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SADDLEPOINT APPROXIMATION OF RISK MEASURES

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Lorenz Weiler
01326362

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Institut für Stochastik und Wirtschaftsmathematik
der Fakultät für Mathematik und Geoinformation
der Technischen Universität Wien

betreut durch

Assoc. Prof. Dipl.-Ing. Dr. techn. Stefan Gerhold

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Tyche, Goddess of Chance, chooses a point ω of Ω at random according to the law P in that, for $\mathsf{F} \in \mathcal{F}$, $\mathsf{P}(\mathsf{F})$ represents the probability in the sense understood by our intuition that the point ω chosen by Tyche belongs to F.

Williams, 1991

ACKNOWLEDGEMENT

First, my gratitude goes to Dr Stefan Gerhold for supervising my thesis and to Mag Maria Grabner for sparking my interest in mathematics.

Also, my thanks are due to my girlfriend for her affectionate support and to my parents, who enable me to exercise my personal interests.

Last, I express my appreciation for my friends, whose nonconformity up to the point of eccentricity is an inestimable source of inspiration.

REMARKS

NOVELTY

This thesis introduces the saddlepoint approximation of risk measures from scratch. Sophisticated notation allows to reveal the conceptional structure on which this special kind of approximation is based.

INFORMALITY

To save us from the burden of undue rigor, we do not address every mathematical detail. We shall assume the existence of integrals and interchange differentiation and expectation without further ado.

UNCERTAINTY

We represent our uncertainty about the future state of the world by a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, the steering mechanism for the probabilistic evolution and the domain of all random variables we introduce.

OUTLINE

Knowledge about the distribution of a random variable is of practical importance. Unfortunately, its density and distribution function are often unknown. In principle, knowledge about the moment-generating or cumulant-generating function permits to obtain both functions using certain integral inversion formulas. In practice, though, the complexity of the integration involved may be unduly costly. Fortunately, the inversion integrals can be approximated, which is where the saddlepoint approximation comes into play. We follow the schematic strategy

$$\text{function} \longrightarrow \text{representation} \longrightarrow \text{approximation}.$$

Although we initially consider saddlepoint approximations for arbitrarily large sample sizes, they finally prove efficient even for only a single observation. This rather surprising approach takes the form

$$n \to \infty \quad \longrightarrow \quad n = 1.$$

Chapter 1 introduces the notion of asymptotic power series, whose successive terms may give an increasingly accurate description of the asymptotic behaviour of a function. For that reason, approximations are always to be understood as truncated asymptotic power series.

Chapter 2 deals with integral transforms, namely the Laplace transform on the one hand and the moment-generating and the cumulant-generating function on the other hand. All three are closely related.

Chapter 3 and chapter 4 introduce the basic concepts with which the saddlepoint approximations in chapter 5 are eventually established.

Chapter 6 is about the application of the saddlepoint approximations to common risk measures and chapter 7 considers risk contributions.

Appendix A gives an overview of mathematical expressions related to the thesis and appendix B is about the preparation of the document.

CONTENTS

Part I

THEORY

1 ASYMPTOTIC EXPANSIONS

We follow de Bruijn, 1958, who impresses with an unusual way of presenting a mathematical subject as some excerpts will demonstrate.

Usually in asymptotic analysis, the limit of a function is taken as

$$z \to z_0 \qquad (z \in \mathbb{C}).$$

Note that z_0 can be taken to be ∞ by the change of variable

$$\zeta = \frac{1}{z - z_0} \qquad (z \in \mathbb{C}).$$

Hereafter, we are solely concerned with the limit of a function as

$$n \to \infty \qquad (n \in \mathbb{N}).$$

Thus, let $f(n)$ and $g(n)$ be two functions of an integer variable n.

Definition. We write

$$f(n) = O(g(n)) \qquad (n \to \infty)$$

if and only if

$$\exists A\colon \exists N\colon \forall n > N\colon |f(n)| \leqslant A \cdot |g(n)|.$$

Lemma. If $g(n)$ is nonzero for n sufficiently large, then $f(n) = O(g(n))$ if and only if

$$\limsup_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| < \infty.$$

Definition. We write

$$f(n) = o(g(n)) \qquad (n \to \infty)$$

if and only if

$$\forall A\colon \exists N\colon \forall n > N\colon |f(n)| \leqslant A \cdot |g(n)|.$$

Lemma. If $g(n)$ is nonzero for n sufficiently large, then $f(n) = o(g(n))$ if and only if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Definition. We write

$$f(n) \sim g(n) \qquad (n \to \infty)$$

The reader should notice that so far we did not define what $O(\varphi(s))$ means; we only defined the meaning of some complete formulas. It is obvious that the isolated expression $O(\varphi(x))$ cannot be defined, at least not in such a way that (1.2.5) remains equivalent to (1.2.6). For, $f(s) = O(\varphi(s))$ obviously implies $2f(s) = O(\varphi(s))$. If $O(\varphi(s))$ in itself were to denote anything, we would infer $f(s) = = O(\varphi(s)) = 2f(s)$, whence $f(s) = 2f(s)$.

The trouble is, of course, due to abusing the equality sign $=$. A similar situation arises if someone, because his typewriter lacks the sign $<$, starts to write $= L$ for the words "is less than", and so writes $3 = L(5)$. On being asked: "What does $L(5)$ stand for?", he has to reply "Something that is less than 5". Consequently, he rapidly gets the habit of reading L as "something that is less than", thus coming close to the actual words we used when introducing (1.2.5). After that, he writes $L(3) = L(5)$ (something that is less than 3 is something that is less than 5), but certainly not $L(5) = L(3)$. He will not see any harm in $4 = 2 + L(3)$, $L(3) + L(2) = L(8)$.

The O-symbol is used in exactly the same manner as this person's L-symbol. We give a few examples:

$$O(x) + O(x^2) = O(x) \qquad (x \to 0).$$

This means: for any pair of functions f, g, such that

$$f(x) = O(x) \ \ (x \to 0), \qquad g(x) = O(x^2) \ \ (x \to 0),$$

we have

$$f(x) + g(x) = O(x) \qquad (x \to 0).$$

The common interpretation of all these formulas can be expressed as follows. Any expression involving the O-symbol is to be considered as a class of functions: If the range $0 < x < \infty$ is considered, then $O(1)+O(x^2)$ denotes the class of all functions of the form $f(x) + g(x)$, with $f(x) = O(1)$ $(0 < x < \infty)$, $g(x) = O(x^2)$ $(0 < x < \infty)$. And $x^{-1}O(1) = O(1) + O(x^{-2})$ means that the class $x^{-1}O(1)$ is contained in the class $O(1) + O(x^{-2})$. Sometimes the left-hand-side of a relation is not a class, but a single function, as in (1.2.7). Then the relation means that the function on the left is a member of the class on the right.

de Bruijn, 1958

if and only if

$$f(n) - g(n) = o(g(n)) \qquad (n \to \infty).$$

Lemma. If $g(n)$ is nonzero for n sufficiently large, then $f(n) \sim g(n)$ if and only if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

Example. As is generally known, Stirling's formula states that

$$n! \sim \sqrt{2\pi n}\left[\frac{n}{e}\right]^n \qquad (n \to \infty). \tag{1.1}$$

Among formal power series, we distinguish between convergent power series and asymptotic power series. As one can show, the set of convergent power series is contained in the set of asymptotic power series.

To emphasize the differences between both concepts, we bring the defintions of convergent power series and asymptotic power series face to face. The first is due to B. Taylor, the second due to H. Poincaré.

Note, though, that there is a more general notion of asymptotic power series, with which asymptotic analysis can unfold its full potential.

Definition. We call

$$f(n) = \sum_{j=0}^{\infty} \frac{a_j}{n^j} \qquad (n \to \infty)$$

the convergent expansion of $f(n)$ and say that $f(n)$ can be expanded into a convergent power series around ∞ if there are coefficients $a_j \in \mathbb{R}$ such that

$$\exists\, N\colon\ \forall n > N\colon\ f(n) - \sum_{j=0}^{m} \frac{a_j}{n^j} \to 0 \qquad (m \to \infty).$$

Definition. We call

$$f(n) \approx h(n) \sum_{j=0}^{\infty} \frac{a_j}{n^j} \qquad (n \to \infty)$$

the asymptotic expansion of $f(n)$ and say that $f(n)$ can be expanded into an asymptotic power series around ∞ if there are coefficients $a_j \in \mathbb{R}$ and there is a function $h(n)$ such that

$$\forall\, m\colon\ f(n) - h(n) \sum_{j=0}^{m} \frac{a_j}{n^j} = o\left(\frac{h(n)}{n^m}\right) \qquad (n \to \infty).$$

Approximation of the factorial using Stirling's series (1.2).

order	n	$n!$	approximation	relative error [%]
1	1	1	0.9221	7.7863
	2	2	1.9190	4.0498
	3	6	5.8362	2.7298
	4	24	23.5062	2.0576
	5	120	118.0192	1.6507
	6	720	710.0782	1.3780
2	1	1	0.9990	0.1018
	2	2	1.9990	0.0519
	3	6	5.9983	0.0279
	4	24	23.9959	0.0171
	5	120	119.9862	0.0115
	6	720	719.9404	0.0083
3	1	1	1.0022	0.2184
	2	2	2.0006	0.0314
	3	6	6.0006	0.0096
	4	24	24.0010	0.0041
	5	120	120.0025	0.0021
	6	720	720.0089	0.0012

The latter condition holds true if and only if

$$\forall\, m\colon f(n) - h(n)\sum_{j=0}^{m}\frac{a_j}{n^j} \sim \frac{a_M}{n^M} \qquad (n\to\infty),$$

where the coefficient a_M denotes the first nonzero coefficient after the coefficient a_m. The first nonzero term of an asymptotic expansion is called the dominant term. Under the m-th order approximation of the function $f(n)$ we understand the truncated asymptotic expansion

$$f(n) \sim h(n)\sum_{j=0}^{m-1}\frac{a_j}{n^j} \qquad (n\to\infty).$$

Example. Stirling's formula (1.1) clearly being the dominant term, the factorial can be expanded into Stirling's series which is given by

$$n! \approx \sqrt{2\pi n}\left[\frac{n}{e}\right]^n\left[1+\frac{1}{12n}+\frac{1}{288n^2}+\cdots\right] \qquad (n\to\infty). \tag{1.2}$$

Now, if

$$\forall\, m\colon g(n) = o\left(\frac{h(n)}{n^m}\right) \qquad (n\to\infty),$$

then

$$g(n) \approx h(n)\sum_{j=0}^{\infty}\frac{0}{n^j} \qquad (n\to\infty).$$

Hence, if

$$f(n) \approx h(n)\sum_{j=0}^{\infty}\frac{a_j}{n^j} \qquad (n\to\infty),$$

then

$$f(n)+g(n) \approx h(n)\sum_{j=0}^{\infty}\frac{a_j}{n^j} \qquad (n\to\infty).$$

This is why the function $f(n)$ and any function which deviates only by such a so-called recessive function $g(n)$ can both be expanded into the same asymptotic power series. The coefficients of the asymptotic power series, however, are uniquely determined by the recursion

$$\forall\, m\colon a_m = \lim_{n\to\infty} n^m\left[f(n) - \sum_{j=0}^{m-1}\frac{a_j}{n^j}\right].$$

In most cases, the final estimates obtained in this way are rather weak, with constants a thousand times, say, greater than they could be. The reason is, of course, that such estimates are obtained by means of a considerable number of steps, and in each step a factor 2 or so is easily lost. Quite often it is possible to reduce such errors by a more careful examination.

But even if the asymptotic result is presented in its best possible explicit form, it need not be satisfactory from the numerical point of view. The following dialogue between Miss N.A., a Numerical Analyst, and Dr A.A., an Asymptotic Analyst, is typical in several respects.

N.A.: I want to evaluate my function $f(x)$ for large values of x, with a relative error of at most 1%.

A.A.: $f(x) = x^{-1} + O(x^{-2}) \qquad (x \to \infty)$.

N.A.: I am sorry, but I don't understand.

A.A.: $|f(x) - x^{-1}| < 8x^{-2} \qquad (x > 10^4)$.

N.A.: But my value of x is only 100.

A.A.: Why did not you say so? My evaluations give

$$|f(x) - x^{-1}| < 57000x^{-2} \qquad (x \geq 100).$$

N.A.: This is no news to me. I know already that $0 < f(100) < 1$.

A.A.: I can gain a little on some of my estimates. Now I find that

$$|f(x) - x^{-1}| < 20x^{-2} \qquad (x \geq 100).$$

N.A.: I asked for 1%, not for 20%.

A.A.: It is almost the best thing I possibly can get. Why don't you take larger values of x?

N.A.: !!!. I think it's better to ask my electronic computing machine.

Machine: $f(100) = 0{\cdot}01137\ 42259\ 34008\ 67153$.

A.A.: Haven't I told you so? My estimate of 20% was not very far from the 14% of the real error.

N.A.: !!! . . . !.

Some days later, Miss N.A. wants to know the value of $f(1000)$. She now asks her machine first, and notices that it will require a month, working at top speed. Therefore, she returns to her Asymptotic Colleague, and gets a fully satisfactory reply.

de Bruijn, 1958

Example. We consider the Exponential integral

$$\mathrm{Ei}(n) := \int_n^\infty \frac{e^{-x}}{x}\,dx \qquad (n \in \mathbb{N}).$$

After integrating by parts m times, we get

$$\mathrm{Ei}(n) = e^{-n} \sum_{j=1}^{m} (-1)^{j-1} \frac{(j-1)!}{n^j} + (-1)^m m! \int_n^\infty \frac{e^{-x}}{x^{m+1}}\,dx.$$

Let $S_m(n)$ denote the partial sum of the first m terms,

$$S_m(n) := e^{-n} \sum_{j=1}^{m} (-1)^{j-1} \frac{(j-1)!}{n^j},$$

and let $R_m(n)$ denote the remainder after m terms,

$$R_m(n) := (-1)^m m! \int_n^\infty \frac{e^{-x}}{x^{m+1}}\,dx.$$

On the one hand, $S_m(n)$ diverges since the m-th term of S_m is growing in magnitude as m increases. On the other hand, $S_m(n) + R_m(n)$ is bounded since $\mathrm{Ei}(n)$ is bounded. Consequently, R_m is unbounded.

The inequality

$$|R_m(n)| = m! \int_n^\infty \frac{e^{-x}}{x^{m+1}}\,dx < \frac{m!}{n^{m+1}} \int_n^\infty e^{-x}\,dx = \frac{m!\,e^{-n}}{n^{m+1}}$$

yields

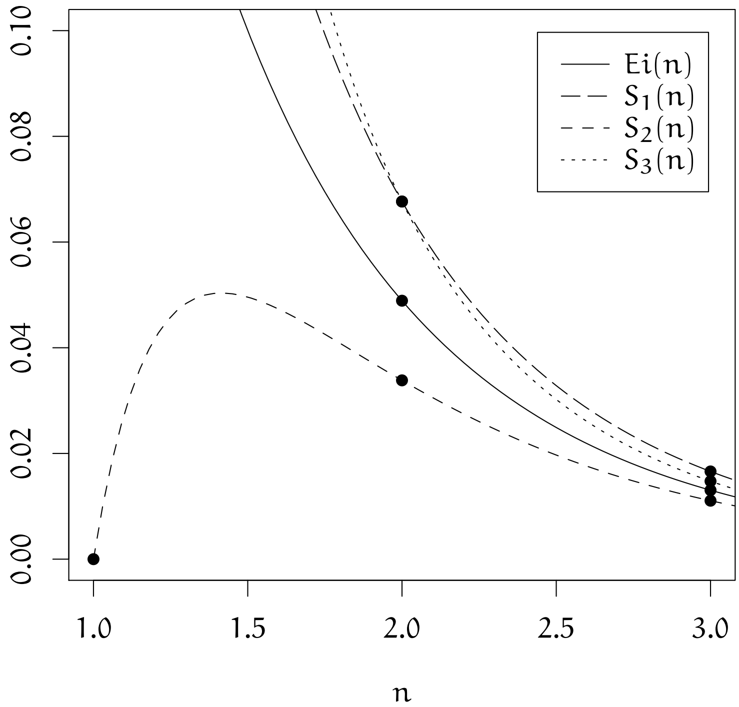
$$\mathrm{Ei}(n) - S_m(n) = R_m(n) = o\left(\frac{e^{-n}}{n^m}\right) \qquad (n \to \infty).$$

Thus,

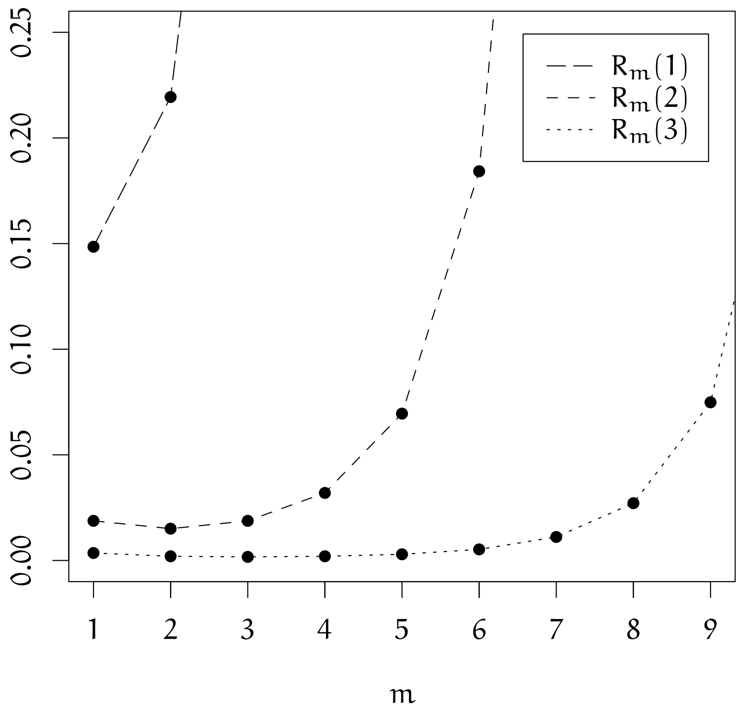
$$\mathrm{Ei}(n) \approx e^{-n} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{(j-1)!}{n^j} \qquad (n \to \infty).$$

Note that $\mathrm{Ei}(n)$ cannot be expanded into a convergent power series but into an asymptotic power series. The dominant term is given by

$$\mathrm{Ei}(n) \sim \frac{e^{-n}}{n} \qquad (n \to \infty).$$



$\mathrm{Ei}(n)$ and $S_m(n)$ for fixed m and variable n.



$R_m(n)$ for variable m and fixed n.

In chapter 5, we shall make use of the following versions of Watson's and Temme's lemma for deriving asymptotic expansions of certain integrals. We refer to Daniels, 1954 and Broda and Paolella, 2010.

Lemma (Watson)**.** Let $\zeta(z)$ be holomorphic in a neighbourhood of $z = 0$ and bounded for $z = \psi$ in $\gamma \leqslant \psi \leqslant \delta$ with $\gamma < 0 < \delta$. Then

$$\sqrt{\frac{n}{2\pi}} \int_{\gamma}^{\delta} \exp\Big(-\frac{n}{2}\psi^2\Big)\zeta(\psi)\,d\psi \approx \sum_{j=0}^{\infty} \frac{\zeta^{(2j)}(0)}{2^j j!\,n^j} \qquad (n \to \infty). \tag{1.3}$$

Lemma (Temme)**.** Let $A_n(z)$ be holomorphic in a strip containing the real axis in its interior. For $j \in \mathbb{N}$, if there are $\gamma_j, \delta \in \mathbb{R}$ such that

$$\forall j\colon A_n^{(j)}(z) = O\left(|z|^{\gamma_j} e^{\delta z^2}\right) \qquad (\mathrm{Re}(z) \to \pm\infty)$$

and if there are holomorphic functions $a_{z,j}$ such that

$$A_n(z) \approx \sum_{j=0}^{\infty} \frac{a_{z,j}}{n^j} \qquad (n \to \infty),$$

then, considering the integral function

$$B_n(t) \coloneqq \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{t} \exp\Big(-\frac{n}{2}s^2\Big)A_n(s)\,ds \qquad (t \in \mathbb{R}),$$

one has

$$B_n(t) = \Phi(t\sqrt{n})B_n(\infty) + \frac{\varphi(t\sqrt{n})}{\sqrt{n}}C_n(t) \qquad (n \to \infty), \tag{1.4}$$

where

$$B_n(\infty) \approx \sum_{j=0}^{\infty} \frac{b_j}{n^j} \qquad (n \to \infty)$$

and

$$C_n(t) \approx \sum_{j=0}^{\infty} \frac{c_{t,j}}{n^j} \qquad (n \to \infty)$$

with

$$c_{t,0} = b_0 - a_{t,0} \qquad (j = 0)$$

and

$$tc_{t,j} = b_j - a_{t,j} + \left[\frac{d}{ds}c_{s,j-1}\right]_{s=t} \qquad (j > 0).$$

Here, $\varphi(t) = \Phi'(t)$ denotes the standard normal density function.

2 INTEGRAL TRANSFORMS

We first establish the link between the Laplace transform and generating functions and then present some handy inversion formulas.

LAPLACE TRANSFORM

Usually, the unilateral Laplace transform or the Laplace transform of a real variable is considered. In the following, we introduce the bilateral Laplace transform of a complex variable. We refer to Widder, 1941 and Doetsch, 1958 for a rather extensive treatment of related issues.

Let $f(t)$ be a real-valued function of a real variable t.

Definition. The bilateral Laplace transform $L_f(z)$ of $f(t)$ is defined by

$$L_f(z) := \int_{-\infty}^{+\infty} e^{-tz} f(t)\, dt \qquad (z \in \mathbb{C}). \tag{2.1}$$

In case the complex variable is purely imaginary, meaning that $z = iy$, then $L_f(z)$ becomes the common Fourier transform $F_f(y)$ of $f(t)$.

We next obtain a formula which recovers $f(t)$ from its Laplace transform $L_f(z)$. A historical remark can be found in Tamarkin, 1926.

Lemma. Let $f(t)$ be a continuous function of bounded variation. For $\gamma \in \mathbb{R}$, if $L_f(\gamma)$ converges absolutely, one has the inversion formula

$$f(t) = \lim_{\delta \to \infty} \frac{1}{2\pi i} \int_{\gamma - i\delta}^{\gamma + i\delta} e^{tz} L_f(z)\, dz. \qquad (t \in \mathbb{R}). \tag{2.2}$$

This formula is called Bromwich's, Mellin's or Fourier and Mellin's inversion formula. Moreover, we speak of Bromwich's integral and of Bromwich's contour. If γ is zero, then the formula becomes Fourier's inversion formula. In this case, the contour clearly coincides with the imaginary axis and the integral is said to be Fourier's integral.

If Bromwich's integral, which is given as Cauchy principal value, even exists as improper integral, then both integrals coincide, that is

$$\lim_{\delta \to \infty} \int_{\gamma - i\delta}^{\gamma + i\delta} e^{tz} L_f(z)\, dz = \int_{\gamma - i\infty}^{\gamma + i\infty} e^{tz} L_f(z)\, dz.$$

It is worth mentioning that there are other inversion formulas for the Laplace transform such as Post's inversion formula, which involves differentiation instead of integration. See Cohen, 2007 for details.

GENERATING FUNCTIONS

From now on, let X be a real-valued random variable with distribution function $F_X(t)$ and suppose that X has a density function $f_X(t)$.

Definitions

At first, the following definitions and notations might seem needlessly complicated. However, they will turn out to come in very useful.

Definition. The moment-generating function $M_X(z)$ of X is defined by

$$M_X(z) \coloneqq \mathrm{E}(e^{zX}) = \int_{-\infty}^{+\infty} e^{zX}\,dP \qquad (z \in \mathbb{C}).$$

In case the complex variable is purely imaginary, that is $z = iy$, then $M_X(z)$ becomes the common characteristic function $C_X(y)$ of X.

The n-th moment $m_{X,n,t}$ of X at $z_t \in \mathbb{C}$ is defined by

$$m_{X,n,t} \coloneqq M_X^{(n)}(z_t) \qquad (n \in \mathbb{N})$$

and the n-th moment $m_{X,n}$ of X is defined by

$$m_{X,n} \coloneqq M_X^{(n)}(0) \qquad (n \in \mathbb{N}).$$

For the sake of clarity, we write

$$\mu_{X,t} \coloneqq m_{X,1,t}, \quad \mu_X \coloneqq m_{X,1}$$

and, using this notation,

$$\sigma_{X,t}^2 \coloneqq m_{X-\mu_X,2,t}, \quad \sigma_X^2 \coloneqq m_{X-\mu_X,2}.$$

Clearly,

$$\mathrm{E}(X) = \mu_X, \quad \mathrm{V}(X) = \sigma_X^2.$$

The n-th standard moment $\nu_{X,n,t}$ of X at $z_t \in \mathbb{C}$ is defined by

$$\nu_{X,n,t} \coloneqq \frac{m_{X-m_X,n,t}}{\sigma_{X,t}^n} \qquad (n \in \mathbb{N})$$

and the n-th standard moment $\nu_{X,n}$ of X is defined by

$$\nu_{X,n} \coloneqq \frac{m_{X-m_X,n}}{\sigma_X^n} \qquad (n \in \mathbb{N}).$$

Definition. The cumulant-generating function $K_X(z)$ of X is defined by

$$K_X(z) \coloneqq \log \mathrm{E}(e^{zX}) = \log \int_{-\infty}^{+\infty} e^{zX}\,dP \qquad (z \in \mathbb{C}).$$

The n-th cumulant $k_{X,n,t}$ of X at $z_t \in \mathbb{C}$ is defined by

$$k_{X,n,t} \coloneqq K_X^{(n)}(z_t) \qquad (n \in \mathbb{N})$$

and the n-th cumulant $k_{X,n}$ of X is defined by

$$k_{X,n} \coloneqq K_X^{(n)}(0) \qquad (n \in \mathbb{N}).$$

The n-th standard cumulant $\lambda_{X,n,t}$ of X at $z_t \in \mathbb{C}$ is defined by

$$\lambda_{X,n,t} \coloneqq \frac{k_{X,n,t}}{\sqrt{k_{X,2,t}^n}} \qquad (n \in \mathbb{N})$$

and the n-th standard cumulant $\lambda_{X,n}$ of X is defined by

$$\lambda_{X,n} \coloneqq \frac{k_{X,n}}{\sqrt{k_{X,2}^n}} \qquad (n \in \mathbb{N}).$$

Relationships

As indicated,

$$M_X(-z) = L_{f_X}(z), \quad K_X(-z) = \log L_{f_X}(z).$$

Furthermore,

$$m_{X,n} = \mathrm{E}(X^n) = \int_{-\infty}^{+\infty} t^n f_X(t)\, dt \tag{2.3}$$

and

$$\nu_{X,n} = \mathrm{E}\left(\left[\frac{X-\mu_X}{\sigma_X}\right]^n\right) = \int_{-\infty}^{+\infty} \left[\frac{t-\mu_X}{\sigma_X}\right]^n f_X(t)\, dt. \tag{2.4}$$

Note that

$$\mu_X = k_{X,1}, \quad \sigma_X^2 = k_{X,2}.$$

Consider the n-th incomplete exponential Bell polynomial

$$B_{n,m}(x_1, \ldots, x_{n-m+1}) \coloneqq \sum \prod_{j=1}^{n-m+1} \frac{n!}{l_j} \left[\frac{x_j}{j!}\right]^{l_j},$$

where the sum is taken over all nonnegative integer solutions to

$$\sum_{j=1}^{n-m+1} l_j = m, \quad \sum_{j=1}^{n-m+1} j l_j = n.$$

The first moments and cumulants.

n	$m_{X,n}$	$\nu_{X,n}$	$k_{X,n}$	$\lambda_{X,n}$
1	mean	0	mean	–
2	–	1	variance	1
3	–	skewness	–	skewness
4	–	kurtosis	–	excess

Firstly,

$$m_{X,n} = \sum_{j=1}^{n} B_{n,j}(k_{X,1}, \dots, k_{X,n-j+1}) \tag{2.5}$$

and

$$k_{X,n} = \sum_{j=1}^{n} (-1)^{j-1}(j-1)!\, B_{n,j}(m_{X,1}, \dots, m_{X,n-j+1}). \tag{2.6}$$

Secondly,

$$\nu_{X,n} = \sum_{j=1}^{n} B_{n,j}(0, \lambda_{X,2}, \dots, \lambda_{X,n-j+1}) \tag{2.7}$$

and

$$\lambda_{X,n} = \sum_{j=1}^{n} (-1)^{j-1}(j-1)!\, B_{n,j}(0, \nu_{X-\mu_X,2}, \dots, \nu_{X-\mu_X,n-j+1}). \tag{2.8}$$

See appendix A for the first few instances of these relationships.

Independence

Let the sum of n independent random variables X_j be given by

$$S_n := \sum_{j=1}^{n} X_j,$$

then

$$M_{S_n}(z) = \prod_{j=1}^{n} M_{X_j}(z), \quad K_{S_n}(z) = \sum_{j=1}^{n} K_{X_j}(z) \qquad (n \in \mathbb{N}). \tag{2.9}$$

INVERSION FORMULAS

As to the following lemmas, note that Fubini's theorem justifies interchanging the order of integration as long as all points of the integration contour have a positive real part which is bounded away from zero.

Lemma. If $\gamma \in \mathbb{R}$, then

$$f_X(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{K_X(z) - tz}\, dz \qquad (t \in \mathbb{R}). \tag{2.10}$$

Proof. By Bromwich's inversion formula,

$$f_X(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{tz} L_{f_X}(z)\, dz,$$

but

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{tz} L_{f_X}(z)\, dz = \int_{\gamma-i\infty}^{\gamma+i\infty} e^{K_X(z)-tz}\, dz.$$

□

Lemma. If $\gamma > 0$, then

$$P(X > t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{K_X(z)-tz}}{z}\, dz \qquad (t \in \mathbb{R}). \tag{2.11}$$

Proof. By the inversion formula (2.10),

$$\int_t^\infty f(s)\, ds = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{K_X(z)} \int_t^\infty e^{-sz}\, ds\, dz,$$

but

$$\int_t^\infty e^{-sz}\, ds = \frac{e^{-tz}}{z}.$$

□

A more general formula in Daniels, 1987 states the following.

Lemma. If $\gamma \in \mathbb{R}$, then

$$P(X > t) = H(\gamma) + \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{K_X(z)-tz}}{z}\, dz \qquad (t \in \mathbb{R}),$$

where

$$H(\gamma) := \begin{cases} 1 & \text{if } \gamma < 0, \\ 1/2 & \text{if } \gamma = 0, \\ 0 & \text{if } \gamma > 0. \end{cases}$$

Lemma. If $\gamma > 0$, then

$$\mathrm{E}\big([X - t]\mathbb{1}_{\{X>t\}}\big) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{K_X(z)-tz}}{z^2}\,dz \qquad (t \in \mathbb{R}). \tag{2.12}$$

Proof. By the inversion formula (2.10)

$$\int_t^{\infty} [s-t]f(s)\,ds = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{K_X(z)} \int_t^{\infty} [s-t]e^{sz}\,ds\,dz,$$

but

$$\int_t^{\infty} [s-t]e^{sz}\,ds = \frac{e^{-tz}}{z^2}.$$

□

Note that

$$\mathrm{E}\big([X - t]\mathbb{1}_{\{X>t\}}\big) = \mathrm{E}\big([X-t]^+\big) \qquad (t \in \mathbb{R}).$$

Lemma. If $\gamma > 0$, then

$$\mathrm{E}\big(X\mathbb{1}_{\{X>t\}}\big) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} K_X'(z)\frac{e^{K_X(z)-tz}}{z}\,dz \qquad (t \in \mathbb{R}). \tag{2.13}$$

Proof. First, assume that X has a positive lower bound. Let

$$Q(F) := \int_F \frac{X}{\mu_X}\,dP \qquad (F \in \mathcal{F}).$$

Writing the expectation as an integral, one has

$$\mathrm{E}\big(X\mathbb{1}_{\{X>t\}}\big) = \mu_X Q(X > t).$$

By the inversion formula (2.11),

$$Q(X > t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{K_X^Q(z)-tz}}{z}\,dz.$$

However,

$$M_X^Q(z) = \int_{-\infty}^{+\infty} e^{zX}\frac{X}{\mu_X}\,dP = \frac{M_X'(z)}{\mu_X} = K_X'(z)\frac{M_X(z)}{\mu_X},$$

so

$$K_X^Q(z) = \log K_X'(z) + K_X(z) - \log \mu_X.$$

Hence,

$$E\big(X\mathbb{1}_{\{X>t\}}\big) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} K_X'(z)\frac{e^{K_X(z)-tz}}{z}\,dz.$$

Secondly, assume that X has a negative lower bound $-a < 0$. Then

$$Y := X + a$$

has a positive lower bound. As one can easily see, we have that

$$K_Y(z) = K_X(z) + az, \quad K_Y'(z) = K_X'(z) + a.$$

Clearly,

$$E\big(X\mathbb{1}_{\{X>t\}}\big) = E\big(Y\mathbb{1}_{\{Y-a>t\}}\big) - a\,P(Y - a > t).$$

By the first step,

$$E\big(Y\mathbb{1}_{\{Y-a>t\}}\big) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} [K_X'(z) + a]\frac{e^{K_X(z)+az-z[t+a]}}{z}\,dz,$$

but, by the inversion formula (2.11),

$$\frac{a}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{K_X(z)-tz}}{z}\,dz = a\,P(Y - a > t).$$

Combining both equations, we consequently get

$$E\big(X\mathbb{1}_{\{X>t\}}\big) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} K_X'(z)\frac{e^{K_X(z)-tz}}{z}\,dz.$$

Thirdly, assume that X is unbounded. For any constant $-a < 0$,

$$Z := \max(X, -a)$$

is bounded from below. Clearly, we have

$$E\big(X\mathbb{1}_{\{X>t\}}\big) = E\big(Z\mathbb{1}_{\{Z>t\}}\big) \qquad (t > a).$$

By the second step,

$$\mathsf{E}\big(Z\mathbb{1}_{\{Z>t\}}\big) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} K_Z'(z)\frac{e^{K_Z(z)-tz}}{z}\,dz,$$

hence

$$\mathsf{E}\big(Z\mathbb{1}_{\{Z>t\}}\big) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} M_Z'(z)\frac{e^{tz}}{z}\,dz.$$

As one can show, one has the relationship

$$M_Z'(z) \uparrow M_X'(z) \qquad (a \to \infty).$$

Thus, by monotone convergence,

$$\mathsf{E}\big(X\mathbb{1}_{\{X>t\}}\big) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} M_X'(z)\frac{e^{-tz}}{z}\,dz \qquad (t \in \mathbb{R}),$$

which finally yields

$$\mathsf{E}\big(X\mathbb{1}_{\{X>t\}}\big) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} K_X'(z)\frac{e^{K_X(z)-tz}}{z}\,dz \qquad (t \in \mathbb{R}).$$

□

3 SADDLEPOINT METHOD

The saddlepoint method or steepest descent method deals with the asymptotic expansion of integrals on the complex plane. It is an extension of Laplace's method, which is merely concerned with integrals on the real axis. The method goes back to B. Riemann and P. Debye.

To start with, we recall some basic facts from real and complex analysis about the analytic description of some surface. Let $u(x, y)$ be a smooth real-valued function of two real variables x and y. The gradient

$$\nabla u(x_t, y_t) = (u_x, u_y)(x_t, y_t) \qquad (x_t, y_t \in \mathbb{R})$$

points in the direction of the greatest rate of increase of $u(x, y)$ at the point (x_t, y_t). We hence speak of a steepest ascent contour or of a steepest descent contour if its tangent at each point (x_t, y_t) has the direction $\nabla u(x_t, y_t)$ or the direction $-\nabla u(x_t, y_t)$, respectively. The contour is steepest contour if it is either a steepest ascent or a steepest descent contour. The gradient is orthogonal to the level contour

$$u^{-1}(u(x_t, y_t)) = \{(x, y) \colon u(x, y) = u(x_t, y_t)\},$$

along which $u(x, y)$ takes on the given constant value $u(x_t, y_t)$. A point (x_t, y_t) is said to be a critical or a stationary point of $u(x, y)$ if

$$\nabla u(x_t, y_t) = 0,$$

whereas a critical point of $u(x, y)$ which is, however, no local extremum of $u(x, y)$ is said to be a saddlepoint of $u(x, y)$. Next, let

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

be a holomorphic function. The real part $u(x_t, y_t)$ and the imaginary part $v(x_t, y_t)$ are linked through the equations

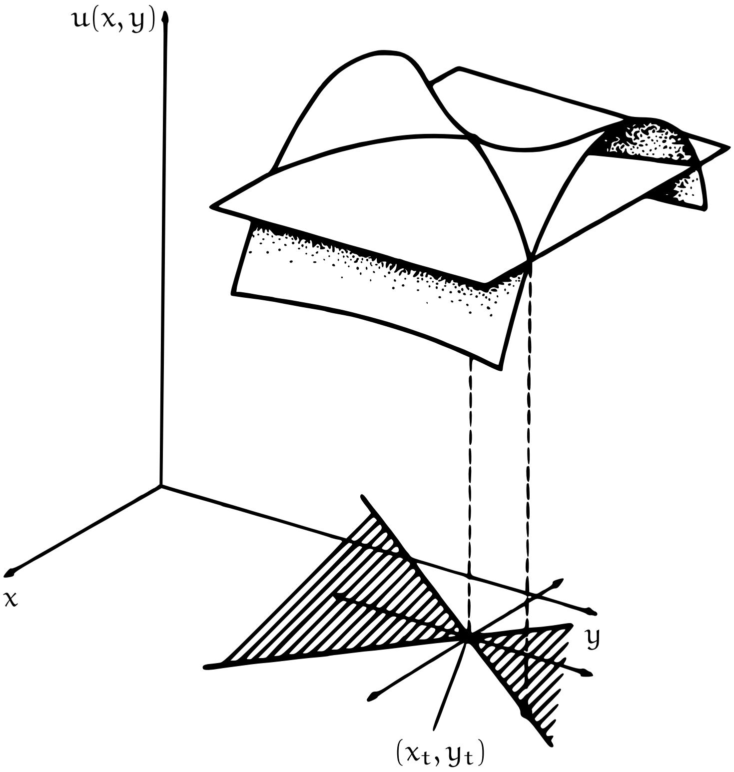
$$u_x(x_t, y_t) = u_y(x_t, y_t), \quad u_y(x_t, y_t) = -v_x(x_t, y_t),$$

being the analytic expression for the geometrical insight that $f(x + iy)$ has the same limit as $x + iy$ approaches $x_t + iy_t$ along the real axis and the imaginary axis, respectively.

Lemma. A critical point of $u(x, y)$ is a saddlepoint of $u(x, y)$.

Proof. Let H denote the Hessian matrix of u. On the one hand, we have

$$\det H = u_{xx}u_{yy} - (u_{xy})^2 = -(v_{xy}^2) - (u_{xy})^2 < 0.$$



Some function $u(x, y)$ near its saddlepoint (x_t, y_t).

On the other hand, $\det H$ is the product of both eigenvalues of H. As a consequence, the eigenvalues of H have a different sign so that H is indefinite. This is why a critical point of u is a saddlepoint of u. □

Lemma. The level contour of $u(x, y)$ is orthogonal to the level contour of $v(x, y)$. In particular, the steepest contour of $u(x, y)$ coincides with the level contour of $v(x, y)$ and vice versa.

Proof. Since

$$\nabla u \cdot \nabla v = u_x v_x + u_y v_y = 0,$$

the gradient ∇u is orthogonal to the gradient ∇v, which clearly means that the level contour of u is orthogonal to the level contour of v. □

Next, we informally state Cauchy's integral theorem as a lemma.

Lemma. Let two different contours connect the same two points. If a function is holomorphic everywhere in between the two contours, then the two contour integrals of the function are the same.

By referring to de Bruijn, 1958, Copson, 1965 and Murray, 1974, we address ourselves to the discussion of the steepest descent method.

Our concern is the asymptotic behaviour of the integral

$$\int_C e^{nf(z)} g(z)\, dz \qquad (n \to \infty), \tag{3.1}$$

where $f(z)$ and $g(z)$ are holomorphic functions and C is some integration contour. Let $f(z)$ be given by its real and imaginary part as

$$f(z) = f(x + iy) = u(x, y) + iv(x, y).$$

Relying on Cauchy's integral theorem, we aim to deform C without crossing any singularity in the integrand. The reason is threefold.

First of all, we want the deformed contour to pass through a critical point of $f(z)$ for it facilitates expanding into a convergent power series as the first derivative vanishes. Secondly, we want $v(x, y)$ to take on a constant value along the deformed contour. In this way, we obviate the need for considering rapid oscillations of the integrand when n becomes large. Thirdly, we want $u(x, y)$ to drop from the critical point of $f(z)$ at the fastest possible rate so that the largest values of the integrand become concentrated in a small contour segment.

Therefore, we first choose a critical point z_t of $f(z)$. Following the initial reasoning, z_t must be a saddlepoint of $f(z)$. Surprisingly, the latter two demands can both be fulfilled since the contour, along which $v(x, y)$ is constant, coincides with the contour, where $u(x, y)$ exhibits the fastest rate of change. We thus move from the saddlepoint z_t along the steepest descent contours.

Along the deformed contour, we define a positive real function by

$$\phi(z) := f(z_t) - f(z) = u(x_t, y_t) - u(x, y) > 0, \tag{3.2}$$

whose inverse function determines z as a function of ϕ. We write

$$\phi = \phi(z), \quad z = z(\phi).$$

Employing integration by substitution for contour integrals, all that remains to consider is the asymptotic behaviour of the expression

$$e^{nf(z_t)} \int_0^\infty e^{-n\phi} g(z(\phi)) \frac{dz}{d\phi} \, d\phi \qquad (n \to \infty). \tag{3.3}$$

In chapter 5, we shall present an application of the steepest descent method. Instead of approximating $f(z)$ in the exponential, where the error would clearly be beyond control, one can approximate $dz/d\phi$ by expanding it into a convergent power series near the saddlepoint z_t.

4 DENSITY EXPANSIONS

In this chapter, without going into exhaustive detail, we introduce Gram and Charlier's and Edgeworth's expansion, following the presentation in Butler, 2007 and referring to Brenn and Anfinsen, 2017 for an overview of different approaches. Let X denote a real-valued random variable with expectation μ, variance σ^2 and density function $f_X(t)$.

In the following, consider the normal density function

$$\varphi(t;\mu,\sigma) \coloneqq \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{[t-\mu]^2}{2\sigma^2}\right) \qquad (t \in \mathbb{R})$$

and the standard normal density function

$$\varphi(t) \coloneqq \varphi(t;0,1) \qquad (t \in \mathbb{R}),$$

which are related by

$$\varphi(t;\mu,\sigma) = \frac{1}{\sigma}\varphi(\bar{t}) \qquad (t \in \mathbb{R}),$$

where

$$\bar{t} \coloneqq \frac{t-\mu}{\sigma} \qquad (t \in \mathbb{R}).$$

Considering the differential operator

$$D^n \coloneqq \frac{d^n}{dt^n} \qquad (n \in \mathbb{N}),$$

the n-th Hermite polynomial

$$H_n(t) = \sum_{j=0}^{n} a_{n,j} t^j \qquad (n \in \mathbb{N})$$

is defined by

$$H_n(t) \coloneqq \varphi(-t)[-D]^n \varphi(t) \qquad (n \in \mathbb{N}).$$

Subject to certain conditions, we have the representation

$$f_X(t) = \sum_{j=0}^{\infty} b_j [-D]^j \varphi(t;\mu,\sigma) \qquad (t \in \mathbb{R}).$$

Since the Hermite polynomials are orthogonal in the sense that

$$\int_{-\infty}^{+\infty} H_n(t)H_m(t)\varphi(t)\,dt = \begin{cases} 0 & \text{if } n \neq m, \\ n! & \text{if } n = m \end{cases} \qquad (n \in \mathbb{N}),$$

it follows that

$$b_j = \frac{\sigma^n}{j!} \int_{-\infty}^{+\infty} H_j(\bar{s}) f_X(s)\,ds \qquad (j \in \mathbb{N}).$$

As

$$[-D]^n \varphi(t; \mu, \sigma) = \frac{\varphi(t; \mu, \sigma)}{\sigma^n} H_n(\bar{t}) \qquad (n \in \mathbb{N}),$$

we arrive at

$$f_X(t) = \varphi(t; \mu, \sigma) \sum_{j=0}^{\infty} c_j H_j(\bar{t}) \qquad (t \in \mathbb{R}), \tag{4.1}$$

where

$$c_j = \frac{1}{j!} \int_{-\infty}^{+\infty} H_j(\bar{s}) f_X(s)\,ds \qquad (j \in \mathbb{N})$$

so that, by (2.4),

$$c_j = \frac{1}{j!} \sum_{l=0}^{j} a_{j,l} \nu_{X,l} \qquad (j \in \mathbb{N}) \tag{4.2}$$

or, equivalently, by (2.7),

$$c_j = \frac{1}{j!} \sum_{l=0}^{j} a_{j,l} \sum_{r=1}^{l} B_{l,r}(0, \lambda_{X,2}, \ldots, \lambda_{X,l-r+1}) \qquad (j \in \mathbb{N}). \tag{4.3}$$

The expansion (4.1) is either referred to as Gram and Charlier's expansion in case the coefficients (4.2) are considered or referred to as Edgeworth's expansion in case the coefficients (4.3) are considered.

A list of the first few Hermite polynomials $H_n(t)$ and of the first few coefficients (4.2) as well as (4.3) can be found in appendix A. Note that the Hermite polynomials $H_n(t)$ can be calculated explicitly through

$$H_n(t) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{[-1]^j t^{n-2t}}{[n-2j]!\, j!\, 2^j} \qquad (n \in \mathbb{N}). \tag{4.4}$$

5 SADDLEPOINT APPROXIMATIONS

Consider

$$\overline{X}_n := \frac{1}{n}\sum_{j=1}^{n} X_j \qquad (n \in \mathbb{N}),$$

the sample mean of n independent and identically distributed random variables X_j with moment-generating function $M_X(z)$ and cumulant-generating function $K_X(z)$. By independence, using (2.9), we get

$$M_{\overline{X}_n} = \mathrm{E}\Big(\exp\Big(\frac{z}{n}\sum_{j=1}^{n} X_j\Big)\Big) = \prod_{j=1}^{n} \mathrm{E}\Big(\exp\Big(\frac{z}{n}X_j\Big)\Big) = M_X^n\Big(\frac{z}{n}\Big).$$

Hence,

$$K_{\overline{X}_n} = nK_X\Big(\frac{z}{n}\Big)$$

and

$$\lambda_{\overline{X}_n,j} = \frac{n}{\sqrt{n^j}}\lambda_{X,j} \qquad (j \in \mathbb{N}). \tag{5.1}$$

In the following, we examine the asymptotic behaviour of both the density function $f_{\overline{X}_n}(t)$ and the distribution function $F_{\overline{X}_n}(t)$ for a given value of t. To be precise, we approximate $f_{\overline{X}_n}(t)$ by the first terms of its asymptotic expansion, whereas we approximate $F_{\overline{X}_n}(t)$ by the first terms of the sum of two seemingly unrelated asymptotic expansions. Since both approximations were originally established by the use of the saddlepoint method, they are commonly called saddlepoint approximations.

DENSITY FUNCTION

The inversion formula (2.10) yields the integral representation

$$f_{\overline{X}_n}(t) = \frac{n}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \exp\big(n\big[K_X(z) - tz\big]\big)\,dz \qquad (t \in \mathbb{R}), \tag{5.2}$$

from which one can basically pursue two ways. Both the saddlepoint method and Esscher's method lead to the same approximation.

The saddlepoint approximation of the density function originates with the pioneering Daniels, 1954, which is why it is sometimes called Daniels' formula. Our exposition is based on Field and Ronchetti, 1990.

Saddlepoint method

The integral (5.2) is clearly of the form (3.1) with the functions

$$f(z) = K_X(z) - tz, \quad g(z) = \frac{n}{2\pi i},$$

and a single real saddlepoint z_t which satisfies the saddlepoint equation

$$K_X'(z_t) = t$$

can be shown to exist under mild conditions so that the steepest ascent contour goes along the real axis. To move from z_t along the steepest descent contour, we now deform the integration contour as follows.

First of all, after constructing a circle of radius ε around z_t, follow the steepest descent contour D_0 inside the circle, starting from z_t in the direction orthogonal to the real axis. Along this contour, the imaginary part $v(x, y)$ vanishes. Secondly, at the points z_1 and z_2, where D_0 enters the circle, continue in the direction orthogonal to D_0 on the contours D_1 and D_2 of constant real part $u(x, y)$. Thirdly, at the points z_3 and z_4, where D_1 and D_2, as can be shown, cross the original vertical contour going through z_t, proceed along the vertical contours D_3 and D_4.

Hence, consider

$$\frac{n}{2\pi i} \int_D \exp\big(n\big[K_X(z) - tz\big]\big)\, dz,$$

where

$$D = D_0 \cup D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5.$$

Along the straight line $z = z_t + iy$, we have

$$e^{f(z)} = e^{-tz} \int_{-\infty}^{+\infty} \exp\big(\big[z_t + iy\big]s\big)\, dF_X(s).$$

However, writing the integral as

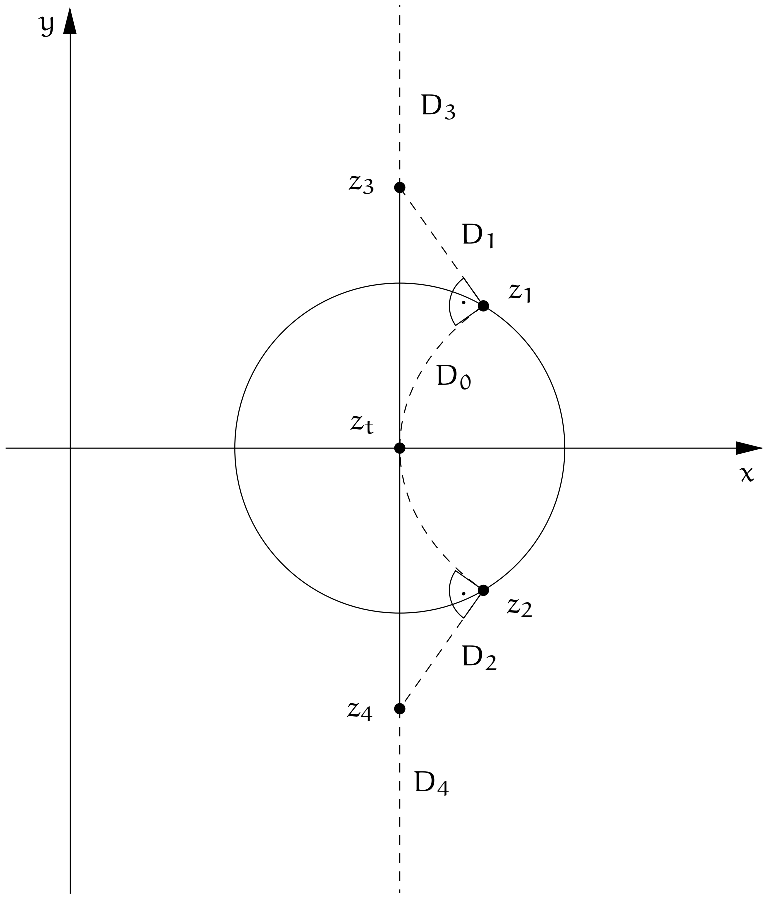
$$\int_{-\infty}^{+\infty} \exp\big(\big[z_t + iy\big]s\big)\, dF_X(s) = M_X(z_t) \int_{-\infty}^{+\infty} e^{iys} \frac{e^{z_t s}}{M_X(z_t)}\, dF_X(s),$$

we conclude that

$$e^{f(z)} = e^{-tz} M_X(z_t) C(y),$$

where $C(y)$ is some characteristic function. Since, in general, one has

$$\forall\, y \neq 0\colon C(y) < 1,$$



The deformed contour D.

hence

$$\exists\, a < 1\colon \left|e^{f(z)}\right| = e^{u(x,y)} \leqslant a e^{f(z_t)},$$

the contribution along $D \setminus D_0$ outside the circle can be neglected.

It remains to consider D_0 inside the circle. By (3.2), define

$$\phi(z) := K_X(z) - tz - \left[K_X(z_t) - tz_t\right].$$

By expanding,

$$\phi(z) = \sum_{j=2}^{\infty} \frac{k_{X,j,t}}{j!} \left[z - z_t\right]^j,$$

writing

$$-\frac{1}{2}\psi^2(z) := \phi(z)$$

and setting

$$\zeta := \left[z - z_t\right]\sqrt{k_{X,2,t}},$$

we get

$$-\frac{1}{2}\psi^2(\zeta) = \sum_{j=2}^{\infty} \frac{\lambda_{X,j,t}}{j!}\zeta^j.$$

Consequently,

$$i\psi(\zeta) = \zeta\sqrt{1 + 2\sum_{j=1}^{\infty} \frac{\lambda_{X,j+1,t}}{[j+1]!}\zeta^j}.$$

By the generalised binomial theorem, the first terms are

$$i\psi(\zeta) = \zeta + \frac{\lambda_{X,3,t}}{6}\zeta^2 + \left[\frac{\lambda_{X,4,t}}{24} - \frac{\lambda_{X,3,t}^2}{72}\right]\zeta^3 + O(\zeta^4),$$

which can be expressed as a function of ζ. Threefold recursion gives

$$\zeta(\psi) = i\psi + \frac{\lambda_{X,3,t}}{6}\psi^2 + \left[\frac{\lambda_{X,4,t}}{24} - \frac{5\lambda_{X,3,t}^2}{72}\right]i\psi^3 + O(\psi^4).$$

Following (3.1), we consider the asymptotic behaviour of

$$\frac{n}{2\pi i\sqrt{k_{X,2,t}}}\exp\left(n\left[K_X(z_t) - tz_t\right]\right)\int_{\psi(z_1)}^{\psi(z_2)} \exp\left(-\frac{n}{2}\psi^2\right)\frac{d\zeta}{d\psi}\, d\psi,$$

which yields, by Watson's lemma (1.3), the second-order approximation

$$f_{\overline{X}_n}(t) \sim h_t(n)\left[\alpha_{t,0} + \frac{\alpha_{t,1}}{n}\right] \qquad (n \to \infty), \tag{5.3}$$

where

$$h_t(n) := \sqrt{\frac{n}{2\pi k_{X,2,t}}}\exp\big(n\left[K_X(z_t) - tz_t\right]\big)$$

or, equivalently,

$$h_t(n) := \sqrt{\frac{n}{2\pi k_{X,2,t}}}\exp\big(-\frac{n}{2}p_t^2\big)$$

with

$$p_t := \operatorname{sgn}(z_t)\sqrt{2\left[tz_t - K_X(z_t)\right]}.$$

One has

$$\alpha_{t,0} := 1, \quad \alpha_{t,1} := \frac{\lambda_{X,4,t}}{8} - \frac{5\lambda_{X,3,t}^2}{24}.$$

Regarding the inversion of $i\psi(\zeta)$, one can use the following lemma about Lagrange's inversion formula as well. See Charalambides, 2001.

Lemma. If

$$\psi(\zeta) = \sum_{j=1}^{\infty}\frac{\psi_j}{j!}\zeta^j, \quad \zeta(\psi) = \sum_{j=1}^{\infty}\frac{\zeta_j}{j!}\psi^j \qquad (\psi_1 \neq 0)$$

such that

$$\psi = \psi(\zeta) = \psi(\zeta(\psi)), \quad \zeta = \zeta(\psi) = \zeta(\psi(\zeta)),$$

then

$$\left[\frac{d^j}{d\psi^j}\zeta(\psi)\right]_{\psi=0} = \left[\frac{d^{j-1}}{d\zeta^{j-1}}\left[\frac{\zeta}{\psi(\zeta)}\right]^j\right]_{\zeta=0} \qquad (j \in \mathbb{N}). \tag{5.4}$$

Lagrange's inversion formula (5.4) and Watson's lemma (1.3) yield

$$f_{\overline{X}_n}(t) \approx h_t(n)\sum_{j=0}^{\infty}\frac{\alpha_{t,j}}{n^j} \qquad (n \to \infty),$$

where

$$\alpha_{t,j} := \frac{1}{2^j j!}\left[\frac{d^{2j}}{d\zeta^{2j}}\left[\frac{\zeta}{i\psi(\zeta)}\right]^{2j+1}\right]_{\zeta=0} \qquad (j \in \mathbb{N}).$$

Esscher's method

The following definition and method originate with F. Esscher.

Definition. The Esscher transform f_{X_s} of X for $s \in \mathbb{R}$ is defined by

$$f_{X_s}(t) := \exp\bigl(st - K_X(s)\bigr) f_X(t) \qquad (t \in \mathbb{R}). \tag{5.5}$$

Note that f_{X_s} is a density function. The set of Esscher transforms f_{X_s} indexed by s is known as natural exponential family of X. One has

$$\mu_{X_s} = k_{X_s,1} = k_{X,1,s}, \quad \sigma^2_{X_s} = k_{X_s,2} = k_{X,2,s}.$$

Now, Esscher's method for establishing the saddlepoint approximation of $f_{\overline{X}_n}(t)$ comprises three steps. First of all, s_t is chosen such that

$$\mu_{\overline{X}_{n,s_t}} = t,$$

which amounts to solving the saddlepoint equation

$$K'_X(s_t) = t.$$

Secondly, we write

$$f_{\overline{X}_n}(t) = \exp\bigl(K_{\overline{X}_n}(s) - st\bigr) f_{\overline{X}_{n,s}}(t).$$

Thirdly, by Edgeworth's expansion and (5.1),

$$f_{\overline{X}_{n,s_t}}(t) \sim \sqrt{\frac{n}{2\pi k_{X,2,t}}}\left[1 + \frac{1}{n}\left[\frac{\lambda_{X,4,t}}{8} - \frac{5\lambda^2_{X,3,t}}{24}\right]\right] \quad (n \to \infty).$$

Altogether, we again arrive at the saddlepoint approximation

$$f_{\overline{X}_n}(t) \sim h_t(n)\left[\alpha_{t,0} + \frac{\alpha_{t,1}}{n}\right] \qquad (n \to \infty).$$

Roughly speaking, Edgeworth's expansion is used locally at the point where the approximation is to be obtained. For this reason, Esscher's method is sometimes called Edgeworth's indirect expansion. In the tails of the distribution, the first proves surprisingly accurate, while the latter can be inaccurate. In other words, one replaces a global high-order approximation by a local low-order approximation around each point of interest. Martin, 2004 provides a daring but intuitive derivation.

Remarks

Naturally, the question arises whether the first-order approximation given by $h_t(n)$ is exact or exact up to normalisation. Interestingly, this is the case for three probability densities $f_X(t)$ only, namely for the probability density of the normal distribution, in which case the dominant term is exact and the higher order terms are zero, and for the probability density of the gamma and of the inverse normal distribution, in which case the dominant term is exact up to a constant and the higher order terms, which are nonzero but independent of t, can be included in the normalisation constant. We refer to Daniels, 1980 for details.

DISTRIBUTION FUNCTION

Originating with Lugannani and Rice, 1980, the saddlepoint approximation of the distribution function $F_{\overline{X}_n}(t)$ is commonly called Lugannani and Rice's formula. The second-order approximation takes the form

$$F_{\overline{X}_n}(t) \sim \Phi(\sqrt{n}p_t) + \frac{\varphi(\sqrt{n}p_t)}{\sqrt{n}}\left[\beta_{t,0} + \frac{\beta_{t,1}}{n}\right] \qquad (n \to \infty), \tag{5.6}$$

where $\Phi(t)$ denotes the standard normal distribution function and $\varphi(t)$ the standard normal density function. The coefficients are given by

$$\beta_{t,0} := \frac{1}{p_t} - \frac{1}{q_t}, \quad \beta_{t,1} := -\frac{1}{p_t^3} + \frac{1}{q_t^3} + \frac{\lambda_{X,3,t}}{2q_t^2} - \frac{\alpha_{t,1}}{q_t}$$

and

$$q_t := z_t\sqrt{k_{X,2,t}}.$$

Approximation (5.6) can be derived from Daniels' second-order formula (5.3) using Temme's lemma (1.4). See Reid and Fraser, 1992.

Consider

$$B_n(t) := \int_{-\infty}^{t} f_{\overline{X}_n}(s)\, ds,$$

then

$$B_n(t) = \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{p_t} \exp\left(-\frac{n}{2}p_s^2\right) A_n(p_s)\, dp_s,$$

where

$$A_n(p_s) \sim \frac{ds}{dp_s}\frac{1}{\sqrt{k_{X,2,s}}}\left[1 + \frac{\alpha_{s,1}}{n}\right] \qquad (n \to \infty).$$

Since

$$\frac{ds}{dp_s} = \frac{p_s}{z_s},$$

it follows that

$$A_n(p_s) \sim a_{p_s,0} + \frac{a_{p_s,1}}{n} \qquad (n \to \infty),$$

where

$$a_{p_s,0} = \frac{p_s}{q_s}, \quad a_{p_s,1} = \frac{p_s \alpha_{s,1}}{q_s}.$$

As

$$B_n(\infty) = 1,$$

we have

$$b_0 = 1, \quad b_1 = 0.$$

From

$$p_t c_{p_t,0} = b_0 - a_{p_s,0}$$

it follows that

$$c_{p_t,0} = \frac{1}{p_t} - \frac{1}{q_t}.$$

As

$$p_t c_{p_t,1} = b_1 - a_{p_t,1} + \left[\frac{d}{dp_s} c_{p_s,0}\right]_{p_s=p_t},$$

we get

$$p_t c_{p_t,1} = -\frac{p_t \alpha_{t,1}}{q_t} + \left[-\frac{1}{p_s^2} - \left[-\frac{1}{q_s^2}\frac{dq_s}{dp_s}\right]\right]_{p_s=p_t},$$

but

$$\frac{dq_s}{dp_s} = \frac{p_s}{q_s} + \frac{p_s \lambda_{X,3,s}}{2}$$

so that

$$c_{p_t,1} = -\frac{1}{p_t^3} + \frac{1}{q_t^3} + \frac{\lambda_{X,3,t}}{2q_t^2} - \frac{\alpha_{t,1}}{q_t}.$$

Taking a closer look at Lugannani and Rice's formula (5.6), it reveals the flavour of a series expansion. Indeed, one has the approximation

$$F_{\overline{X}_n}(t) \sim \Phi\left(\sqrt{n}p_t - \frac{1}{\sqrt{n}p_t}\log\left(\frac{1}{\sqrt{n}q_t}\right)\right) \qquad (n \to \infty), \tag{5.7}$$

which originates with Barndorff-Nielsen, 1991, after whom the formula is called. Note that it is guaranteed to take values between 0 and 1.

EXAMPLES

We finally give two examples which both deploy Daniels' first-order formula (5.3) and Lugannani and Rice's first-order formula (5.6).

The first example shows that the saddlepoint approximation of both the normal density and the normal distribution function is exact.

Example. In case the random variables X_j are normally distributed with mean μ_X and variance σ_X^2, then the corresponding sample mean is normally distributed with mean μ_X and σ_X^2/n. Hence, one has

$$f_{\overline{X}_n}(t) = \sqrt{\frac{n}{2\pi\sigma_X^2}} \exp\left(-\frac{n[t-\mu_X]^2}{2\sigma_X^2}\right) \qquad (n \in \mathbb{N})$$

and

$$F_{\overline{X}_n}(t) = \Phi\left(\frac{\sqrt{n}[t-\mu_X]}{\sigma_X}\right) \qquad (n \in \mathbb{N}).$$

Since

$$K_X(z) = \mu_X z + \frac{1}{2}\sigma_X^2 z^2,$$

we get

$$K_X'(z) = \mu_X + \sigma_X^2 z$$

so that

$$z_t = \frac{t-\mu_X}{\sigma_X^2}$$

and

$$k_{X,2,t} = \sigma_X^2.$$

Hence,

$$p_t = q_t = \frac{t-\mu_X}{\sigma_X}.$$

Therefore, the first-order approximations are exact, meaning that

$$f_{\overline{X}_n}(t) = h_n(t) \qquad (t \in \mathbb{R})$$

and

$$F_{\overline{X}_n}(t) = \Phi(\sqrt{n}p_t) \qquad (t \in \mathbb{R}).$$

The second example considers the Gumbel distribution, which is one out of three special cases of the generalised extreme value distribution from extreme value theory. See, for example, McNeil et al., 2005.

Although our saddlepoint approximations are based on the asymptotic behaviour of the sample mean as $n \to \infty$, it turns out that the approximations are surprisingly accurate even in the case $n = 1$.

Example. The standard Gumbel density function is given by

$$f_X(t) = e^{-t} e^{-e^{-t}} \qquad (t \in \mathbb{R}) \tag{5.8}$$

and the distribution function takes the similar form

$$F_X(t) = e^{-e^{-t}} \qquad (t \in \mathbb{R}). \tag{5.9}$$

By a simple transformation, it follows that

$$M_X(z) = \int_{-\infty}^{+\infty} e^{zt} e^{-t} e^{-e^{-t}} \, dt = \int_0^{+\infty} s^{-z} e^{-s} \, ds = \Gamma(1 - z),$$

where $\Gamma(z)$ is the gamma function

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \, dt.$$

Consequently,

$$K_X(z) = \log \Gamma(1 - z)$$

and

$$K_X'(z) = -\psi_0(1 - z), \quad K_X''(z) = \psi_1(1 - z),$$

where $\psi_0(z)$ and $\psi_1(z)$ are the digamma and the trigamma function.
Therefore,

$$f_X(t) \sim \sqrt{\frac{1}{2\pi\psi_1(1 - z_t)}} \exp\Big(\log\big(\Gamma(1 - z_t)\big) - tz_t\Big) \qquad (n = 1) \tag{5.10}$$

and

$$F_X(t) \sim \Phi(p_t) + \varphi(p_t)\left[\frac{1}{p_t} - \frac{1}{q_t}\right] \qquad (n = 1), \tag{5.11}$$

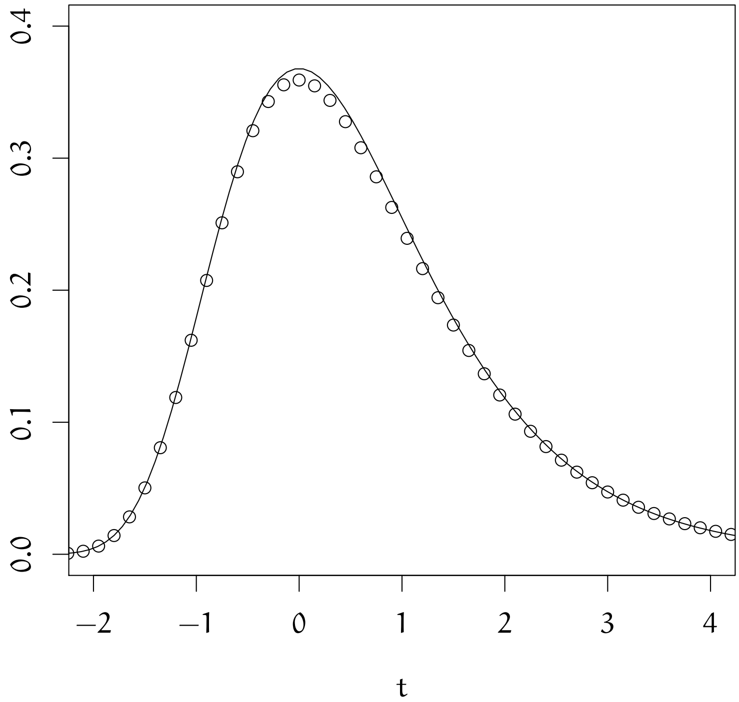
where

$$p_t = \operatorname{sgn}(z_t)\sqrt{2\big[tz_t - \log\big(\Gamma(1 - z_t)\big)\big]}, \quad q_t = z_t\sqrt{\psi_1(1 - z_t)}$$

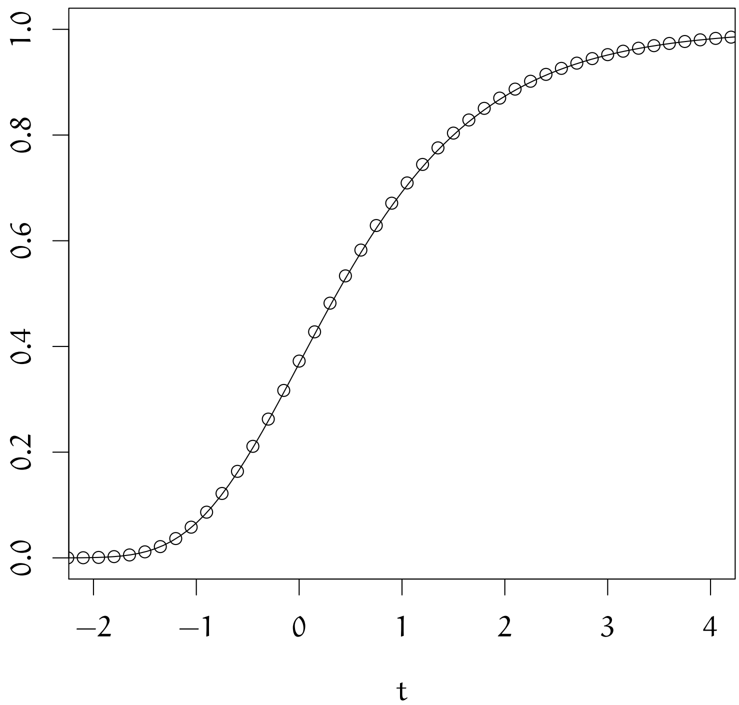
and z_t is defined as the solution of the saddlepoint equation given by

$$-\psi_0(1 - z_t) = t.$$

See appendix B for the implementation of the following plots.



First-order approximation (5.10) of the Gumbel density function (5.8).



First-order approximation (5.11) of the Gumbel distribution function (5.9).

Part II

APPLICATION

6 RISK MEASURES

We consider the total profit and loss variable X with density function $f_X(t)$ and continuous distribution function $F_X(t)$ of some portfolio.

In the following, we introduce two common risk measures, namely the value at risk and the expected shortfall, and show how the saddlepoint approximations can be applied. Regarding the approximation, we confine ourselves to $n = 1$, thus considering X as a single random variable.

VALUE AT RISK

Definition

The value at risk of X at the confidence level α is defined by

$$\eta := \mathrm{VR}_{X,\alpha} := \inf\{t \in \mathbb{R} : F_X(t) \geqslant \alpha\} \qquad (0 < \alpha < 1), \tag{6.1}$$

which is simply the α-quantile of X. It is the smallest number such that the probability that X exceeds it is no larger than $1 - \alpha$. Clearly,

$$F_X(\mathrm{VR}_{X,\alpha}) = \alpha$$

so that computing the value at risk at some confidence level α is the inverse task of computing the tail probability at some loss level t.

Approximation

For fixed α, Lugannani and Rice's formula (5.6) yields

$$\alpha \sim \Phi(p_\eta) + \varphi(p_\eta)\left[\frac{1}{p_\eta} - \frac{1}{q_\eta}\right] \qquad (n = 1), \tag{6.2a}$$

where z_η satisfies the saddlepoint equation

$$K_X'(z_\eta) = \eta. \tag{6.2b}$$

The coefficients are

$$p_\eta = \mathrm{sgn}(z_\eta)\sqrt{2\big[z_\eta \eta - K_X(z_\eta)\big]}, \quad q_\eta = z_\eta\sqrt{k_{X,2,\eta}}.$$

As p_η and q_η involve z_η, one is faced with the task of solving the coupled equations (6.2a) and (6.2b) simultaneously for z_η and η.

EXPECTED SHORTFALL

Definition

The expected shortfall of X at the confidence level α is defined by

$$\vartheta := \mathrm{ES}_{X,\alpha} := \mathrm{E}(X \mid X > \mathrm{VR}_{X,\alpha}) \qquad (0 < \alpha < 1) \tag{6.3}$$

and one can show that

$$\mathrm{ES}_{X,\alpha} = \frac{1}{1-\alpha}\,\mathrm{E}\big(X \mathbb{1}_{\{X>\mathrm{VR}_{X,\alpha}\}}\big) = \frac{1}{1-\alpha}\int_{\alpha}^{1} \mathrm{VR}_{X,\beta}\; d\beta. \tag{6.4}$$

Hence, $\mathrm{ES}_{X,\alpha}$ is the expected loss beyond the threshold $\mathrm{VR}_{X,\alpha}$. Instead of considering $\mathrm{VR}_{X,\alpha}$ for some fixed value of α, we average over all $\mathrm{VR}_{X,\beta}$ with $\beta > \alpha$, thus looking further into the tail distribution.

Approximation

The following approximation of the expected shortfall can be established in two different ways. We refer to both Martin, 2006 and Broda and Paolella, 2010. For fixed α, the inversion formula (2.13) yields

$$E\big(X\mathbb{1}_{\{X>\eta\}}\big) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} K_X'(z)\frac{e^{K_X(z)-z\eta}}{z}\,dz,$$

where the singularity in the integrand can be rewritten as

$$\frac{K_X'(z)}{z} = \left[\frac{\mu_X}{z} + \frac{K_X'(z)-\mu_X}{z}\right].$$

By the inversion formula (2.11), the first term gives

$$\frac{\mu_X}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{K_X(z)-z\eta}}{z}\,dz = \mu_X\,\mathrm{P}(X>\eta).$$

By the saddlepoint method, one can show that

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \big[K_X'(z)-\mu_X\big]\frac{e^{K_X(z)-z\eta}}{z}\,dz \sim \frac{\eta-\mu_X}{z_\eta} f_X(\eta) \quad (n=1).$$

Altogether, we get the approximation

$$E\big(X\mathbb{1}_{\{X>\eta\}}\big) \sim \mu_X\,\mathrm{P}(X>\eta) + \frac{\eta-\mu_X}{z_\eta} f_X(\eta). \qquad (n=1).$$

Replacing $P(X > \eta)$ by Daniels' second-order formula (5.3) and $f_X(\eta)$ by Lugannani and Rice's second-order formula (5.6), we arrive at

$$E\left(X\mathbb{1}_{\{X>\eta\}}\right) \sim \mu_X\left[1-\Phi(p_\eta)\right] - \varphi(p_\eta)\left[\gamma_0+\gamma_1\right] \qquad (n=1), \tag{6.5}$$

where

$$\gamma_0 := \frac{\mu_X}{p_\eta} - \frac{\eta}{q_\eta}$$

and

$$\gamma_1 := -\frac{\mu_X}{p_\eta^3} + \frac{\eta}{q_\eta^3} + \frac{\eta\lambda_{X,\eta,3}}{2q_\eta^2} - \frac{\eta\alpha_{\eta,1}}{q_\eta} - \frac{1}{z_\eta q_\eta}.$$

In a similar way to how Lugannani and Rice's formula (5.6) was established, approximation (6.5) can be derived from Daniels' second-order formula (5.3) by Temme's lemma (1.4) as follows. Consider

$$B_1(\eta) := \int_{-\infty}^{\eta} s f_X(s)\, ds,$$

then

$$B_1(\eta) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{p_\eta} \exp\left(-\frac{1}{2}p_s^2\right) A_1(p_s)\, dp_s,$$

where

$$A_1(p_s) \sim \frac{ds}{dp_s}\frac{s}{\sqrt{k_{X,s,2}}}\left[1+\alpha_{s,1}\right] \qquad (n=1).$$

Since

$$\frac{ds}{dp_s} = \frac{p_s}{z_s},$$

it follows that

$$A_1(p_s) \sim a_{p_s,0} + a_{p_s,1} \qquad (n=1),$$

where

$$a_{p_s,0} = \frac{p_s s}{q_s}, \quad a_{p_s,1} = \frac{p_s s\alpha_{s,1}}{q_s}.$$

As

$$B_1(\infty) = \mu_X,$$

we have

$$b_0 = \mu_X, \quad b_1 = 0.$$

From

$$p_\eta c_{p_\eta,0} = b_0 - a_{p_s,0}$$

it follows that

$$c_{p_\eta,0} = \frac{\mu_X}{p_\eta} - \frac{\eta}{q_\eta}.$$

As

$$p_\eta c_{p_\eta,1} = b_1 - a_{p_\eta,1} - \left[\frac{d}{dp_s} c_{p_s,0}\right]_{p_s=p_\eta},$$

we get

$$p_\eta c_{p_\eta,1} = -\frac{p_\eta \eta \alpha_{\eta,1}}{q_\eta} + \left[-\frac{\mu_X}{p_s^2} - \left[\frac{1}{q_s}\frac{ds}{dp_s} - \frac{s}{q_s^2}\frac{dq_s}{dp_s}\right]\right]_{p_s=p_\eta},$$

but

$$\frac{ds}{dp_s} = \frac{p_s}{z_s}$$

and

$$\frac{dq_s}{dp_s} = \frac{p_s}{q_s} + \frac{p_s \lambda_{X,s,3}}{2}$$

so that

$$c_{p_\eta,1} = -\frac{\mu_X}{p_\eta^3} + \frac{\eta}{q_\eta^3} + \frac{\eta \lambda_{X,\eta,3}}{2q_\eta^2} - \frac{\eta \alpha_{\eta,1}}{q_\eta} - \frac{1}{z_\eta q_\eta}.$$

Note that

$$E\big(X \mathbb{1}_{\{X>\eta\}}\big) = \mu_X - B_1(\eta).$$

7 EULER CONTRIBUTIONS

Suppose that the total profit and loss variable X arises from d independent single profit and loss variables Y_j. We therefore consider

$$X \coloneqq \sum_{j=1}^{d} Y_j.$$

The single variables Y_j may have a different economic interpretation depending on the area of application. They may, for instance, represent different lines of business, investments in assets or perhaps loans.

Furthermore, we introduce weight variables a_j so that

$$X(a) \coloneqq \sum_{j=1}^{d} a_j Y_j,$$

which suggests the interpretation of a_j as amount of money invested in the asset Y_j or as loan exposure if Y_j is some Bernoulli variable.

Optimising the risk of the portfolio, one might be interested how each component contributes to the overall risk. We tackle the question how risk measures can be decomposed into its different risk sources.

Hereafter, let $\rho(a)$ denote some risk measure of $X(a)$. For details on the following definition, see McNeil et al., 2005 or Tasche, 2008.

Definition. The Euler contribution $\pi_{\rho,j}(a)$ to $\rho(a)$ is defined by

$$\pi_{\rho,j}(a) \coloneqq a_j \frac{\partial \rho}{\partial a_j}(a).$$

One can interpret $\pi_{\rho,j}(a)$ as the amount of economic capital allocated to the position $a_j Y_j$ when the total position has profit and loss $X(a)$.

PROPERTIES

In case $\rho(a)$ is a positive homogeneous risk measure such that

$$\forall x > 0 \colon \rho(xa) = x\rho(a),$$

then it follows, by use of the multivariable chain rule, that

$$\rho(a) = \left[\frac{d}{dx} x\rho(a)\right]_{x=1} = \left[\frac{d}{dx}\rho(xa)\right]_{x=1} = \sum_{j=1}^{d} \pi_{\rho,j}(a)$$

which is commonly known as Euler's theorem.

Both the value at risk and the expected shortfall are positive homogeneous risk measures so that the Euler contributions add up to the risk measure itself. To stress the dependence on the weight variable, let

$$\eta(a) \coloneqq \mathrm{VR}_{X(a),\alpha}, \quad \vartheta(a) \coloneqq \mathrm{ES}_{X(a),\alpha}.$$

Subject to certain conditions, Tasche, 1999 shows that

$$\frac{\partial\eta}{\partial a_j}(a) = \mathrm{E}\big(Y_j \,|\, X(a) = \eta(a)\big), \quad \frac{\partial\vartheta}{\partial a_j}(a) = \mathrm{E}\big(Y_j \,|\, X(a) > \eta(a)\big).$$

so that, in accordance with Euler's theorem,

$$\sum_{j=1}^{n} a_j \,\mathrm{E}\big(Y_j \,|\, X(a) = \eta(a)\big) = \mathrm{E}\big(X(a) \,|\, X(a) = \eta(a)\big) = \eta(a)$$

and, similarly,

$$\sum_{j=1}^{n} a_j \,\mathrm{E}\big(Y_j \,|\, X(a) > \eta(a)\big) = \mathrm{E}\big(X(a) \,|\, X(a) > \eta(a)\big) = \vartheta(a).$$

APPROXIMATION

We proceed with deriving saddlepoint approximations for the Euler contributions $\pi_{\eta,j}$ and $\pi_{\vartheta,j}$, where the confidence level α is fixed.

For the sake of clarity, we drop both the weight variable a and the integration contour in case it corresponds to Bromwich's contour.

Value at risk

We follow Martin et al., 2001. By the inversion formula (2.11),

$$F_X(\eta) = \frac{1}{2\pi i}\int \frac{e^{K_X(z)-\eta z}}{z}\,dz.$$

Differentiating both sides with respect to a_j then yields

$$\frac{\partial F_X}{\partial a_j}(\eta) = \frac{1}{2\pi i}\int \frac{e^{K_X(z)-\eta z}}{z}\left[\frac{\partial K_X}{\partial a_j}(z) - z\frac{\partial\eta}{\partial a_j}\right]dz.$$

Since the confidence level α is fixed, we have

$$F_X(\eta) = \alpha,$$

so the derivative with respect to a_j is

$$\frac{\partial F_X}{\partial a_j}(\eta) = 0.$$

After rearranging, we get

$$\frac{\partial \eta}{\partial a_j} = \int \frac{e^{K_X(z)-\eta z}}{z} \frac{\partial K_X}{\partial a_j}(z)\, dz \bigg/ \int e^{K_X(z)-\eta z}\, dz.$$

By the saddlepoint method, one can show that

$$\frac{\partial \eta}{\partial a_j} \sim \frac{1}{z_\eta} \frac{\partial K_X}{\partial a_j}(z_\eta) \qquad (n = 1), \tag{7.1}$$

thus yielding the approximation

$$\pi_{\eta,j} \sim \frac{a_j}{z_\eta} \frac{\partial K_X}{\partial a_j}(z_\eta) \qquad (n = 1), \tag{7.2}$$

where z_η satisfies

$$\frac{\partial K_X}{\partial z}(z_\eta) = \eta.$$

Note that the cumulant-generating function has the property

$$\frac{\partial K_X}{\partial z}(z) = \sum_{j=1}^{d} \frac{a_j}{z} \frac{\partial K_X}{\partial a_j}(z),$$

from which we arrive at the possibly surprising result that

$$\eta = \sum_{j=1}^{n} \frac{a_j}{z_\eta} \frac{\partial K_X}{\partial a_j}(z_\eta).$$

This means that the approximations (7.2) of the Euler contributions $\pi_{\eta,j}$ to η add exactly up to η, despite merely being approximations.

Writing

$$Y := (Y_1, \ldots, Y_n),$$

then it follows from

$$\frac{1}{z} \frac{\partial K_X}{\partial a_j}(z) = \int y_j \exp\Big(\sum_{j=1}^{n} z a_j y_j\Big)\, dP_Y(y) \bigg/ \int \exp\Big(\sum_{j=1}^{n} z a_j y_j\Big)\, dP_Y(y)$$

that the Euler contributions satisfy

$$a_j \min Y_j \leqslant \frac{a_j}{z_\eta} \frac{\partial K_X}{\partial a_j}(z_\eta) \leqslant a_j \max Y_j.$$

In case the variables Y_j are Bernoulli variables, then

$$0 \leqslant \frac{1}{z_\eta} \frac{\partial K_X}{\partial a_j}(z_\eta) \leqslant 1,$$

which suggests the interpretation of risk-adjusted default probabilities.

Expected shortfall

We follow Martin, 2009. By the trivial identity

$$E(X \mid X > \eta) = \eta + E(X - \eta \mid X > \eta)$$

and the inversion formula (2.12), we get

$$\vartheta = \eta + \frac{1}{2\pi i[1-\alpha]} \int \frac{e^{K_X(z)-\eta z}}{z^2}\, dz.$$

Differentiation then yields

$$\frac{\partial \vartheta}{\partial a_j} = \frac{\partial \eta}{\partial a_j} + \frac{1}{2\pi i[1-\alpha]} \int \frac{e^{K_X(z)-\eta z}}{z^2} \left[\frac{\partial K_X}{\partial a_j}(z) - z\frac{\partial \eta}{\partial a_j} \right] dz.$$

However,

$$\frac{1}{2\pi i} \int \frac{e^{K_X(z)-\eta z}}{z}\, dz = 1 - \alpha,$$

hence the expression becomes

$$\frac{\partial \vartheta}{\partial a_j} = \frac{1}{2\pi i[1-\alpha]} \int \frac{e^{K_X(z)-\eta z}}{z^2} \frac{\partial K_X}{\partial a_j}(z)\, dz,$$

where the singularity can be rewritten as

$$\frac{1}{z^2}\frac{\partial K_X}{\partial a_j}(z) = \frac{\mu_{Y_j}}{z} + \frac{1}{z}\left[\frac{1}{z}\frac{\partial K_X}{\partial a_j}(z) - \mu_{Y_j} \right].$$

By the inversion formula (2.11), the first term gives

$$\frac{\mu_{Y_j}}{2\pi i} \int \frac{e^{K_X(z)-\eta z}}{z}\, dz = \mu_{Y_j} P(X > \eta).$$

By the saddlepoint method, one can show that

$$\frac{1}{2\pi i} \int \frac{1}{z}\left[\frac{1}{z}\frac{\partial K_X}{\partial a_j}(z) - \mu_{Y_j} \right] e^{K_X(z)-z\eta}\, dz \sim \frac{f_X(\eta)}{z_\eta}\left[\frac{1}{z_\eta}\frac{\partial K_X}{\partial a_j}(z_\eta) - \mu_{Y_j} \right],$$

thus yielding the approximation

$$\pi_{\vartheta,j} \sim \frac{a_j}{1-\alpha}\left[\mu_{Y_j} P(X > \eta) + \frac{f_X(\eta)}{z_\eta}\left[\frac{1}{z_\eta}\frac{\partial K_X}{\partial a_j}(z_\eta) - \mu_{Y_j} \right] \right],$$

where z_η satisfies

$$\frac{\partial K_X}{\partial z}(z_\eta) = \eta.$$

Part III

APPENDIX

A EXPRESSIONS

Moments as cumulants

According to (2.5), one has

$$
\begin{aligned}
m_1 &= k_1, \\
m_2 &= k_2 + k_1^2, \\
m_3 &= k_3 + 3k_2k_1 + k_1^3, \\
m_4 &= k_4 + 4k_3k_1 + 3k_2^2 + 6k_2k_1^2 + k_1^4.
\end{aligned}
$$

Cumulants as moments

According to (2.6), one has

$$
\begin{aligned}
k_1 &= m_1, \\
k_2 &= m_2 - m_1^2, \\
k_3 &= m_3 - 3m_2m_1 + 2m_1^3, \\
k_4 &= m_4 - 4m_3m_1 - 3m_2^2 + 12m_2m_1^2 - 6m_1^4.
\end{aligned}
$$

Standard moments as standard cumulants

According to (2.7), one has

$$
\begin{aligned}
\nu_2 &= \lambda_2, \\
\nu_3 &= \lambda_3, \\
\nu_4 &= \lambda_4 + 3\lambda_2^2, \\
\nu_5 &= \lambda_5 + 10\lambda_3\lambda_2, \\
\nu_6 &= \lambda_6 + 15\lambda_4\lambda_2 + 10\lambda_3^2 + 15\lambda_2^3.
\end{aligned}
$$

Standard cumulants as standard moments

According to (2.8), one has

$$
\begin{aligned}
\lambda_2 &= \nu_2, \\
\lambda_3 &= \nu_3, \\
\lambda_4 &= \nu_4 - 3\nu_2^2, \\
\lambda_5 &= \nu_5 - 10\nu_3\nu_2, \\
\lambda_6 &= \nu_6 - 15\nu_4\nu_2 - 10\nu_3^2 + 30\nu_2^3.
\end{aligned}
$$

Hermite polynomials

According to (4.4), the first polynomials are

$$
\begin{aligned}
H_0(t) &= 1,\\
H_1(t) &= t,\\
H_2(t) &= t^2 - 1,\\
H_3(t) &= t^3 - 3t,\\
H_4(t) &= t^4 - 6t^2 + 3,\\
H_5(t) &= t^5 - 10t^3 + 15t,\\
H_6(t) &= t^6 - 15t^4 + 45t^2 - 15.
\end{aligned}
$$

Gram and Charlier's expansion

According to (4.2), the first coefficients are

$$
\begin{aligned}
c_0 &= 1,\\
c_1 &= 0,\\
c_2 &= 0,\\
c_3 &= \nu_3/6,\\
c_4 &= [\nu_4 - 3]/24,\\
c_5 &= [\nu_5 - 10\nu_3]/120,\\
c_6 &= [\nu_6 - 15\nu_4 + 30]/720.
\end{aligned}
$$

Edgeworth's expansion

According to (4.3), the first coefficients are

$$
\begin{aligned}
c_0 &= 1,\\
c_1 &= 0,\\
c_2 &= 0,\\
c_3 &= \lambda_3/6,\\
c_4 &= \lambda_4/24,\\
c_5 &= \lambda_5/120,\\
c_6 &= [\lambda_6 + 10\lambda_3^2]/270.
\end{aligned}
$$

B PREPARATION

This thesis was created in LaTeX with the template *classicthesis*. Most of the figures were first created externally, then transformed into code blocks and finally embedded into the document by the package *tikz*.

The figures on page 9 and on page 36 were created using *R* with the source code given below and issued into code blocks by the package *tikzDevice*. The figure on page 21 was taken from Bleistein and Handelsman, 1975 and edited in *Inkscape*. The figure on page 28 was plotted with *GeoGebra* following Field and Ronchetti, 1990. Both figures were then transformed into code blocks in *Inkscape* by the extension *tikz*.

Source code of figures on page 9.

```
library("expint")

Ei <- function(n) expint_E1(n) # exponential integral

S_m <- function(n) { # partial sum for fixed m
  sum <- 0
  for (j in 1:m) sum <- sum + (-1)^(j-1)*factorial(j-1)*n^(-j)
  exp(-n)*sum
}

S_n <- function(m) { # partial sum for fixed n
  sum <- 0
  for (j in 1:m) sum <- sum + (-1)^(j-1)*factorial(j-1)*n^(-j)
  exp(-n)*sum
}

R_n <- function(m) abs(Ei(n)-S_n(m)) # remainder for fixed n

# figure 1

N <- 1:3

curve(Ei,from=1,to=4,xlim=c(1,3),ylim=c(0,0.1))
ei <- vector(length=length(N))
for (n in N) ei[n] <- Ei(n)
points(N,ei,pch=16)

m <- 1
curve(S_m,from=1,to=4,lty=5,add=T)
s_1 <- vector(length=length(N))
for (n in N) s_1[n] <- S_m(n)
points(N,s_1,pch=16)

m <- 2
curve(S_m,from=1,to=4,lty=2,add=T)
s_2 <- vector(length=length(N))
```

```
for (n in N) s_2[n] <- S_m(n)
points(N,s_2,pch=16)

m <- 3
curve(S_m,from=1,to=4,lty=3,add=T)
s_3 <- vector(length=length(N))
for (n in N) s_3[n] <- S_m(n)
points(N,s_3,pch=16)

# figure 2

M <- 1:10

n <- 1
r_1 <- vector(length=length(M))
for (m in M) r_1[m] <- R_n(m)
plot(M,r_1,xlim=c(1,9),ylim=c(0,0.25),pch=16)
axis(1,at=1:9,labels=1:9)
lines(M,r_1,lty=5)

n <- 2
r_2 <- vector(length=length(M))
for (m in M) r_2[m] <- R_n(m)
points(M,r_2,pch=16)
lines(M,r_2,lty=2)

n <- 3
r_3 <- vector(length=length(M))
for (m in M) r_3[m] <- R_n(m)
points(M,r_3,pch=16)
lines(M,r_3,lty=3)
```

Source code of figures on page 36.

```
f <- function(t) exp(-t)*exp(-exp(-t)) # density function
F <- function(t) exp(-exp(-t)) #distribution function

spe <- function(z,t) -digamma(1-z)-t # saddlepoint equation

f_spa <- function(t){ # saddlepoint approximation of f
  z_t <- uniroot(spe,c(-20,0.999),t=t)$root
  sqrt(1/(2*pi*trigamma(1-z_t)))*exp(log(gamma(1-z_t))-t*z_t)
}

F_spa <- function(t){ # saddlepoint approximation of F
  z_t <- uniroot(spe,c(-20,0.999),t=t)$root
  p_t <- sign(z_t)*sqrt(2*(t*z_t-log(gamma(1-z_t))))
  q_t <- z_t*sqrt(trigamma(1-z_t))
  pnorm(p_t)+dnorm(p_t)*(1/p_t-1/q_t)
}

T <- seq(from=-3,to=5,by=0.15)
f_T <- F_T <- vector(length=length(T))

for(t in 1:length(T)){
```

```
  f_T[t] <- f_spa(T[t])
  F_T[t] <- F_spa(T[t])
}

# figure 1

curve(f,from=-3,to=5,xlim=c(-2,4),ylim=c(0,0.4))
points(T,f_T)

# figure 2

curve(F,from=-3,to=5,xlim=c(-2,4),ylim=c(0,1))
points(T,F_T)
```

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