log k}{\log k}\right)\right) \\ &= \frac{\sqrt{2\log k}}{\delta} - \frac{\mu}{\delta^2} + \mathcal{O}\left(\frac{\log\log k}{\sqrt{\log k}}\right).\end{aligned}$$

■

Remark 3.7 One sees, that $\hat{s}$ tends to infinity as $k \to \infty$. Hence $\hat{s}$ is bigger than 1 for sufficiently big k and it can be used as a saddle-point in (2.4).

3.2.2. Asymptotics of the cumulant generating function

After having derived an expansion for $\hat{s}$ in Lemma 3.6 , one has to investigate the behaviour of the cumulant generating function $m(\hat{s}, T)$ and its derivatives, in order to determine an interval for the central approximation.

Lemma 3.8 *The cumulant generating function satisfies for* $k \to \infty$

$$\begin{aligned}m(\hat{s}, T) &= \frac{k}{\delta\sqrt{2\log k}}\left(1 + \mathcal{O}\left(\frac{\log\log k}{\log k}\right)\right), \\ m''(\hat{s}, T) &= \delta k\sqrt{2\log k}\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right),\end{aligned}$$

and

$$m'''(\hat{s}+it, T) = 2k\log k\left(1 + \mathcal{O}\left(\frac{1}{\log k}\right)\right),$$

where $|t| < k^{-\alpha},\ \alpha > 0.$

Proof: We have for $s \in \mathbb{R}$

$$m(s,T) = T\left(\frac{\sigma^2}{2}s^2 + b + \lambda\left(\exp\left(\frac{\delta^2}{2}s^2 + \mu s\right) - 1\right)\right),$$
$$m'(s,T) = T\left(\sigma^2 s + b + \lambda\left((\delta^2 s + \mu)\exp\left(\frac{\delta^2}{2}s^2 + \mu s\right)\right)\right),$$
$$m''(s,T) = T\left(\sigma^2 + \lambda\left(\left(\delta^2 s + \mu\right)^2 + \delta^2\right)\exp\left(\frac{\delta^2}{2}s^2 + \mu s\right)\right)$$

and

$$m'''(s,T) = \lambda T \exp\left(\frac{\delta^2}{2}s^2 + \mu s\right)\left((\delta^2 s + \mu)\left((\delta^2 s + \mu)^2 + \delta^2\right) + 2\delta^2(\delta^2 s + \mu)\right).$$

First, one gets for the dominant term by using the asymptotics of $\hat{s}$ from Lemma 3.6

$$\begin{aligned}\exp\left(\frac{\delta^2}{2}\hat{s}^2 + \mu\hat{s}\right) &= \exp\left(\log k - \log\sqrt{\log k} - \log(\lambda T\delta^2) - \log\frac{\sqrt{2}}{\delta} + \mathcal{O}\left(\frac{\log\log k}{\sqrt{\log k}}\right)\right) \\ &= \frac{k}{\lambda\delta T\sqrt{2\log k}}\left(1 + \mathcal{O}\left(\frac{\log\log k}{\sqrt{\log k}}\right)\right).\end{aligned}$$

So we have

$$\begin{aligned}m(\hat{s},T) &= T\left(\frac{\sigma^2}{2}\hat{s}^2 + b\hat{s} + \lambda\left(\exp\left(\frac{\delta^2}{2}\hat{s}^2 + \mu\hat{s}\right) - 1\right)\right) \\ &= \frac{k}{\delta\sqrt{2\log k}}\left(1 + \mathcal{O}\left(\frac{\log\log k}{\sqrt{\log k}}\right)\right)\end{aligned}$$

and

$$\begin{aligned}m''(\hat{s},T) &= T\left(\sigma^2 + \lambda\left(\left(\delta^2\hat{s} + \mu\right)^2 + \delta^2\right)\exp\left(\frac{\delta^2}{2}\hat{s}^2 + \mu\hat{s}\right)\right) \\ &\sim \lambda T\delta^4\hat{s}^2\exp\left(\frac{\delta^2}{2}\hat{s}^2 + \mu\hat{s}\right) \\ &= \delta k\sqrt{2\log k}\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right).\end{aligned}$$

For the third derivative, one has to investigate the term

$$\begin{aligned}&\left((\delta^2(\hat{s} + it) + \mu)\left((\delta^2(\hat{s} + it) + \mu)^2 + \delta^2\right) + 2\delta^2(\delta^2(\hat{s} + it) + \mu)\right) \\ &= \delta^2\hat{s}\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right)\delta^2\hat{s}^2\left(1 + \mathcal{O}\left(\frac{1}{\log k}\right)\right) + 2\delta^4\hat{s}\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right) \\ &= \delta^4\hat{s}^3\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right) + 2\delta^4\hat{s}\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right) = \delta^4\hat{s}^3\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right) \\ &= \delta(2\log k)^{3/2}\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right).\end{aligned}$$

So we get

$$\begin{aligned}
\left|m'''(\hat{s}+it,T)\right| &= \left|\lambda T \exp\left(\frac{\delta^2}{2}(\hat{s}+it)^2+\mu(\hat{s}+it)\right)\delta(2\log k)^{3/2}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right)\right| \\
&= \lambda T \exp\left(\frac{\delta^2}{2}(\hat{s}^2-t^2)+\mu\hat{s}\right)\delta(2\log k)^{3/2}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right) \\
&\le \lambda T \exp\left(\frac{\delta^2}{2}\hat{s}^2+\mu\hat{s}\right)\delta(2\log k)^{3/2}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right) \\
&= 2k\log k\left(1+\mathcal{O}\left(\frac{1}{\log k}\right)\right).
\end{aligned}$$

■

As in the Kou model, the result of this Lemma is crucial, in order to determine the interval for the central approximation.

3.2.3. Central approximation

We have to determine the interval around the saddle-point which satisfies the two conditions as in (2.6). By using the fact that $\log k$ grows slowlier than k^ϵ for any $\epsilon > 0$, we get by using Lemma 3.8

$$\begin{aligned}
m''(\hat{s})k^{-2\alpha} &\xrightarrow[k\to\infty]{} \infty \Leftrightarrow \alpha < 1/2 \\
m'''(\hat{s}+it)k^{-3\alpha} &\xrightarrow[k\to\infty]{} 0 \Leftrightarrow \alpha > 1/3
\end{aligned}$$

We get the interval $(\hat{s}-k^{-\alpha}, \hat{s}+k^{-\alpha})$ with $1/3 < \alpha < 1/2$.
Before doing the central approximation, one needs to examine the denominator. By inserting the asymptotics of $\hat{s}$ from Lemma 3.6 and using $(1+x)^{-1} = 1+\mathcal{O}(x)$ for small x, we obtain

$$\begin{aligned}
&\frac{1}{(\hat{s}+it)(\hat{s}-1+it)} \\
= &\left(\frac{\sqrt{2\log k}}{\delta}-\frac{\mu}{\delta^2}+it+\mathcal{O}\left(\frac{\log\log k}{\sqrt{\log k}}\right)\right)^{-1}\left(\frac{\sqrt{2\log k}}{\delta}-\frac{\mu}{\delta^2}-1+it+\mathcal{O}\left(\frac{\log\log k}{\sqrt{\log k}}\right)\right)^{-1} \\
= &\frac{\delta^2}{2\log k}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right). \qquad (3.6)
\end{aligned}$$

Putting (3.6) and the asymptotics of the cumulant generating function $m(\hat{s},T)$ from Lemma 3.8 together, gives us the following result.

Lemma 3.9 *Let $1/3 < \alpha < 1/2$. Then the integral satisfies*

$$\begin{aligned}
\frac{e^k}{2\pi i}\int_{\hat{s}-ik^{-\alpha}}^{\hat{s}+ik^{-\alpha}} e^{-ks}\frac{M(s,T)}{s(s-1)}ds &= \frac{\delta^2 e^{k(1-\hat{s})}M(\hat{s},T)}{2\log k\sqrt{2\pi m''(\hat{s},T)}}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right) \\
&= \frac{\delta^2 e^{k(1-\hat{s})+T\left(\sigma^2\hat{s}^2/2+b\hat{s}+\lambda\left(e^{\delta^2\hat{s}^2/2+\mu\hat{s}}-1\right)\right)}}{2\log k\sqrt{2\pi T\left(\sigma^2+\lambda(\delta^2\hat{s}+\mu)\left((\delta^2\hat{s}+\mu)+\delta^2\right)e^{\delta^2\hat{s}^2/2+\mu\hat{s}}\right)}}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right),
\end{aligned}$$

where $\hat{s}$ is implicitely defined by

$$m'(\hat{s},T) = k. \tag{3.7}$$

Proof: Applying Taylors theorem and using the results of the Lemma 3.8 , gives us

$$\begin{aligned} m(\hat{s}+it,T) &= m(\hat{s},T) + itm'(\hat{s},T) - \frac{t^2}{2}m''(\hat{s},T) + \mathcal{O}\left(t^3 k \log k\right) \\ &= m(\hat{s},T) + itk - \frac{t^2}{2}m''(\hat{s},T) + \mathcal{O}\left(k^{1-3\alpha}\right). \end{aligned}$$

Using $\exp(x) = 1 + \mathcal{O}(x)$ for small x, we get

$$\begin{aligned} M(\hat{s}+it,T) &= \exp\left(m(\hat{s}+it,T)\right) = \exp\left(m(\hat{s},T) + itk - \frac{t^2}{2}m''(\hat{s},T) + \mathcal{O}\left(k^{1-3\alpha}\right)\right) \\ &= M(\hat{s},T)\exp\left(itk - \frac{t^2}{2}m''(\hat{s},T)\right)\left(1+\mathcal{O}\left(k^{1-3\alpha}\right)\right). \end{aligned}$$

Inserting this in the integral and the asymptotics of denominator from (3.6), yields

$$\begin{aligned} &\frac{e^k}{2\pi i}\int_{\hat{s}-ik^{-\alpha}}^{\hat{s}+ik^{-\alpha}} e^{-ks}\frac{M(s,T)}{s(s-1)}ds \\ &= \frac{e^{k(1-\hat{s})}}{2\pi}\int_{-k^{-\alpha}}^{k^{-\alpha}} e^{-itk}\frac{M(\hat{s}+it,T)}{(\hat{s}+it)(\hat{s}-1+it)}dt \\ &= \frac{\delta^2 e^{k(1-\hat{s})}M(\hat{s},T)}{4\pi\log k}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right)\int_{-k^{-\alpha}}^{k^{-\alpha}} e^{-\frac{t^2}{2}m''(\hat{s},T)}dt. \end{aligned} \tag{3.8}$$

After doing the substitution $\omega = ut$ (with $u := \sqrt{m''(\hat{s},T)}$) and using the fact that Gaussian integrals have exponentially decaying tails, we get

$$\begin{aligned} \int_{-k^{-\alpha}}^{k^{-\alpha}} e^{-\frac{t^2}{2}m''(\hat{s},T)}dt &= \frac{1}{u}\int_{-uk^{-\alpha}}^{uk^{-\alpha}} e^{-\omega^2/2}d\omega = \frac{1}{u}\left(\int_{-\infty}^{\infty} e^{-\omega^2/2}d\omega + \mathcal{O}\left(e^{-u^2/(2k^{2\alpha})}\right)\right) \\ &= \frac{\sqrt{2\pi}}{\sqrt{m''(\hat{s},T)}}\left(1+\mathcal{O}\left(e^{-u^2/(2k^{2\alpha})}\right)\right). \end{aligned}$$

So we have

$$\begin{aligned} &\frac{\delta^2 e^{k(1-\hat{s})}M(\hat{s},T)}{4\pi\log k}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right)\int_{-k^{-\alpha}}^{k^{-\alpha}} e^{-\frac{t^2}{2}m''(\hat{s},T)}dt \\ &= \frac{\delta^2 e^{k(1-\hat{s})}M(\hat{s},T)}{2\sqrt{2\pi m''(\hat{s},T)}\log k}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right) \\ &= \frac{\delta^2 e^{k(1-\hat{s})+T\left(\sigma^2\hat{s}^2/2+b\hat{s}+\lambda\left(e^{\delta^2\hat{s}^2/2+\mu\hat{s}}-1\right)\right)}}{2\log k\sqrt{2\pi T\left(\sigma^2+\lambda\left(\delta^2\hat{s}+\mu\right)\left(\left(\delta^2\hat{s}+\mu\right)+\delta^2\right)e^{\delta^2\hat{s}^2/2+\mu\hat{s}}\right)}}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right). \end{aligned}$$

■

3.2.4. Estimation of the tails

Now one needs to show that the tails are negligible, to complete the approximation.

Lemma 3.10 *Let* $1/3 < \alpha < 1/2$. *Then we have*

$$\frac{e^k}{2\pi i}\int_{\hat{s}+ik^{-\alpha}}^{\hat{s}+i\infty}\frac{e^{-ks}M(s,T)}{s(s-1)}ds \ll \frac{e^{k(1-\hat{s})}M(\hat{s},T)e^{-\delta k^{1-2\alpha}/(2\sqrt{2\log k})}}{\log k}$$

and

$$\frac{e^k}{2\pi i}\int_{\hat{s}-i\infty}^{\hat{s}-ik^{-\alpha}}\frac{e^{-ks}M(s,T)}{s(s-1)}ds \ll \frac{e^{k(1-\hat{s})}M(\hat{s},T)e^{-\delta k^{1-2\alpha}/(2\sqrt{2\log k})}}{\log k}.$$

Proof: One just needs to consider the first integral, as the derivation of the second result is the same due to the symmetry of t^2. Let $s = \hat{s} + it$ with $t \geq k^{-\alpha}$. Then we get

$$|s(s-1)| \geq \mathrm{Re}\,(s(s-1)) = \hat{s}(\hat{s}-1) \gg \log k.$$

Furthermore we obtain

$$\begin{aligned}
|M(s,T)| &= \exp\left[\mathrm{Re}\left(T\left(\frac{\sigma^2}{2}s^2 + bs + \lambda\left(e^{\delta^2 s^2/2+\mu s} - 1\right)\right)\right)\right] \\
&= \exp\left[\left(T\left(\frac{\sigma^2}{2}(\hat{s}^2 - t^2) + b\hat{s} + \lambda\left(\cos\left(\delta^2 t + \mu t\right)e^{\delta^2(\hat{s}^2-t^2)/2+\mu\hat{s}} - 1\right)\right)\right)\right] \\
&\leq \exp\left[\left(T\left(\frac{\sigma^2}{2}(\hat{s}^2 - t^2) + b\hat{s} + \lambda\left(e^{\delta^2\hat{s}^2/2+\mu\hat{s}}e^{-\delta^2 k^{-2\alpha}/2} - 1\right)\right)\right)\right]
\end{aligned}$$

Using

$$e^{-\delta^2 k^{-2\alpha}/2} = 1 - \delta^2 k^{-2\alpha}/2 + \mathcal{O}\left(k^{-4\alpha}\right)$$

and the result from Lemma 3.8

$$e^{\delta^2\hat{s}^2/2+\mu\hat{s}} \sim \frac{k}{\lambda\delta T\sqrt{2\log k}},$$

we get

$$M(s,T) \ll e^{-T\sigma^2 t^2/2}M(\hat{s},T)e^{-\delta k^{1-2\alpha}/(2\sqrt{2\log k})}.$$

These estimates imply

$$\frac{e^k}{2\pi}\left|e^{-ks}\frac{M(s,T)}{s(s-1)}\right| \ll \frac{e^{k(1-\hat{s})}M(\hat{s},T)e^{-\delta k^{1-2\alpha}/(2\sqrt{2\log k})}}{\log k}e^{-T\sigma^2 t^2/2}$$

and so we obtain

$$\begin{aligned}
\frac{e^k}{2\pi i}\int_{\hat{s}+ik^{-\alpha}}^{\hat{s}+i\infty}\frac{e^{-ks}M(s,T)}{s(s-1)}ds &\ll \frac{e^{k(1-\hat{s})}M(\hat{s},T)e^{-\delta k^{1-2\alpha}/(2\sqrt{2\log k})}}{\log k}\int_{k^{-\alpha}}^{\infty}e^{-\sigma^2 t^2 T/2}dt \\
&\ll \frac{e^{k(1-\hat{s})}M(\hat{s},T)e^{-\delta k^{1-2\alpha}/(2\sqrt{2\log k})}}{\log k}.
\end{aligned}$$

■

Now all three main steps of the saddle-point method have been done and we can prove the main result for the call price

Proof (*Theorem 3.1*): The call price satisfies

$$\begin{aligned} C(k,T) &= \frac{e^k}{2\pi i}\int_{\hat{s}-i\infty}^{\hat{s}-k^{-\alpha}} \frac{e^{-ks}M(s,T)}{s(s-1)}ds + \frac{e^k}{2\pi i}\int_{\hat{s}-k^{-\alpha}}^{\hat{s}+k^{-\alpha}} \frac{e^{-ks}M(s,T)}{s(s-1)}ds \\ &+ \frac{e^k}{2\pi i}\int_{\hat{s}+k^{-\alpha}}^{\hat{s}+i\infty} \frac{e^{-ks}M(s,T)}{s(s-1)}ds. \end{aligned}$$

The middle term gives the main term, as we can see by using the central approximation from Lemma 3.9

$$\begin{aligned} C(k,T) &= \frac{\delta^2 e^{k(1-\hat{s})}M(\hat{s},T)}{2\log k\sqrt{2\pi m''(\hat{s},T)}}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)+\mathcal{O}\left(\frac{e^{-\delta k^{1-2\alpha}/(2\sqrt{2\log k})}}{\sqrt{k\log k}}\right)\right) \\ &= \frac{\delta^2 e^{k(1-\hat{s})}M(\hat{s},T)}{2\log k\sqrt{2\pi m''(\hat{s},T)}}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right)\right). \end{aligned}$$

■

3.3. Density function

The density function of S_T satisfies

$$\tilde{f}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-lu}M(u-1,T)du = \frac{e^{-l}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-lt}M(t,T)dt \tag{3.9}$$

with $l := \log x$ and $c \in \mathbb{R}$ (see (B.2)).
Again, we define our saddle-point implicitely by

$$m'(\hat{s},T) = l$$

and in the same way as in Lemma 3.6 we get the asymptotics (for $l \to \infty$)

$$\begin{aligned} \frac{\delta^2\hat{s}^2}{2} &= \log l - \frac{\mu}{\delta}\sqrt{2\log l} - \log\sqrt{\log l} + \frac{\mu^2}{\delta^2} - \log\frac{\sqrt{2}}{\delta} - \log\left(\lambda T\delta^2\right) \\ &+ \mathcal{O}\left(\frac{\log\log l}{\log l}\right) \qquad (3.10) \\ &= \log l\left(1 - \frac{\sqrt{2}\mu}{\delta\sqrt{\log l}} - \frac{\log\sqrt{\log l}}{\log l} + \left(\frac{\mu^2}{\delta^2} - \log\frac{\sqrt{2}}{\delta} - \log(\lambda T\delta^2)\right)\frac{1}{\log l}\right. \\ &\left. + \mathcal{O}\left(\frac{\log\log l}{(\log l)^2}\right)\right) \end{aligned}$$

and

$$\begin{aligned} \hat{s} &= \frac{\sqrt{2\log l}}{\delta}\left(1 - \frac{\mu}{\delta\sqrt{2\log l}} + \mathcal{O}\left(\frac{\log\log l}{\log l}\right)\right) \qquad (3.11) \\ &= \frac{\sqrt{2\log l}}{\delta} - \frac{\mu}{\delta^2} + \mathcal{O}\left(\frac{\log\log l}{\sqrt{\log l}}\right). \end{aligned}$$

Similarly, the asymptotics for the cumulant generating function $m(\hat{s}, T)$, its derivatives do not change and neither does the interval around the saddle-point for the approximation ($\hat{s} \pm l^{-\alpha}$ with $1/3 < \alpha < 1/2$). Hence, the expansion for the density can be derived in the same way as for the call price in Section 3.2.

Proof (*Theorem 3.2*): Let $\tilde{f}(x)$ denote the density of S_T, so we have

$$\tilde{f}(x) = \frac{e^{-l}}{2\pi} \int_{-\infty}^{\infty} e^{-l(\hat{s}+it)} M(\hat{s}+it, T) dt,$$

where $l = \log x$.
Applying the Laplace method on the central interval as in Lemma 3.9 and estimating the remaining tails as in Lemma 3.10 , yields

$$\begin{aligned} \tilde{f}(x) &= \frac{e^{-l(1+\hat{s})} M(\hat{s}, T)}{\sqrt{2\pi m''(\hat{s}, T)}} \left(1 + \mathcal{O}\left(l^{1-3\alpha}\right)\right) \\ &= \frac{e^{-l(1+\hat{s})+T\left(\sigma^2\hat{s}^2/2+b\hat{s}+\lambda\left(e^{\delta^2\hat{s}^2/2+\mu\hat{s}}-1\right)\right)}}{\sqrt{2\pi T\left(\sigma^2 + \lambda(\delta^2\hat{s}+\mu)\left((\delta^2\hat{s}+\mu)+\delta^2\right)e^{\delta^2\hat{s}^2/2+\mu\hat{s}}\right)}} \left(1 + \mathcal{O}\left(l^{1-3\alpha}\right)\right), \end{aligned}$$

where $1/3 < \alpha < 1/2$.
To get the density of X_T (denoted by $f(x)$), we have to use the relation

$$f(x) = e^x \tilde{f}(e^x), \quad x \in \mathbb{R}.$$

So we have

$$\tilde{f}(e^x) = \frac{e^{-x(1+\hat{s})} M(\hat{s}, T)}{\sqrt{2\pi m''(\hat{s}, T)}} \left(1 + \mathcal{O}\left(x^{1-3\alpha}\right)\right),$$

where $\hat{s}$ is implicitely defined by

$$m'(\hat{s}, T) = x.$$

Multiplying the term with e^x and choosing $\alpha = \frac{1}{2} - \frac{\epsilon}{3}$ completes the proof.

■

3.4. Tail probability

From (B.3) in the appendix we have the integral representation for the tail probability of S_T

$$1 - \tilde{F}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{M(u, T)}{u} e^{-lu} du \tag{3.12}$$

where $1 < c < \infty$.
Again, we ignore the denominator in the calculation of the saddle-point, so we can use the same saddle-point as for the density function. The derivation of the expansion is also the same, just the asymptotics of the denominator have to be added

$$\begin{aligned} \frac{1}{(\hat{s}+it)} &= \left(\frac{\sqrt{2\log l}}{\delta} - \frac{\mu}{\delta^2} + it + \mathcal{O}\left(\frac{\log\log l}{\sqrt{\log l}}\right)\right)^{-1} \\ &= \frac{\delta}{\sqrt{2\log l}} \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{\log l}}\right)\right). \end{aligned} \tag{3.13}$$

Lemma 3.11 *The tail probability of S_T satisfies*

$$
\begin{aligned}
1-\tilde{F}(x) &= \frac{\delta e^{-l\hat{s}} M(\hat{s},T)}{2\sqrt{\pi m''(\hat{s},T)\log l}} \left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log l}}\right)\right) \\
&= \frac{\delta e^{-l\hat{s}+T\left(\sigma^2/2\hat{s}^2+b\hat{s}+\lambda\left(e^{\delta^2\hat{s}^2/2+\mu\hat{s}}-1\right)\right)}}{2\sqrt{\pi T \log l\left(\sigma^2+\lambda(\delta^2\hat{s}+\mu)\left((\delta^2\hat{s}+\mu)+\delta^2\right)e^{\delta^2\hat{s}^2/2+\mu\hat{s}}\right)}} \left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log l}}\right)\right),
\end{aligned}
$$

where $l=\log x$ and $\hat{s}$ is implicitely defined by

$$m'(\hat{s},T)=l.$$

Proof:
The proof goes along the same lines as for the density of S_T in the proof of Theorem 3.2, one just has to add (3.13).

■

With this result, it is easy to prove the asymptotics for the tail probability of X_T.

Proof (*Theorem 3.3*): One just needs the result for S_T in Lemma 3.11 and to use the simple tranformation

$$1-F(x)=1-\tilde{F}(e^x),$$

where $F(x)$ is the cumulative distribution function of X_T.

■

Finally, we compare the tails to the Gaussian and exponential distribution to verify a result from [CT04].

Corollary 3.12 The tails of X_T are heavier than Gaussian, but lighter than exponential, i.e.

$$1-F_N(x)=o\left(1-F(x)\right)$$

and

$$1-F(x)=o\left(1-F_{Ex}(x)\right),$$

where F_N and F_{Ex} are Gaussian, respectively Exponential, distribution functions and F is the distribution function of X_T.

Proof: From Theorem 3.3 , Lemma 3.8 and Lemma 3.10, we know that the tail probability of X_T (denoted by $\bar{F}(x)$) satisfies

$$\begin{aligned}\log\left(\bar{F}(x)\right) &= \log\left(\frac{\delta}{2\sqrt{\pi}}\right) - x\hat{s} + m(\hat{s},T) - \frac{1}{2}\log\left(m''(\hat{s},T)\log x\right) + \mathcal{O}\left(\frac{1}{\sqrt{\log x}}\right)\\ &= -\frac{\sqrt{2}}{\delta}x\sqrt{\log x}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{\log x}}\right)\right) + \frac{x}{\delta\sqrt{2\log x}}\left(1+\mathcal{O}\left(\frac{\log\log x}{\log x}\right)\right)\\ &\quad - \frac{1}{2}\log(\delta\log x\sqrt{2\log x}) + \mathcal{O}\left(\frac{1}{\sqrt{\log x}}\right)\\ &\sim -\frac{\sqrt{2}}{\delta}x\sqrt{\log x},\end{aligned}$$

as $x \to \infty$.
The standard Gaussian tails with distribution function $\Phi(x)$ satisfy

$$1-\Phi(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt \le \int_x^\infty \frac{t}{x}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt = \frac{e^{-x^2/2}}{\sqrt{2\pi}x},$$

so we have

$$1-\Phi(x) \ll \frac{e^{-x^2/2}}{x}.$$

For a general Gaussian random variable with mean μ and standard deviation σ, one just needs to use the relation to a standard Gaussian distribution function

$$F_N(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

Hence, we get for Gaussian distributions (with distribution function F_N)

$$\bar{F}_N(x) := 1 - F_N(x) = 1 - \Phi\left(\frac{x-\mu}{\sigma}\right) \ll \frac{e^{-(x-\mu)^2/(2\sigma^2)}}{x-\mu}.$$

The exponential distribution with mean $\frac{1}{\lambda} > 0$ satisfies

$$\bar{F}_{Ex}(x) := 1 - F_{Ex}(x) = e^{-\lambda x} \quad \text{for } x \ge 0.$$

The relation

$$\begin{aligned}\bar{F}_N(x) &\ll \frac{\exp\left(-(x-\mu)^2/(2\sigma^2)\right)}{x-\mu}\\ &\ll \exp\left(\frac{-x\sqrt{2\log x}}{\delta}\right)\\ &\ll \exp\left(-\lambda x\right)\end{aligned}$$

gives us the desired result.

■

Remark 3.13 This result is already mentioned in [CT04, Table 4.3, p.124].

Remark 3.14 In [BF09, Example 5.4], it is already mentioned that some Lévy tail estimate results in [AB] can be used to derive the asymptotics

$$\log \bar{F}(x) \sim -\frac{x}{\delta}\sqrt{2\log x},$$

where $\bar{F}(x) = 1 - F(x)$.

3.5. Implied volatility

To get an expansion for the implied volatility, we need again a result from the paper [GL, Corollary 7.1.]

Lemma 3.15 *If the absolute log of the call price* $L = -\log C(k,T)$ *satisfies*

$$k/L \xrightarrow[k\to\infty]{} 0,$$

then the dimensionless implied volatility satisfies for $k \to \infty$

$$\begin{aligned} \left| G_-\left(k, L - \frac{3}{2}\log L + \log\frac{k}{4\sqrt{\pi}}\right) - V(k)\right| &= \mathcal{O}\left(\frac{k}{L^{3/2}}\right), \\ \left| G_-\left(k, L - \frac{3}{2}\log L + \log\frac{k}{4\sqrt{\pi}} + \frac{9\log L}{4L}\right) - V(k)\right| &= \mathcal{O}\left(\frac{k}{L^{3/2}}\right), \end{aligned}$$

where $G_-(\kappa, u) := \sqrt{2}\left(\sqrt{u+\kappa} - \sqrt{u}\right)$.

With this result, it is easy to proof the asymptotics for the implied volatility.

Proof (*Theorem 3.4*): By using Theorem 3.1 , Lemma 3.8 and the asymptotics of the saddle-point $\hat{s}$ from Lemma 3.7, we get (similarly to the proof of Corollary 3.12)

$$\begin{aligned} L &= -\log\left(C(k,T)\right) \\ &= -\log\left(\frac{\delta^2}{2\sqrt{2\pi}}\right) - k(1-\hat{s}) - m(\hat{s},T) + \frac{1}{2}\log\left(m''(\hat{s},T)\log k\right) + \log\log k + \mathcal{O}\left(\frac{1}{\sqrt{\log k}}\right) \\ &\sim \frac{\sqrt{2}}{\delta}k\sqrt{\log k}, \end{aligned}$$

as $k \to \infty$.

Because $k/L \to 0$ for $k \to \infty$, we can apply Lemma 3.15 and get the result.

∎

3.6. Numerical Tests

The parameters from Table 3.1 are taken from [M11, p.14] (again except for the interest rate r which is set zero).

r	T	σ	λ	μ	δ
0.0	1	0.1	5	-0.001	0.1

Table 3.1.: Parameters for the Merton Jump Diffusion model.

Again the martingale condition (2.2) has to be fulfilled, so we get

$$b = -\left(\sigma^2/2 + \lambda\left(e^{\delta^2/2+\mu} - 1\right)\right) = -0.025.$$

Here the asymptotic expansions are compared with the exact call price (respectively the density, tail probability and implied volatility), which we get by numerical computation. The saddlepoint $\hat{s}$ is calculated numerically with the root-finding procedure `uniroot()`.

3.6.1. Call price

In Table 3.2 are two plots, where the expansion from Theorem 3.1 is compared with the exact value of the call option which we get by numerical integration. The functions are plotted on logarithmic x-axis with different ranges. One can see that the approximation is very close to the exact price of the call option for $k > 3$.

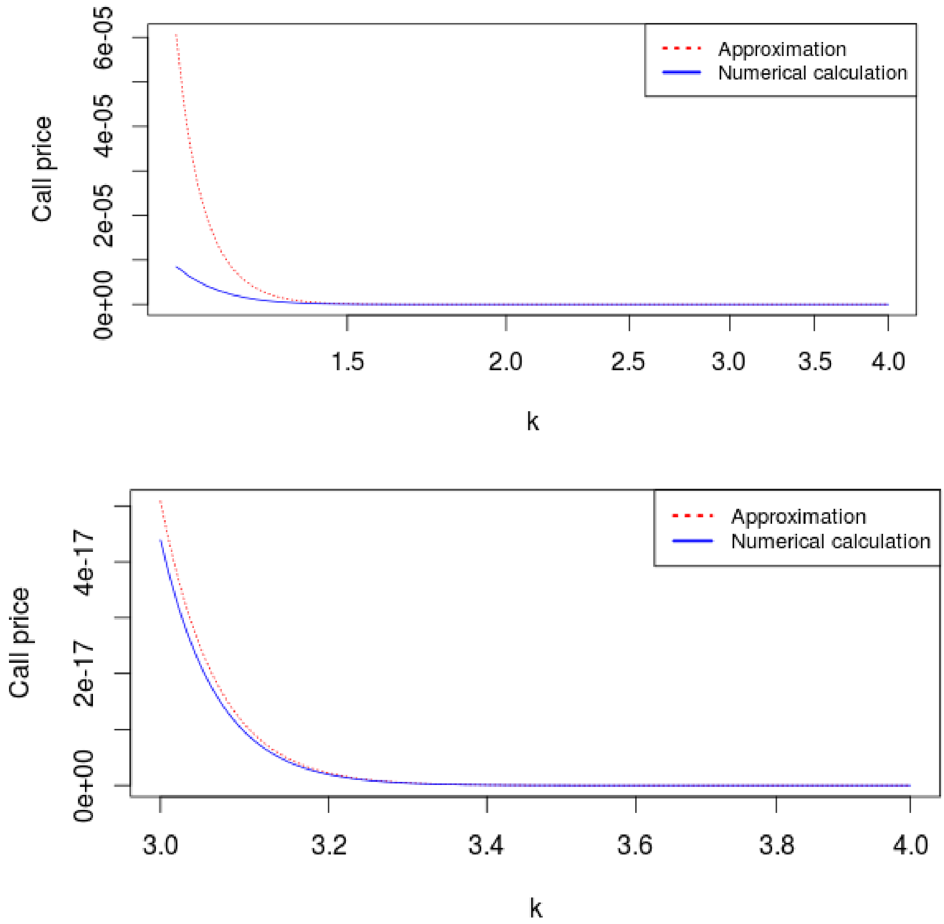


Table 3.2.: Approximation of the price function in terms of log-strikes is compared with exact value.

3.6.2. Density

In Table 3.3, one sees that the density can be approximated with our expansion very well. For values $x > 0.4$ the expansion and the numerically calculated density are almost identical and they cannot be distinguished anymore in the plot.

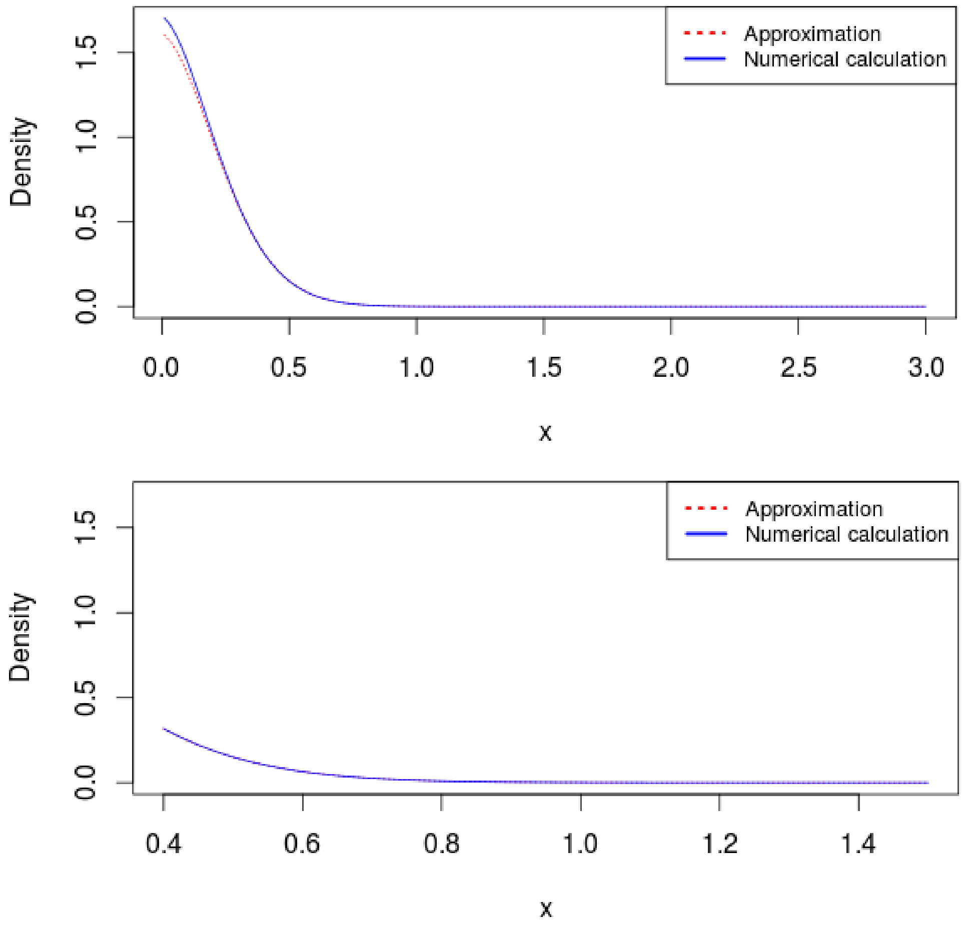


Table 3.3.: Approximation of the density function is compared with exact density.

3.6.3. Tail probability

In Table 3.4 it is illustrated, that the tail probabilities do satisfy Corollary 3.12, which said that the log-price of the stock, $X_T = \log S_T$, has heavier tails than Gaussian distributions, but lighter ones than the exponential distribution. Here was the Gaussian probability calculated with the R-command `pnorm()` with mean 0 and standard deviation 0.1, the exponential tail probability was calculated explicitely (with mean 0.1). It can also be seen that the expansion approximates the tail probability fairly well for $x > 1.5$.

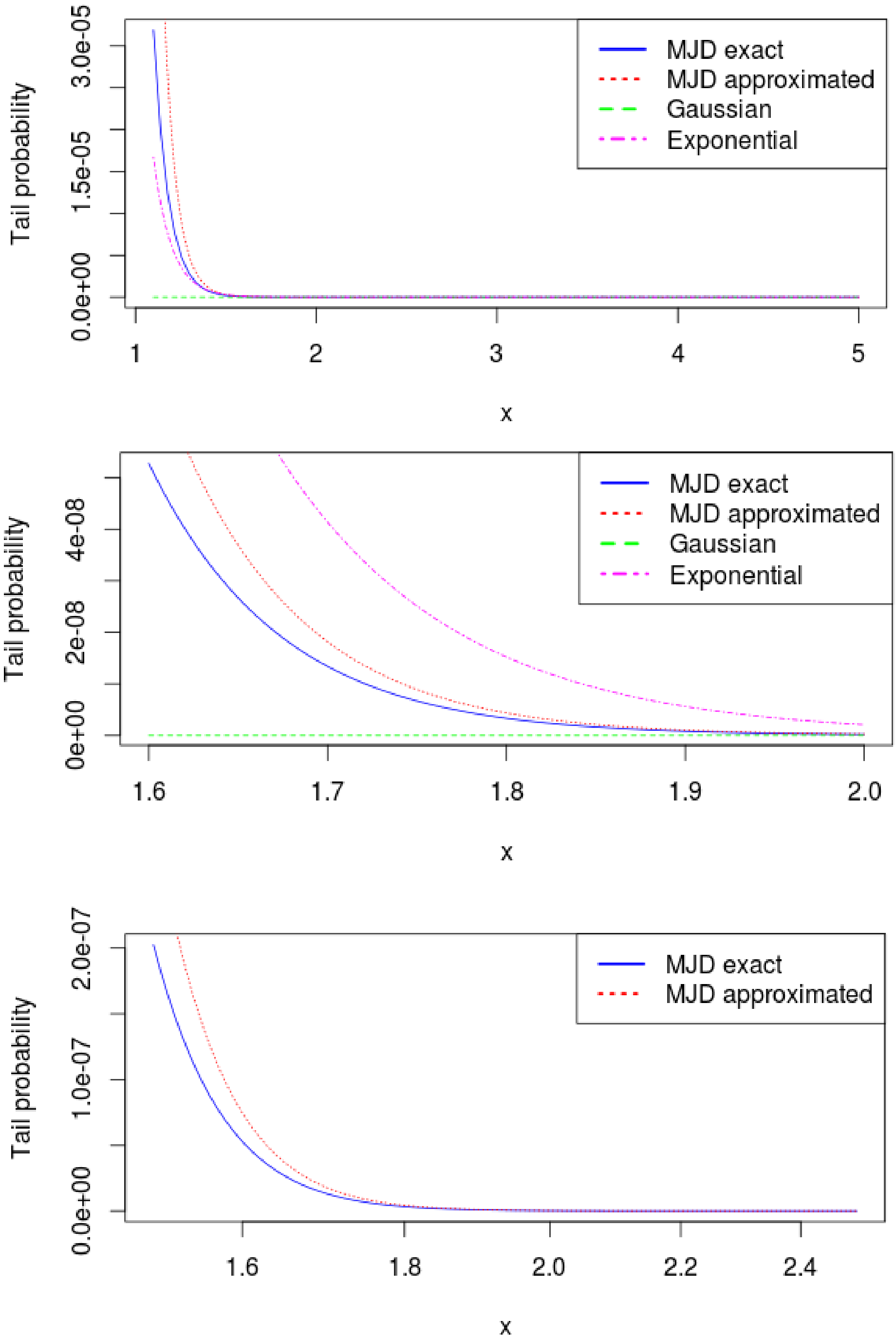
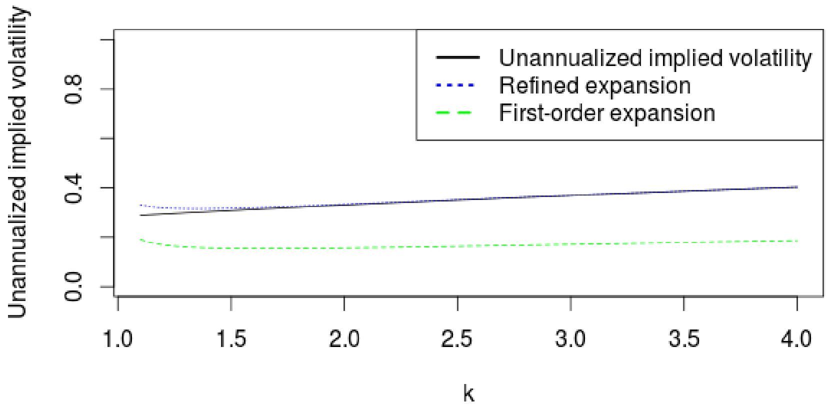


Table 3.4.: Approximation of the tail probability is compared with exact tail probability as well as with Gaussian and Exponential tails.

3.6.4. Implied volatility

Finally the implied volatility is plotted against the expansion from Theorem 3.4 and the approximation given in [BF09] for different parameters and maturities. One sees that the higher-order expansion from Theorem 3.4 yields clearly better results.



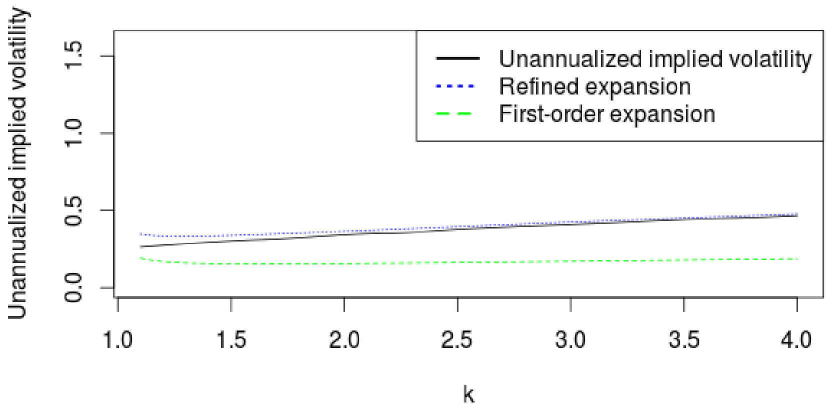


Table 3.5.: Comparison of implied volatility (solid) with the expansion from Theorem 3.4 (dotted) and the expansion from [BF09] (dashed). Parameters used: $T = 1$, $\sigma = 0.1$, $\lambda = 5$, $\mu = -0.001$, $\delta = 0.1$ (first plot); $T = 0.1$, $\sigma = 0.4$, $\lambda = 0.1$, $\mu = 0.3$ and $\delta = 0.4$ (second plot).

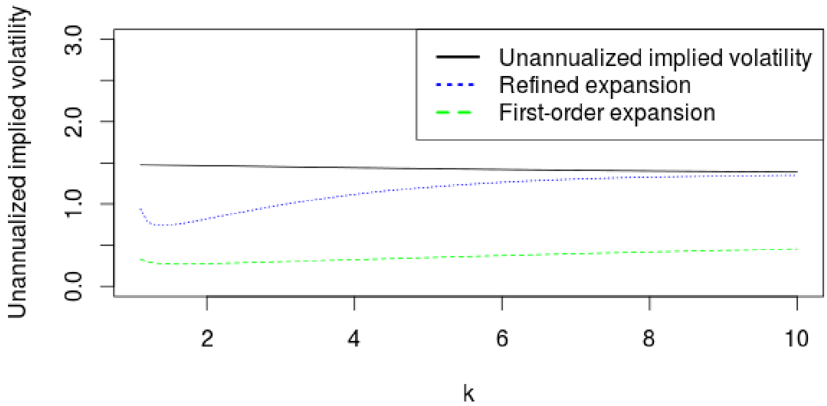


Table 3.6.: Comparison of implied volatility (solid) with the expansion from Theorem 3.4 (dotted) and the expansion from [BF09] (dashed). Parameters used: ; $T = 4$, $\sigma = 0.6$, $\lambda = 3$, $\mu = 0.2$ and $\delta = 0.4$.

4. Conclusion

In the Kou model, the numerical tests for the constants in the expansion for the call price showed very slow convergence. Nevertheless, the results for the implied volatility skew were very good and it was possible to show that the higher-order expansion does yield better approximations than the first-order expansion given in [BF08]. As it was already noted, the saddle-point was approximated in the Kou model in contrast to the Merton Jump diffusion model, where it was defined implicitely and calculated exactly by a root finding procedure for the numerical tests. This explains, why the results in the Merton Jump Diffusion model seem to be sharper. The call price approximation yields fairly good results for $k > 3$. Also the tail probability and especially the density expansion seem to be very accurate for values $x > 2$. Finally, we could also find a sharper approximation of the implied volatility and improve the result for the Merton Jump Diffusion model, given in [BF09].
One should add, that the approximations are becoming accurate for very high log-strike values (respectively stock values), which are not very relevant in the practical world. Nevertheless, this thesis contributes to the qualitative analysis. In the Kou model, one can see that the critical moment λ_+ is the only parameter that is needed to determine the first-order approximation. The jump intensity λ, the maturity T and p are the parameters, which determine together with λ_+ the constant in the second-order expansion for the call price. The other parameters σ, b and λ_- are just in the α_0 term. The same applies to the implied volatility, where the second-order expansion is entirely defined by the four parameters λ_+, λ, T and p.

A. Landau notation

A.1. Basics

Definition A.1 For $f : D \subseteq \mathbb{C} \to \mathbb{C}$ and $g : D \subseteq \mathbb{C} \to \mathbb{C}$, $f(x) = \mathcal{O}(g)$ means that there exist $C > 0$ and $\delta > 0$ such that

$$|f(x)| \leq C\,|g(x)| \quad \text{for } x > \delta.$$

Definition A.2 For $f : D \subseteq \mathbb{C} \to \mathbb{C}$ and $g : D \subseteq \mathbb{C} \to \mathbb{C}$, $f(x) = o(g)$ means that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

Lemma A.3 (see [H09]): *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be analytic on some disk $|x| < r$ for some $r > 0$, i.e.*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k.$$

Then for $n \in \mathbb{N}$ and $|x| < r$

$$f(x) = \sum_{k=0}^{n} a_k x^k + \mathcal{O}\left(|x|^{n+1}\right).$$

Proof: The result follows immediately from Taylors theorem.

■

A.2. Some Taylor series

Here are the most important series listed that are used in thesis.

Definition A.4 (*Binomial series*):

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \tag{A.1}$$

converges for all $x \in (-1, 1)$ and $\alpha \in \mathbb{R}$.

Definition A.5 (*Exponential series*):

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{A.2}$$

converges for all $x \in \mathbb{R}$.

Definition A.6 (*Logarithmic series*):

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \tag{A.3}$$

converges for $|x| \leq 1$ except $x = -1$.

Most of the time these series are used together with Lemma A.3 . Finally, an example is given, in order to illustrate how the Lemma is used in the thesis.

Example A.7 Let $f : \mathbb{C} \to \mathbb{C}$ satisfy

$$f(x) = x + \mathcal{O}\left(x^{-2}\right), \text{ as } x \to \infty.$$

Then we get by using (A.1) and Lemma A.3 , that

$$(1+x)^{1/2} = 1 + \mathcal{O}(x), \quad \text{for small } x$$

and hence

$$\sqrt{f(x)} = \left(x\left(1 + \mathcal{O}\left(x^{-3}\right)\right)\right)^{1/2} = \sqrt{x}\left(1 + \mathcal{O}\left(x^{-3}\right)\right).$$

B. Mellin transform

Most of the following results have been taken from [P96] and [H].

B.1. Basic results

Definition B.1 (*Mellin transform*): For a function (on the positive real axis) f the Mellin transform (named after the Finnish mathematician Hjalmar Mellin) is defined by

$$\mathcal{M}(f;s) \equiv F(s) := \int_0^\infty f(t)t^{s-1}dt.$$

In general, the transform F exists only on a strip (or halfplane) in $\mathbb{C}$.

Remark B.2 There exist slightly different notations (see for example [H, p.2]).

Lemma B.3 *The Fourier transform of a function f*

$$\mathcal{F}(f;s) \equiv \hat{f}(s) := \int_{-\infty}^\infty f(x)e^{-2\pi isx}dx.$$

satisfies the following relationship to the Mellin transform

$$\mathcal{M}(f(x);a+2b\pi i) = \mathcal{F}(f(e^{-x})e^{-ax};b), \quad a,b \in \mathbb{R}.$$

Proof: By doing the substitution $t = e^{-x}$ we get the relation

$$\begin{aligned} \mathcal{M}(f(t);a+2b\pi i) &= \int_0^\infty f(t)t^{a+2\pi bi-1}dt \\ = \int_{-\infty}^\infty f(e^{-x})e^{-ax}e^{-2\pi bxi}dx &= \mathcal{F}(f(e^{-x})e^{-ax};b). \end{aligned} \tag{B.1}$$

■

Lemma B.4 (Mellin Inversion): *Let $F(s) = \mathcal{M}(f(x); s)$ denote the Mellin transform of a function f and let us assume $F \in L^1$, then the following holds*

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) x^{-s} ds,$$

where c has to be on the strip of definition of F.

Proof: By using the Fourier Inversion theorem

$$f(x) \quad = \quad \int_{-\infty}^{\infty} \hat{f}(s) e^{2\pi x s i} ds$$

and the relation (B.1), one gets easily the Mellin Inversion formula

$$\begin{aligned} & f(e^{-x}) e^{ax} = \int_{-\infty}^{\infty} F(s) e^{2\pi \beta x i} d\beta \\ \Leftrightarrow \quad & f(t) = t^{-a} \int_{-\infty}^{\infty} F(s) t^{-2\pi\beta i} d\beta \\ \Leftrightarrow \quad & f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(s) t^{-s} ds. \end{aligned}$$

■

B.2. Applications of the Mellin transformation

B.2.1. Call price

First it is easy to see for the payoff function of a European call option $f(x) := (x - e^k)_+$ with log-strike $k > 0$, that

$$\mathcal{M}\left(\left(x - e^k\right)_+ ; s\right) = \int_0^{\infty} (x - e^k)_+ x^{s-1} dx = \int_{e^k}^{\infty} (x - e^k) x^{s-1} dx = \frac{e^{k(s+1)}}{s(s+1)} \quad \text{for } \mathrm{Re}(s) < 1.$$

Lemma B.5 *Let X_T be the log-return of a stock and suppose that its moment generating function $M(t)$ exists for some $t > 1$. Then a call price can be written as*

$$C(k, T) = \frac{e^k}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{-ks} \frac{M(s, T)}{s(s-1)} ds,$$

whenever $1 < c < T^* := \sup\{t \geq 0 : \ M(t) < \infty\}$.

Proof: By applying the inversion theorem we get

$$(x - e^k)_+ = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{k(s+1)}}{s(s+1)} x^{-s} ds \quad \text{for } \mathrm{Re}(s) < 1.$$

Setting $X_T := \log S_T$, $k := \log K$ and $\tilde{c} < 1$, applying Fubini's theorem yields (the expectation is taken under the martingale measure) and doing the substitution $s = -t$, yields

$$\begin{aligned}
\mathbb{E}\left[(e^{X_T} - e^k)_+\right] &= \mathbb{E}\left[\frac{e^k}{2\pi i}\int_{\tilde{c}-i\infty}^{\tilde{c}+i\infty} \frac{e^{ks}}{s(s+1)} e^{-sX_T} ds\right] = \frac{e^k}{2\pi i}\int_{\tilde{c}-i\infty}^{\tilde{c}+i\infty} \frac{e^{ks}}{s(s+1)} \mathbb{E}\left[e^{-sX_T}\right] ds \\
&= \frac{e^k}{2\pi i}\int_{\tilde{c}-i\infty}^{\tilde{c}+i\infty} \frac{e^{ks}}{s(s+1)} M(-s) ds = \frac{e^k}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{M(t,T)e^{-kt}}{t(t-1)} dt.
\end{aligned}$$

■

B.2.2. Probability density function

If the moment generating function of X_T exists on an interval, one can recover the density of S_T , denoted by $\tilde{f}(x)$, easily from

$$\mathcal{M}(\tilde{f}(x); s) = \int_0^\infty x^{s-1}\tilde{f}(x)dx = \mathbb{E}\left[S_T^{s-1}\right] = \mathbb{E}\left[e^{(s-1)X_T}\right] = M_{X_T}(s-1)$$

and by using the Mellin inversion theorem, we get

$$\tilde{f}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(u-1)x^{-u}du, \tag{B.2}$$

where c has to be on the strip of definition of $M(c-1)$.

B.2.3. Tail probability

Assuming that $M(t)$ exists for some $t > 1$, then one gets for the tail probability of S_T (by applying Fubini's theorem) for $\mathrm{Re}(s) > 1$

$$\begin{aligned}
\mathcal{M}(1-\tilde{F}(x); s) &= \int_0^\infty \left(1-\tilde{F}(t)\right) t^{s-1}dt = \int_0^\infty \left(\int_t^\infty \tilde{f}(x)dx\right) t^{s-1}dt \\
&= \int_0^\infty \left(\int_0^x t^{s-1}dt\right) \tilde{f}(x)dx \\
&= \int_0^\infty \frac{x^s}{s}\tilde{f}(x)dx = \frac{\mathcal{M}(\tilde{f}(x); s+1)}{s} = \frac{M_{X_T}(s)}{s}
\end{aligned}$$

and hence the tail probability is given by

$$1-\tilde{F}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{M_{X_T}(u)x^{-u}}{u} du, \tag{B.3}$$

where $1 < c < T^*$.

To get the density and tail probability of $X_T = \log S_T$ (denoted by $f(x)$ and its cumulative distribution function by $F(x)$), we use

$$\begin{aligned}
\tilde{F}(e^x) &= \mathbb{P}(S_T < e^x) = \mathbb{P}(X_T < x) = F(x) \\
\tilde{f}(e^x)e^x &= f(x)
\end{aligned} \tag{B.4}$$

C. Implementation in R

Finally, the most important parts of the implementation in R are given.

C.1. Kou model

First the parameters and all constants of the expansions are defined as global variables. Futhermore the value `k_critical` is defined, which gives the k when the saddle-point $\hat{s}$ is greater than 1. This is implemented in `kou_param()`.
The code carries out the numerical checks for the constants, given in the equations (A1)-(A3) and (B1)-(B4). Here is the call price numerically calculated by using the `quadinf()` function on the integral representation of the call price

$$C(k,T) = \frac{e^k}{2\pi}\int_{-\infty}^{\infty} e^{-k(\hat{s}+is)} \frac{M(\hat{s}+is,T)}{(\hat{s}+is)(\hat{s}-1+is)} ds.$$

The implied volatility can be calculated by using a root-finding procedure. Here was the `uniroot()` function used.

```
#parameter definitions as global variables
kou_param<-function(param=c(0.15,0,1,10,25,50,0.3)){

  param1<<-c(0.1,0,1,5,15,15,0.5)
  param2<<-c(0.2,0,0.1,10,25,50,0.3)
  param3<<-c(0.4,0,6,1,2,3,0.2)

  sigma<<- param[1]
  r<<- param[2]
  bigT<<-param[3]
  lambda<<-param[4]
  lambdaminus<<-param[5]
  lambdaplus<<-param[6]
  p<<-param[7]


  b<<--(sigma^2/2+lambda*(lambdaplus*p/(lambdaplus-1)
  +lambdaminus*(1-p)/(lambdaminus+1)-1))
  i<<-complex(real=0,imaginary=1)
  xi<<-lambda*lambdaplus*p*bigT

  alpha1<<-lambdaplus-1
```

```
alpha0.5<<- -2*sqrt(xi)
alpha0<<-log((exp((bigT*sigma^2*lambdaplus^2)/2+
b*lambdaplus*bigT+(bigT*lambda*lambdaminus*(1-p))/(lambdaminus+lambdaplus)
-lambda*bigT)*xi^(1/4))/(2*sqrt(pi)*lambdaplus*(lambdaplus-1)))
gamma_kou<<-1/(sqrt(2*alpha1+2))-1/sqrt(2*alpha1)

beta0 <<- gamma_kou*alpha0.5
beta0.5 <<- -2*gamma_kou*sqrt(alpha1^2+alpha1)
betaell <<- gamma_kou/4
betaminus <<- (alpha0+log((1-(1+1/alpha1)^(-1/2))/sqrt(4*pi*alpha1)))
*gamma_kou+(1/(2*(2*alpha1)^(3/2))-1/(2*(2*alpha1+2)^(3/2)))*alpha0.5^2

source("s_star.R")

f<-function(k) s_star(k) - 1
k_critical <<- uniroot(f,c(0.000001,50))$root
}

#approximated saddle-point
s_star<-function(k)
{
  lambdaplus-xi^(1/2)*k^(-1/2)
}

#numerically calculated exact saddle-point
s_exact<-function(k){

  f<-function(t) m1(t)/(t*(t-1))-k
  uniroot(f,c(1.5,lambdaplus*0.99999))$root
}

#mgf
M<-function(s) {
  as.complex(exp(bigT*((sigma^2*s^2)/2+b*s+
  lambda*((lambdaplus*p)/(lambdaplus-s)+
  (lambdaminus*(1-p))/(lambdaminus+s)-1))))
}

#first derivative of cumulant generating function
m1<-function(s){
  bigT*(sigma^2*s+b+lambda*(lambdaplus*p/(lambdaplus-s)^2
  -lambdaminus*(1-p)/(lambdaminus+s)^2))
}

#numerically calculated call price
Kou_Price<-function(k) {
  exp(k)/(2*pi)*quadinf(function(s) Re(exp(-k*(s_star(k)+i*s))
```

```
  *M(s_star(k)+ i*s)/((s_star(k)+i*s)*(s_star(k)-1+i*s))),-Inf, Inf)
}

#asymptotic expansion of the call price
kou_price_approx<-function(k){
  exp(-alpha1*k - alpha0.5*sqrt(k)-alpha0)*k^(-3/4)
}

#normalized Black-Scholes price of a call option
#sig...denotes the unannualized volatility!
BS_Price<-function(k,sig=sigma,mat=bigT){
  d1 = -k/sig+sig/2
  d2 = -k/sig-sig/2

  pnorm(d1)-exp(k)*pnorm(d2)
}

impvola<-function(k,mat=bigT){

  f<-function(t) BS_Price(k,sig=t)-Kou_Price(k)
  uniroot(f,c(0.00001,8))$root
}

#test of the constant alpha_0
Testalpha0<-function(kmin=k_critical+0.4,kmax=50){

  test<-function(k){-log(Kou_Price(k)
  *exp(alpha1*k+alpha0.5*sqrt(k))*k^(3/4))}

  plot(Vectorize(test),kmin,kmax,ylim=c(alpha0,alpha0+30),
  lty=3,xlab="k",ylab="",col="red")
  abline(h=alpha0,col="blue")
  legend("topright",c("Numerical test",expression(alpha[0])),
  lty=c(3,1),lwd=c(2.5,2.5),col=c("red","blue"))
}

#numerical test of alpha_1/2
Testalpha0.5<-function(kmin=k_critical+0.4,kmax=50){

  test<-function(k){-log(exp(alpha1*k)*Kou_Price(k))/sqrt(k)}

  plot(Vectorize(test),kmin,kmax,xlab = "k", ylab ="",
  ylim=c(alpha0.5,alpha0.5+15),lty=3,col="red")
  abline(h=alpha0.5,col="blue")
  legend("topright",c("Numerical test", expression(alpha[1/2])),
  col=c("red","blue"),lty=c(3,1),lwd=c(2.5,2.5))
}
```

```
#numerical test of constant alpha_1
Testalpha1<-function(kmin=k_critical+0.4,kmax=50){

  test<-function(k) -log(Kou_Price(k))/k

  plot(Vectorize(test),kmin,kmax,xlab = "k", ylab ="",
  ylim=c(alpha1-10,alpha1+5), lty=3,col="red")
  abline(h=alpha1,col="blue")
  legend("topright",c("Numerical test",expression(alpha[1])),
  col=c("red","blue"),lty=c(3,1),lwd=c(2.5,2.5))
}

#numerical test of the constand beta_0
Testbeta0<-function(kmin=k_critical+0.4,kmax=4){

  test<-function(k) impvola(k)-k^(1/2)*beta0.5

  plot(Vectorize(test),kmin,kmax,xlab = "k", ylab ="",
  ylim=c(beta0,beta0+0.3),lty=3,col="red")
  abline(h=beta0,col="blue")
  legend("topright",c("Numerical test", expression(beta[0])),
  col=c("red","blue"),lty=c(3,1),lwd=c(2.5,2.5))
}

#numerical test of the constant beta_(1/2)
Testbeta0.5<-function(kmin=k_critical+0.4,kmax=4){

  test<-function(k) impvola(k)/k^(1/2)

  plot(Vectorize(test),kmin,kmax,xlab = "k", ylab ="",
  ylim=c(beta0.5-0.5,beta0.5+0.5),lty=3,col="red")
  abline(h=beta0.5,col="blue")
  legend("topright",c("Numerical test",expression(beta[1/2])),
  col=c("red","blue"),lty=c(3,1),lwd=c(2.5,2.5))
}

#numerical test of the constant beta_(l-1/2)
Testbetaell<-function(kmin=1.1,kmax=3){

  test<-function(k) (impvola(k)-k^(1/2)*beta0.5-beta0)*k^(1/2)/log(k)

  plot(Vectorize(test),kmin,kmax,xlab = "k", ylab ="",
  ylim=c(betaell-1,betaell+1),lty=3,col="red")
  abline(h=betaell,col="blue")
  legend("topright",c("Numerical test",expression(beta[l-1/2])),
  col=c("red","blue"),lty=c(3,1),lwd=c(2.5,2.5))
```

```
}

#numerical test of the constant beta_(-1/2)
Testbetaminus<-function(kmin=k_critical+0.4,kmax=4){

  test<-function(k) (impvola(k)-k^(1/2)*beta0.5-
  beta0-betaell*log(k)/k^(1/2))*k^(1/2)

  plot(Vectorize(test),kmin,kmax,xlab = "k", ylab ="",
  ylim=c(betaminus-0.5,betaminus+0.5),lty=3,col="red")
  abline(h=betaminus,col="blue")
  legend("topright",c("Numerical test", expression(beta[-1/2])),
  col=c("red","blue"),lty=c(3,1),lwd=c(2.5,2.5))
}

#plot of volatility smile and comparison of expansions
TestImpVola<-function(kmin=1.1,kmax=7,ymax=1.6){

  Psi <-function(x) 2-4*(sqrt(x^2+x)-x)
  fourth_order<-function(k) beta0.5*k^(1/2)
  +beta0+betaell*log(k)*k^(-1/2)
  +betaminus*k^(-1/2) #4th-order expansion
  first_order<-function(k) sqrt(Psi(lambdaplus-1))*k^(1/2)
  #first-order expansion

  plot(Vectorize(impvola),kmin,kmax,xlab = "k",
  ylab ="Unannualized implied volatility", ylim=c(0,ymax))
  curve(fourth_order,add=TRUE, col="blue",lty=3) #,
  curve(first_order, add=TRUE, col="green", lty=2) #

  legend("topright",c("Unannualized implied volatility",
  "First-order approximation ", "Fourth-order approximation"),lty=c(1,2,3),
  lwd=c(2.5,2.5,2.5),col=c("black","green","blue"))
}

#comparison of approximated and exactly calculated saddle-point
Test_s<-function(kmin=k_critical+0.4,kmax=50){

  plot(Vectorize(s_exact),kmin,kmax, xlab = "k", ylab ="",ylim=c(0,25))
  curve(s_star, col="red",lty=3,add=TRUE)
  legend("topright",c("approximated saddle-point","exact saddle-point"),
  lty=c(3,1),lwd=c(2.5,2.5),col=c("red","black"))
}
```

C.2. Merton Jump Diffusion model

,The implementation in the Merton Jump diffusion model is very similar. First all constants are defined as global variables which is realised in `mjd_param()`. The saddle-point $\hat{s}$ has to be calculated numerically for every k. Here was again the `uniroot()` function used on the implicit definition of $\hat{s}$

$$m'(\hat{s}, T) = k.$$

The rest of the code plots the call price and the probabilities and compares the approximations of the Theorems 3.1-3.4 with the exact values. The `quadinf()` function is used to calculate numerically the values of the call price, density and tail probability. The implied volatility can be calculated with `uniroot()`.
For the tail probability also Corollary 3.12 is verified. Here was the exponential tail probability calculated explicitely and the Gaussian tail probability with the function `pnorm()`.

```
#parameter definition as global variables
mjd_param<-function(param){

  param1<<-c(0,1,0.1,5,-0.001,0.1)
  param2<<-c(0,0.1,0.4,0.1,0.3,0.4)
  param3<<-c(0,4,0.6,3,0.2,0.2)

  r<<-param[1]
  bigT<<-param[2]
  sigma<<-param[3]
  lambda<<-param[4]
  mu<<--param[5]
  delta<<-param[6]

  b<<- -(sigma^2/2+lambda*(exp(delta^2/2+mu)-1))

  i<<-complex(real=0,imaginary=1)
}

#mgf
M<-function(s,mat=bigT){
  as.complex(exp(mat*(sigma^2/2*s^2+b*s +
   lambda*(exp(delta^2/2*s^2+mu*s)-1))))
}

#first derivative of the cumulant generating function
m1<-function(s,mat=bigT){
  mat*(sigma^2*s +b+ lambda*exp(delta^2/2*s^2+mu*s)*(delta^2*s+mu))
}

#second derivative of the cumulant generating function
m2<-function(s){
  bigT*(sigma^2 + lambda*exp(delta^2/2*s^2+mu*s)
```

```
  *((delta^2*s+mu)^2+delta^2))
}

#saddle-point
s_hat<-function(k){
  f<-function(t) m1(t)-k
  uniroot(f,c(-100,100))$root
}

#expansion for the call price
mjd_price<-function(k){
  S<-s_hat(k)
  Re(delta^2*exp(k*(1-S))*M(S))/(2*log(k)*sqrt(2*pi*m2(S)))
}

#numerical calculation of the call price
mjd_price_numerically<-function(k){
  exp(k)/(2*pi)*quadinf(function(s) Re(exp(-k*(s_hat(k)+i*s))
  *M(s_hat(k)+ i*s)/((s_hat(k)+i*s)*(s_hat(k)-1+i*s))),-Inf, Inf)
}

#approximated density of X_T
mjd_density<-function(x){
  Re(exp(-x*s_hat(x))*M(s_hat(x))/sqrt(2*pi*m2(s_hat(x))))
}

#numerically calculated density of X_T
mjd_density_numerically<-function(x){
  i<-complex(real=0,imaginary=1)
  1/(2*pi)*(quadinf(function(s) Re(exp(-x*(s_hat(x)+i*s))
  *M(s_hat(x)+ i*s)),-Inf, Inf))
}

#approximated tail probability of X_T
mjd_tail<-function(x){
  Re(delta/2*exp(-x*s_hat(x))*M(s_hat(x))/sqrt(log(x)*pi*m2(s_hat(x))))
}

#numerically calculated tail probability of X_T
mjd_tail_numerically<-function(x){
  1/(2*pi)*(quadinf(function(s) Re(exp(-x*(s_hat(x)+i*s))
  *M(s_hat(x)+ i*s)/(s_hat(x)+i*s)),-Inf, Inf))
}

#approximated implied volatility
mjd_impvola_approx<-function(k){
  L<- -(log(mjd_price(k)))
```

```
  G<-function(kappa,u) sqrt(2)*(sqrt(u+kappa)-sqrt(u))
  approx<-G(k,L-3/2*log(L)+log(k/(4*sqrt(pi)))+9*log(L)/(4*L))

  approx
}

#Black-Scholes price
#sig denotes the unannualized volatility!
BS_Price<-function(k,sig=sigma,s0=1,r=0,mat=bigT){
  d1 = -k/sig+sig/2
  d2 = -k/sig-sig/2

  s0*pnorm(d1)-exp(k)*pnorm(d2)
}

#exact implied volatility
mjd_impvola<-function(k){

  f<-function(t) BS_Price(k,sig=t)-mjd_price_numerically(k)
  uniroot(f,c(0.00001,8))$root
}

#numerical test of call price
test_prices<-function(kmin=1.1, kmax=4,n=100){

  delta = (kmax-kmin)/n
  x<-seq(kmin,kmax, by=delta)
  y<-rep(0,n+1)

  for(i in 0:n){
    y[i+1]<-mjd_price_numerically(kmin + i/n*(kmax-kmin))
  }

  plot(Vectorize(mjd_price),kmin,kmax,col="red",xlab="k",
  ylab="Call price",lty=3,log="x")
  lines(x,y,col="blue")
  legend("topright",c("Approximation","Numerical calculation"),
  cex=0.8,col=c("red","blue"),lty=c(3,1),lwd=c(2.5,2.5))
}

#numerical test of density
test_density<-function(Lmin=0.4, Lmax=1.5,n=1000){

  delta = (Lmax-Lmin)/n
  x<-seq(Lmin,Lmax, by=delta)
  y<-rep(0,n+1)
```

```
  for(i in 0:n){
    y[i+1]<-mjd_density_numerically(Lmin + i*delta)
  }

  plot(Vectorize(mjd_density),Lmin,Lmax,col="red",xlab="x",
  ylab="Density",lty=3,ylim=c(0,1.7))#


  lines(x,y,col="blue",lty=1) #log(x)
  legend("topright",c("Approximation","Numerical calculation")
  ,cex=0.8,col=c("red","blue"),lty=c(3,1),lwd=c(2.5,2.5))
}

#numerical test of tail probability
test_tail<-function(Lmin=1.1, Lmax=5,n=1000,sdev=sigma,mean=0.1){
  #sdev... standard deviation of normal distribution
  #mean...mean of exponential distribution
  delta = (Lmax-Lmin)/n
  x<-seq(Lmin,Lmax, by=delta)
  y<-rep(0,n+1)
  nor<-function(x) 1-pnorm(x,sd=sdev)
  ex<-function(x)  exp(-x/mean)
  for(i in 0:n){
    y[i+1]<-mjd_tail(Lmin + i*delta)
  }

  plot(Vectorize(mjd_tail_numerically),Lmin,Lmax,
  col="blue",xlab="x",ylab="Tail probability")
  curve(nor,add=TRUE,col="green",lty=2)
  curve(ex,add=TRUE,col="magenta",lty=4)
  lines(x,y,col="red",lty=3)

  legend("topright",c("MJD exact","MJD approximated","Gaussian",
  "Exponential"), col=c("blue","red","green","magenta"),
  lty=c(1,3,2,4),lwd=c(2.5,2.5,2.5,2.5))
}

#numerical test of implied volatility
test_impvola<-function(min=1.1,max=4,n=100,ymax=1.6){

  delta=(max-min)/n
  x<-seq(min,max,by=delta)

  firstorder<-function(k) sqrt(delta*k)/sqrt(2*sqrt(2*log(k)))
  vola<-rep(0,n+1)
  for(i in 0:n){
    vola[i+1]<-mjd_impvola_approx(min+i*delta)
```

```
  }

  plot(Vectorize(mjd_impvola),min,max,xlab = "k",
  ylab ="Unannualized implied volatility",ylim=c(0,ymax))
  curve(firstorder, add=TRUE, col="green",lty=2)
  lines(x,vola,lty=3,col="blue")
  legend("topright",c("Unannualized implied volatility",
   "Refined expansion", "First-order expansion"),
   lty=c(1,3,2),col=c("black","blue","green"),lwd=c(2.5,2.5,2.5))
}
```

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