Diplomarbeit

zum Thema

Potential Future Exposure of Path-Dependent Financial Derivatives

ausgeführt am
Institut für Wirtschaftsmathematik
der Technischen Universität Wien

unter der Anleitung von
Privatdoz. Dipl.-Ing. Dr.techn. Stefan Gerhold

durch
Philipp Schmöger, BSc.
0625800
Rochusberg 148
2261 Mannersdorf an der March

Vienna, on December 31, 2012

Statutory Declaration
I declare in lieu of an oath that I have written this master thesis myself and that I have not used any sources or resources other than stated for its preparation. This master thesis has not been submitted elsewhere for examination purposes.

Vienna, on December 31, 2012

Abstract

This thesis is about calculating and comparing metrics for credit exposures of path-dependent financial derivatives, which are used to quantify counterparty credit risk. Thereby, the work of Lomibao and Zhu [2005] is taken as a basis for calculating future mark-to-market values, which includes the use of Brownian bridges to describe the evolution of underlying risk factors over time. Several path-dependent derivatives are examined in order to calculate their exposure profiles via scenario generation.

keywords: potential future exposures, path-dependent derivatives, Brownian bridge, Black-Scholes model

Contents

1 Introduction

For the last two centuries, the volume of the OTC market, and consequently credit exposure, has grown dramatically even though the market has experienced some of the most devastating financial crises during this time-span. The total combined notional amount outstanding of interest rate, credit, and equity derivatives has peaked to 466.8 trillion USD up to 2010*.

The first form of financial instruments were bonds, whose value mainly depended on the market's view of how creditworthy the issuer of these bonds were. Up to 2012, the complexity of such instruments has risen to a level where the process of estimating counterparty credit risk requires highly sophisticated mathematical models. In case of financial derivatives, it is as important to accurately model and estimate the value of future transactions as having a way to assess the counterparty's ability to meet its obligations.

It is crucial for a risk department to obtain a clear insight about prices of transactions in potential future scenarios, measuring the risk of the derivative to default. Consequently, there are obvious parallels between exposure calculations and computing value-at-risk with stress testing. Instead of measuring how much the value of a transaction could drop, counterparty risk departments are interested in calculating how high its value could go, or how much a counterparty would owe respectively.

Therefore, the main goal of this thesis is to calculate potential future exposure profiles of path depending derivatives - similar to a value-at-risk approach for potential losses. Thereby, scenarios are generated within the Black-Scholes model in order to calculate future transaction values of those derivatives by simulating the according risk factor for several future points in time. The idea behind this is to calculate the exposure, which depends on the whole evolution of the risk factor, by evaluating the expected value conditioned on a single future value. Lomibao and Zhu [2005] provide the basis for this idea by introducing the concept of the Brownian bridge to the existing models - the following calculations are largely based on their work. Therefore, this thesis can also be interpreted as a stress test for their ideas and results, where all the calculations are broken down and set right. In the course of this work, drawbacks and unfavorable assumptions are pointed out and suggestions for improvement are given in order to make the framework work more reliably.

Before calculating exposure profiles, however, it is crucial to understand the subject of credit

*ISDA Margin Survey 1987-2010, see published surveys on ISDA's website, www.isda.org exposure first. Therefore, the opening chapter deals with the general idea of counterparty credit risk and introduces the simulation framework, as well as risk measures that are going to be used later on.

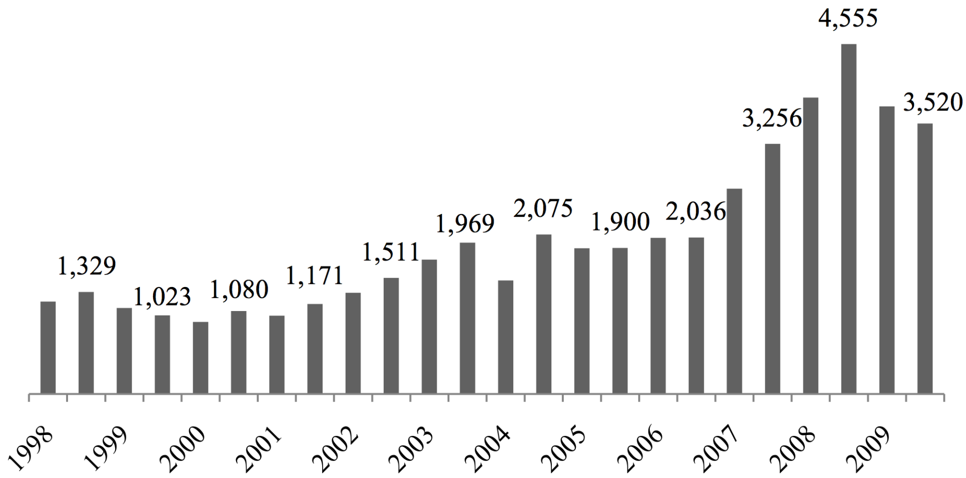

Figure 1.1: Gross Credit Exposure of OTC Derivatives (USD billions)

Source: ISDA Margin Survey 2010

2 Counterparty Credit Risk

Counterparty risk is the risk that a party involved in an over-the-counter (OTC) derivative contract may fail to perform on its contractual obligations, resulting in losses to the other party. As such contracts a bilateral, every party runs the risk of its counterparty not being able to make all the payments it is obliged to do. This eventually leads to losses depending on the value of the positions they hold against each other. In order to measure such losses, the costs of replacing the defaulted derivative are derived and taken into account. The idea behind this method is to mirror the losses with the costs of getting into the same position again as before the default.

Obviously, counterparty risk mainly occurs in the OTC market. Unlike security financing transactions (SFT), OTC contracts do not make use of any clearinghouse like an exchange that not only controls the execution of each obligation, but also guarantees the cash flows defined in the derivative's contract. Over-the-counter derivatives are non standard but individual contracts that are negotiated privately between two parties. So, if a counterparty of an OTC trade defaults, there is no guarantee that the other party will ever get its outstanding payments back, which is contrary to a trade on a stock exchange.

The first step of a market participant to quantify credit risk accurately is to get an idea of the exposures that result from all the trades with its counterparty. For this purpose, a lot of effort has to be put into the collection of all positions. Having consolidated all transactions, the institution can start to apply its models and measures. Of course, it is important to know ones current exposure, however, the key to have a reliable indicator for counterparty credit risk is to measure potential future exposures, as De Prisco and Rosen [2005] explain. Exposures are likely to change substantially over time, along with its underlying derivatives. For that reason, a solid stochastic model has to be adapted in order to describe future development. Moreover, exposure curves do not have to be continuous, because expiry dates, exercise dates and discrete payoff patterns lead to a discontinuous value of the derivative over time, and consequently, to a discontinuous credit exposure.
After establishing the mathematical framework, Monte Carlo simulation is often applied as it remains the most general and reliable approach to capture the complex stochastic nature of exposure profiles. For the purpose of this thesis, explicit probabilities of default, recovery rates, credit spreads and correlations of credit events are going to be largely ignored, since it merely

concentrates on calculating exposure profiles of single path-dependent OTC derivatives. Like Gregory [2010] points out, exposure is the loss, as defined by the replacement cost that would be incurred assuming zero recovery value. Hence, credit exposure is conditional on counterparty default.

2.1 Counterparty Exposures

As mentioned in the previous section, the loss of an OTC trade due to default has to be quantified in order to capture the counterparty credit risk. Assuming that a party would have to close out its position with its counterparty, it would be supposed to enter a similar contract in order to maintain its market position. Therefore, the loss of the party would be the value of the contracts replacement costs at the time of default.

If the value of the derivative is positive, it corresponds to a claim on the defaulting counterparty, whereas a negative value does not release the party of its liabilities. This means that an institution will incur loss if it is owed money and its counterparty defaults. In the reverse situation, however, they will not be able to gain from the default if they have remaining obligations. At this point, it is again referred to Gregory [2010] for a deeper insight into the subject. Since a portfolio of several positions is generally considered, the maximum loss for a party is equal to the sum of the contract-level credit exposures.

Definition *The counterparty credit exposure of contract i at time t is given by*

$$E_i(t) = \max\left(MtM_i(t), 0\right), \tag{2.1.1}$$

where $MtM_i(t)$ is the mark-to-market value of the i^{th} contract at time t.

This is a very general definition that should help to understand the concept of credit exposure. The mark-to-market value (MtM) is a measure of the fair value of the contract that can change over time and that aims to provide a realistic estimate over the development of the future value. It can be described as the present value of all the payments an institution is expecting to receive minus those it has to make. As those payments can be rather uncertain - depending on future market variables - the mark-to-market value can either be positive or negative. In context to the topic of counterparty credit risk, it represents the replacement costs of an institution to even out its position with a counterparty after a default. Thereby, bid-offer spreads and transaction cost are ignored since they are more appropriately treated as liquidity risk. So, exposure is based solely on the current MtM value of a transaction in this thesis.

2.1.1 Metrics for Credit Exposures

As the goal of this thesis is to calculate stochastic future exposures via Monte-Carlo simulation, reliable metrics to measure credit exposure have to be introduced. The following section is closely tied to the explanations of Gregory [2010], who followes the definitions of the Basel Committee on Banking Supervision [2005] himself. Following points have to be considered when choosing a measure for exposures:

- The general idea behind counterparty credit risk measures is to monitor the future exposure over multiple time horizons in order to fully capture the impact on ones portfolio. On the other hand, more traditional risk measures like the value-at-risk merely concentrates on one single date in the future - mostly with maturities of a couple of days/weeks.
- Both departments of risk management and pricing take a look at counterparty credit risk, however, either of them has different applications. The second one is able to put lots of effort into pricing credit risk as precise as possible, whereas the first one generally deals with an overwhelming number of portfolios, which forces the department to make use of more efficient measures to decrease the runtime of the calculations.

The potential future exposure (PFE) is a useful measure to derive potential losses and is well explained by De Prisco and Rosen [2005]. Consider a portfolio of transactions with a single counterparty and a discrete set of times $\{t_0, t_1, \ldots, t_N = T\}$. Furthermore, $S_j(t_k)$ describes the state of scenario j at time t_k as the path of the scenario up time t_k, $k \in \{1, 2, \ldots, N\}$, which contains all the information up to this date.

Definition *For a single position p with a counterparty, the potential future exposure at time t_k and scenario S_j is defined as the maximum of zero and the contract value (if it was replaced in the market)*

$$\text{PFE}(p, S_j, T_k) = \max\big(V(p, S_j(t_k), t_k), 0\big), \tag{2.1.2}$$

where $V(p, S_j(t_k), t_k)$ is the mark-to-market value of the contract at t_k.

The Scenario S_j is based on all market conditions realised by the time t_k and is going to refer to the underlying of the derivative that is going to be modeled and simulated. Further, netting is going to be largely ignored as this thesis solely focuses on individual contracts, and the notation of the potential future exposure at time t is going to be shortened to $\mathrm{PFE}(t)$.

Often, the PFE is connected to a certain high degree of statistical confidence, making it similar to a value-at-risk approach, but with longer time horizon and an association with a gain instead of a loss. Figure 2.1 shows the potential future exposure for several scenarios generated. Only the shaded area is taken into account as exposures are strictly positive. In practice, worst case scenarios with a high confidence level are usually drawn to capture the worst exposure the party could have at a certain time in the future. For example, the PFE at a 95%-confidence level will define an exposure that would be exceeded with a probability of no more than 5%. Generally, it is going to be computed through simulation models for each future date t_k , which the following chapters are going to show.

Definition *The α-percentile of the PFE at time t is the value* $\mathrm{PFE}(t)^\alpha$ *such that*

$$\mathbb{P}(\mathrm{PFE}(t) \geq \mathrm{PFE}(t)^\alpha) \leq 1 - \alpha \,. \tag{2.1.3}$$

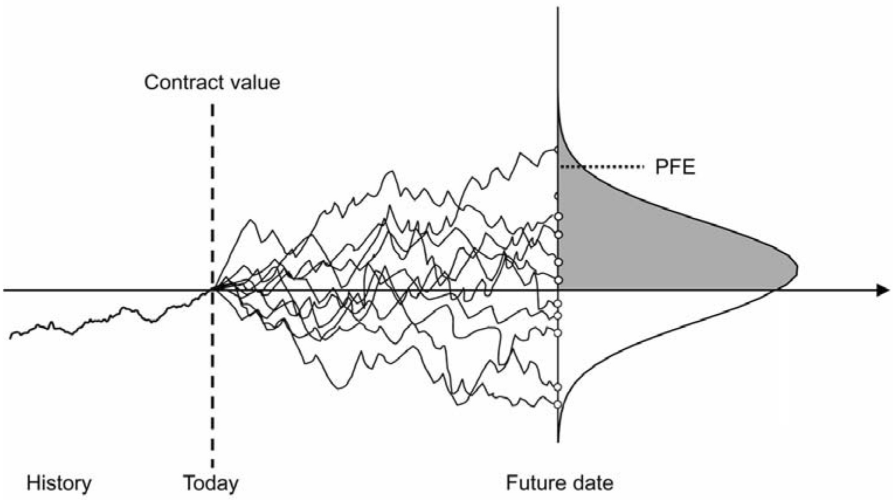

Figure 2.1: Illustration of potential future exposure (Source: Gregory [2010])

The expected exposure (EE) represents the amount expected to be lost if the counterparty defaults. As the exposure is the positive part of the MtM, it is obvious that the EE is always greater than the expected MtM. It differs greatly from the current exposure due to the specifics of cash flows.

Definition *The expected exposure at time t is defined as*

$$\mathrm{EE}(t) = \mathbb{E}[\mathrm{PFE}(t)] \, , \tag{2.1.4}$$

where the expectation is taken on all the scenarios of the PFE at time t.

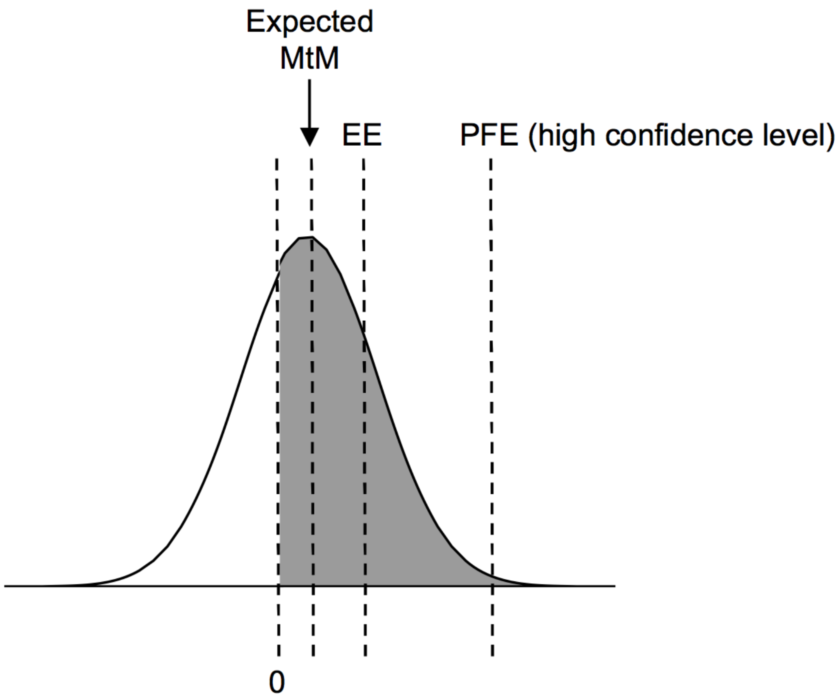

Figure 2.2: Example for the exposure metrics EE and PFE (Source: Gregory [2010])

Figure 2.2 illustrates the differences between MtM, EE and PFE with high confidence interval α (usually α is 95% or 99%).

2.2 Modelling Credit Exposure

This section deals with establishing a framework for calculating the expected exposures and potential future exposures on OTC derivatives.

The mark-to-future framework is an adaptable, multi-step simulation framework to measure risk and reward introduced by Dembo et al. [2000]. They propose the so-called mark-to-future (MtF) value, which equals the future MtM value, as a tool to link market, credit and liquidity risk.

This approach focuses on simulated future scenarios rather than being constrained by narrow assumptions on the shape of the distributions.

2.2.1 Mark-to-Future Methodology

This framework introduces the three-dimensional MtF Cube which stores simulated MtFs from each instrument at each time step under every scenario generated. Figure 2.3 from Dembo et al. [2000] shows a simplified representation of this concept.

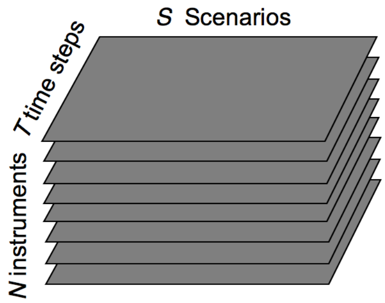

Figure 2.3: Illustration of MtF Cube (Source: Dembo et al. [2000])

In this thesis, each instrument is going to be taken care of individually, so the cube is going to be turned into a table for each instrument, containing the MtF values for each scenario at any point in time. After computing the cube/table, the risk analysis and exposure measures are derived by post-processing the generated numbers. So, the framework can basically be divided into two stages:

- The pre-cube stage
- The post-cube stage

Each of them consists of three different steps. Dembo et al. [2000] list them as follows:

Pre-cube stage:

1. Define the scenario paths and time steps.
2. Define the basis instruments.
3. Simulate the instruments over the scenarios and time steps to generate a MtF Cube.

Post-cube stage:

1. Map the MtF Cube into portfolios to produce a portfolio MtF table.

2. Aggregate across dimensions of the portfolio MtF table to produce risk/reward measures.

3. Incorporate portfolio MtF tables into advanced applications.

Thereby, the pre-cube stage is the CPU-intensive part: Simulating thousands of scenarios of a certain risk factor over many years and a small day count fraction and computing the MtF of complex OTC derivatives afterwards is fairly time-consuming. However, it merely needs to be performed once - the remaining steps can be performed with minimal additional processing.

2.2.2 Definition of Scenarios

Scenarios contain all the information of the market and its risk factors up to a certain point in time and describe the development of the market over time, giving a joint realisation of all the relevant financial risk factors at a given discrete time grid. Therefore, it is crucial to find and define a model that captures all the circumstances that influence the market. The future uncertainty with its joint evolution of risk factors through time is the key point for modelling counterparty credit exposure. Interest rates, securities, exchange rates and other potential underlyings have to be fit into a framework that not only is able to capture all their properties, but is also "handy" enough to be simulated within reasonable time. Therefore, a proper representation of the future relies on the selection of the distribution of the underlying risk factors.

Hence, three trains of thought basically have to be considered when choosing a framework for the scenario:

1. The scenario must take all relevant historical data into account. Parameters have to be estimated from past scenarios in order to calibrate the model appropriately - history does not mirror the future, but is a good indication for the events that might happen. Thereby, the period of historical data should be chosen with regard to its relevance and purpose for the scenario.

2. The scenario should be able to render past events that may be plausible under the current circumstances.

3. The scenario generation method must be tested out of sample. Values that were derived the day before can be compared with today's mark-to-market values. This back-testing is a very important regulatory requirement and determines the performance of the method.

Both pricing and risk management make use of models that describe the evolution of counterparty risk factors, however, a single model will not meet the needs of both departments. Generally, risk management tend to use true probabilities and has to calibrate scenarios using historical and current data. Pricing models, on the other hand, work under the risk-neutral measure that provides a no-arbitrage condition. The discounted expected values under this measure represents the price of the derivative. Implied volatilities and other parameters that are calculated from current prices and market perception are used to calibrate these kind of models. Another difference between those models is the number of dimensions they use. Whereas risk management tends to use high-dimensional models in order to capture the joint behavior of several instruments in a portfolio, pricing models usually consist of one or two risk factors.
Finally, it has to be said that risk management has a bigger focus on the tails of the portfolio distribution. Rare but extreme events are the biggest concern in risk departments, as they account for the biggest part of an annual loss. Those events can make any financial institution struggle because of their rarity and uncertainty. Pricing derivatives, though, requires the computation of expected values for future pay-offs and may be less concerned with extreme events. In order to achieve mathematical tractability, restrictions and simplified assumptions have to be made traditionally. For the purpose of modelling OTC derivatives, a single risk factor is going to be modelled within the Black-Scholes framework. The underlying, a stock price for instance, is going to follow a geometric Brownian motion, which is a stochastic process that is widely used, even though it is arguably not the most accurate description of certain risk factors over a longer period of time.

2.2.3 Defining the Basis Instrument

After establishing the scenario, the next step is to define the basis instrument. As stated before, the MtF cube consists of several MtF tables, each representing an individual basis instrument. There is a nearly unlimited amount of OTC products on the market, however, most of them can fortunately be divided into a composition of more basic instruments such as stocks, bonds or options.

In order to go along with previous notation, the MtF value at time t_k of scenario S_j can be written as a function f of risk factors $u_m(S_j(t_k), t_k)$, $m \in \{1, \ldots, N\}$, where N is the number of risk factors in the model and p_i the position of the i^{th} instrument:

$$V\big(p_i, S_j(t_k), t_k\big) = f_{p_i}\big(u_1(S_j(t_k), t_k), \ldots, u_N(S_j(t_k), t_k)\big) \,. \tag{2.2.1}$$

In the case of a synthetic instrument like certain OTC derivatives, it is merely a function of a single risk factor. Then the MtF value of the portfolio is quantified by applying a function g after simulating all the MtF values. If the portfolio simply consisted of the basis instrument, g would obviously be the identity. This routine has to be provided for every derivative and is perfectly displayed by Dembo et al. [2000], which is the source of figure 2.4.

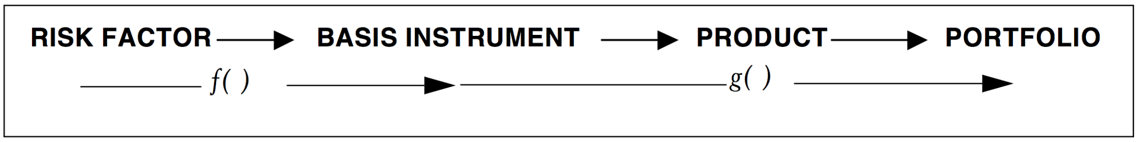

Figure 2.4: Relationship between risk factors, basis instruments, financial products and portfolios

2.3 Simulating MtF Cube and Post-Processing

In order to calculate potential future exposures of credit derivatives, the next step is the actual simulation itself. Monte-Carlo simulation is the most reliable approach and therefore widely used in the financial sector. Its benefit is that it can capture complex stochastic structures without knowing the actual distribution of the MtF value over time by repeatedly generating possible scenarios and applying qualified measures afterwards. Those methods form a class of computational algorithms that rely on repeated random sampling in order to compute their results. Risk management certainly is an area of application, where Monte-Carlo methods are useful for modelling phenomena with significant uncertainty in inputs. The general idea behind these methods is to solve a problem by directly simulating the process and then calculating the average results. They simulate the value of an instrument, portfolio or underlying by including the uncertainty that affects them, and therefore very suitable for calculating profiles of PFEs. The trade-off of these methods is its highly computational complexity if the algorithms used to simulate risk factors and calculate MtF values are not well optimised.

Again, consider a discrete set of times $\{t_0, t_1, \ldots, t_N = T\}$. Further, $S_j(t_k)$ describes the state of scenario j at time t_k as the path of the scenario up time t_k, $k \in \{1, 2, \ldots, N\}$. The set of scenarios up to T is $\{S_1(T), S_2(T), \ldots, S_M(T)\}$, where M is the number of scenarios generated (usually several thousands).

Thus, each derivative is evaluated over every scenario for any time step, which makes the use of efficient methods and algorithms crucial. After simulating the MtF tables, which then are filled with all the MtF values at any given date, the measures established in section 2.1.1 can be applied in order to evaluate and estimate future credit exposure.

3 Mathematical Theory

This chapter of this thesis deals with the mathematical background that is needed to implement the theory discussed and explained in previous sections. The best idea of a model is worthless if it cannot be backed up with a solid mathematical framework.

First of all, it has to be mentioned that most of the theory discussed below refers to the work and books of Øksendal [2007] and Karatzas and Shreve [1988]. All the definitions and theorems in this section are set in probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with set Ω, σ-algebra $\mathfrak{F}$ and probability measure $\mathbb{P}$. It is called a complete probability space if $\mathfrak{F}$ contains all subsets G of Ω with $\mathbb{P}$-outer measure zero, i.e. with

$$\mathbb{P}^*(G) := \inf\{\mathbb{P}(F);\ F \in \mathfrak{F}, G \subset F\} = 0 \quad .$$

Further, a filtration $\{\mathfrak{F}\}_{t\geq 0}$ is added to complete a filtrated probability space.

3.1 Brownian Motion

The Brownian motion is the name given to a random movement first discovered by Scottish botanist Robert Brown. He observed in 1828 that pollen grains suspended in liquid performed an irregular motion. However, the application was not restricted to the studying of microscopic particles, but has been used to model stock prices, thermal noise in electrical circuits and other fields of application ever since discovered.
This movement was put into a mathematical concept, introducing a stochastic process W_t called Brownian motion, alternatively known as Wiener process.

Definition *Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtrated probability space. Then, an adaptive stochastic process with values in $\mathbb{R}^d$ is called a d-dimensional standard Brownian motion (SBM) with respect to $\{\mathfrak{F}\}_{t\geq 0}$ and $\mathbb{P}$, if*

a) $\mathbb{P}(W_0 = 0) = 1$

b) For all $s, t \in [0, \infty)$, $W_t - W_s$ is independent of $\mathfrak{F}_s$
(independent increments)

c) For all $s, t \in [0, \infty)$*,* $W_{s+t} - W_s \sim W_t - W_0 \sim W_t$
(stationary increments)

d) For all $t \in [0, \infty)$*,* $W_t - W_0 \sim N(0, I_d)$ *with* $(d \times d)$ *identity matrix* I_d

e) $\{W_t\}_{t \geq 0}$ *has continuous paths* $\mathbb{P}$*-a.s.*

If $\mathfrak{F}_t = \sigma(B_s : s \in [0, t])$, $t \geq 0$, *b)* in the definition of the Brownian Motion can be replaced by

b') For all $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \ldots < t_n$, $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1}$, ... , $W_{t_n} - W_{t_{n-1}}$ are independent.

If *d)* is dropped and *e)* altered to $\{W_t\}_{t \geq 0}$ to be continuous from the right at 0 (in probability), the process becomes a Lévy process. Furthermore, it is important to know that the Brownian motion W_t in $\mathbb{R}^n$ is a martingale with respect to the σ-algebras $\mathfrak{F}_t$ generated by $\{B_s : s \leq t\}$.

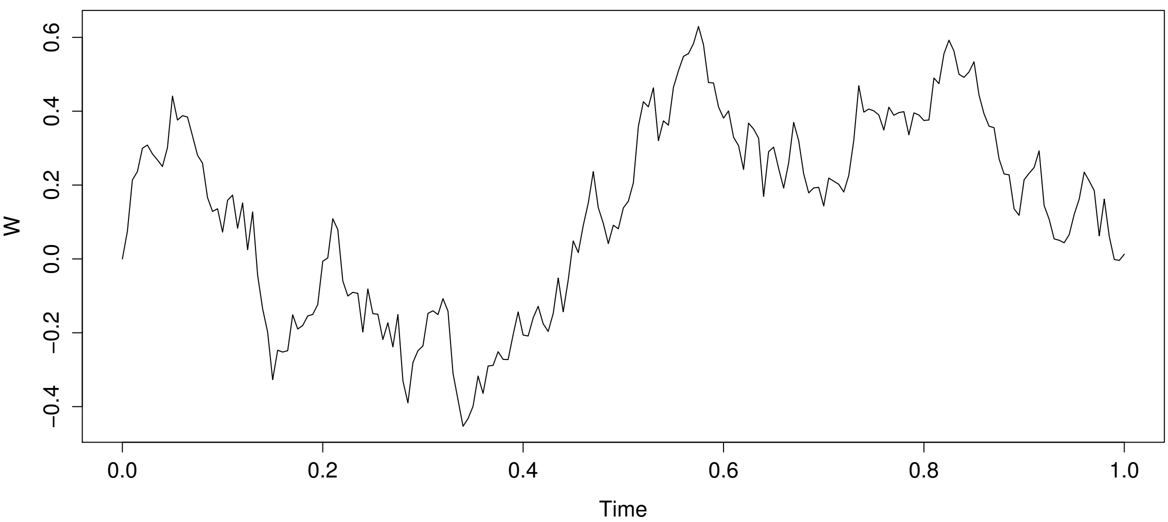

Figure 3.1: Path of a standard Brownian motion

For more information about the Brownian motion and proofs of its existence, it is referred to either Øksendal [2007] or Karatzas and Shreve [1988].

One of the most useful attributes of a Brownian motion it that is a Markov process, as Karatzas and Shreve [1988] show. The process is defined as follows:

Definition *Let d be a positive integer and μ a probability measure on $\big(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d)\big)$. Then, an $\{\mathfrak{F}_t\}_{t\geq 0}$-adapted, d-dimensional stochastic process $X = \{X_t\}_{t\geq 0}$ on some probability space $(\Omega, \mathfrak{F}, \mathbb{P}^\mu)$ is said to be a Markov process with initial distribution μ if*

a) $\mathbb{P}^\mu(X_0 \in B) = \mu(B), \quad \forall B \in \mathfrak{B}(\mathbb{R}^d)$

b) For $s, t \geq 0$ and $B \in \mathfrak{B}(\mathbb{R}^d)$

$$\mathbb{P}^\mu\big(X_{t+s} \in B \mid \mathfrak{F}_s\big) = \mathbb{P}^\mu\big(X_{t+s} \in B \mid X_s\big)$$

The second requirement is called Markov property and ensures that a Markov process merely depends on its last state instead of the whole history of the process. For an in-depth view on this topic, it is referred to David Meintrup and Stefan Schäffler [2005].

3.1.1 Brownian Bridge

The Brownian bridge (BB) is a very useful transformation of the Brownian motion and can be interpreted as a Brownian motion $W(t)$ conditioned on its start value x at time t_0 and its end value y at time T, $t_0 \leq t \leq T$. It is the process $\{W_t \mid W_{t_0} = x, W_T = y\}_{t_0 \leq t \leq T}$ and is defined as

$$W_{t_0,x}^{T,z}(t) = x + W_{t-t_0} - \frac{t-t_0}{T-t_0}(W_{T-t_0} - z + x) \quad , \tag{3.1.1}$$

where $W(t)$ is a Brownian motion and $t_0 \leq t \leq T$ (see Iacus [2008]).

The density of the Brownian bridge for $t \in [t_0, T]$ is calculated as follows

$$\begin{aligned}
\mathbb{P}[W_t = y \mid W_{t_0} = x, W_T = z] &= \mathbb{P}\Big[W_{t_0,x}^{T,z}(t) = y\Big] = \mathbb{P}\left[x + W_{t-t_0} - \frac{t-t_0}{T-t_0}(W_{T-t_0} - z + x) = y\right] \\
&= \mathbb{P}\left[\sqrt{t-t_0}W_1 - \frac{t-t_0}{\sqrt{T-t_0}}W_1 = y - x\left(1 - \frac{t-t_0}{T-t_0}\right) - \frac{t-t_0}{T-t_0}z\right] \\
&= \frac{1}{\sqrt{2\pi(t-t_0)\left(1-\frac{t-t_0}{T-t_0}\right)}} \exp\left[-\frac{1}{2}\frac{\left(y - (x + \frac{t-t_0}{T-t_0}(z-x))\right)^2}{(t-t_0)\left(1-\frac{t-t_0}{T-t_0}\right)}\right] \quad .
\end{aligned}$$

Consequently, $W_{t_0,x}^{T,z}(t) \sim N\left(x + \frac{t-t_0}{T-t_0}(z-x), (t-t_0)\left(1-\frac{t-t_0}{T-t_0}\right)\right)$.

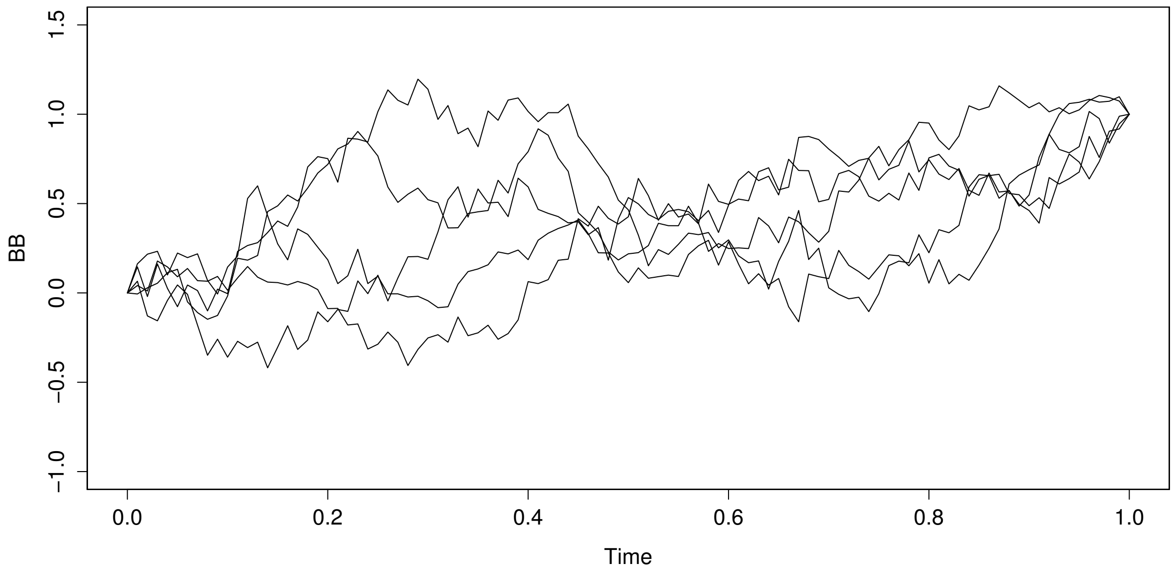

Figure 3.2: Paths of multiple Brownian bridges with initial value 0 and end value 1

For the purpose of this thesis, it is especially focused on Brownian Bridges that starts at 0 with value 0. The following result is a very useful one as it is going to be used fairly often in the course of evaluating exposures later on. For any $\sigma > 0$,

$$
\begin{aligned}
\mathbb{E}[\exp[\sigma W_{0,0}^{T,z}]] = \mathbb{E}[\exp[\sigma W_t] \mid W_0 = 0, W_t = z] &= \mathbb{E}[\exp[\sigma\big(W_t - \frac{t}{T}(W_T - z)\big)]] \\
&= \exp\Big[\sigma\frac{zt}{T}\Big]\,\mathbb{E}\bigg[\exp\Big[\sigma W_t\Big]\,\exp\Big[-\frac{\sigma t}{T}(W_{T-t} + W_t)\Big]\bigg] \\
&= \exp\Big[\sigma\frac{zt}{T}\Big]\,\mathbb{E}\bigg[\exp\Big[\sigma W_t\Big(1 - \frac{t}{T}\Big)\Big]\,\exp\Big[-\frac{\sigma t}{T}W_{T-t}\Big]\bigg] \\
&= \exp\Big[\sigma\frac{zt}{T}\Big]\,\mathbb{E}\bigg[\exp\Big[\sigma W_t\Big(1 - \frac{t}{T}\Big)\Big]\bigg]\,\mathbb{E}\bigg[\exp\Big[-\frac{\sigma t}{T}W_{T-t}\Big]\bigg] \\
&= \exp\Big[\sigma\frac{zt}{T} + \sigma^2\frac{t\big(1-\frac{t}{T}\big)^2}{2} + \sigma^2\frac{t^2(T-t)}{2T^2}\Big] \\
&\quad \times \underbrace{\int_{\mathbb{R}}\phi\left(x - \sigma\sqrt{t}\Big(1-\frac{t}{T}\Big)\right)\,dx\int_{\mathbb{R}}\phi\left(y + \sigma\frac{t\sqrt{T-t}}{T}\right)\,dy}_{=1}
\end{aligned}
$$

$$= \exp\left[\frac{\sigma^2 t}{2}\Big(1 - \frac{t}{T}\Big) + \frac{\sigma t z}{T}\right] \quad . \tag{3.1.2}$$

The expected value is allowed to be split up because of $W_{T-t} \perp\!\!\!\perp W_t$. From now on, it is always assumed that $t_0 = 0$ and $W_{t_0} = 0$ when speaking of Brownian bridges. Hence, the condition on $W_{t_0} = 0$ is ultimately dropped within the notation of the conditional expected value.

3.1.2 Reflection Principle

This section deals with the reflection principle and its use in the calculation of the distribution of the maximum and minimum of a Brownian motion. The results of the following paragraphs turn out to be very useful for further valuations of path-dependent derivatives.
The Brownian motion has many useful qualities that make it suitable for describing certain processes. An important one is the fact that the distribution of many of its functionals can be derived in closed form. One of it is the passage time T_b to a level $b \in \mathbb{R}$, defined by

$$T_b(\omega) = \inf\{t \geq 0;\ W_t(\omega) = b\}\ , \tag{3.1.3}$$

which is a stopping time (see Karatzas and Shreve [1988]). Its density function can be obtained by using the reflection principle, which was first introduced by Désiré André (Lévy [1948]).
Let $\{W_t\}_{0 \leq t < \infty}$ be a standard one-dimensional Brownian motion with respect to filtration $\mathfrak{F}_t$ on $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t \geq 0}, \mathbb{P})$. For $b > 0$,

$$\mathbb{P}[T_b < t] = \mathbb{P}[T_b < t,\ W_T > b] + \mathbb{P}[T_b < t,\ W_T < b]\ . \tag{3.1.4}$$

Obviously, $\mathbb{P}[T_b < t,\ W_T > b] = \mathbb{P}[W_t > b]$. Looking at the second term, one can argue that if $T_b < t$ and $B_t < b$, then the path of the Brownian motion has reached level b sometime before time t and has ended up at a point $c < b$. Since the Brownian motion starting at b is symmetric with respect to b, the probability of it falling off to c is the same as the probability of travelling to point $2c - b$. Karatzas and Shreve [1988] call this second path the "shadow path", which is just a reflection about b of the initial path - both having the same probability. Keep in mind that this is merely a heuristic argument, since the probability for the occurrence of any particular path is zero. Anyway, this principle leads to

$$\mathbb{P}[T_b < t,\ W_t < b] = \mathbb{P}[T_b < t,\ W_t > b] = \mathbb{P}[W_t > b] \tag{3.1.5}$$

and

$$\mathbb{P}[T_b < t] = 2\,\mathbb{P}[W_t > b] = \sqrt{\frac{2}{\pi}} \int\limits_{bt^{-1/2}}^{\infty} \exp\left[-\frac{x^2}{2}\right] dx \quad . \tag{3.1.6}$$

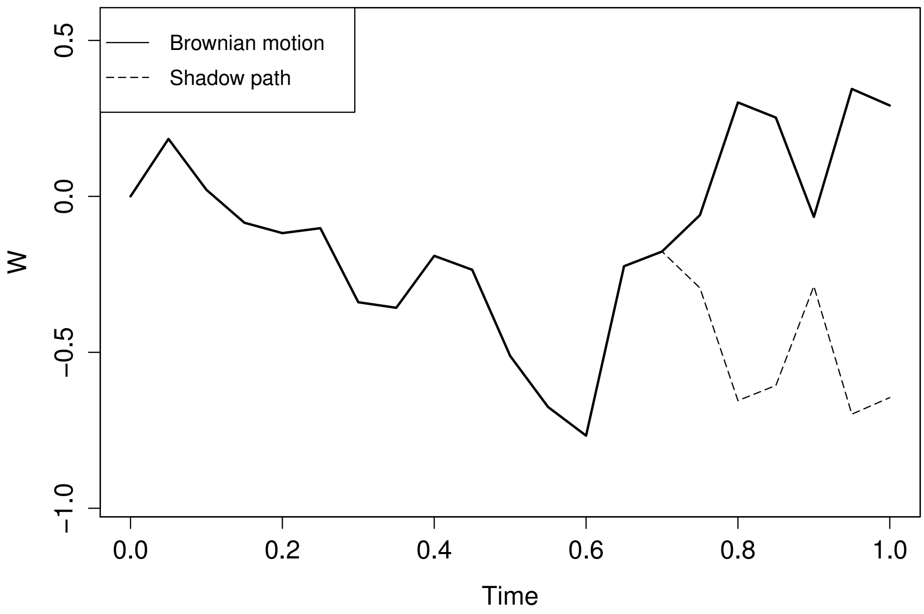

Figure 3.3: Brownian motion and its "shadow path"

The Brownian motion "starts fresh" at stopping time T_b, meaning that the process $\{W_{t+T_b} - W_{T_b}\}_{0\leq t<\infty}$ is independent of filtration $\mathfrak{F}_{T_B}$. This is a consequence of the strong Markov property of the Brownian motion. Chen, Wang and Shyu [2010] sum it up in the following theorem:

Theorem *Let the stochastic process* $\{W_t\}_{0\leq t<\infty}$, $t \in \mathbb{R}_+$, *follow a Brownian motion on* $(\Omega, \mathfrak{F})$ *with stopping time* $T_b(\omega) = \inf\{t \geq 0;\ W_t(\omega) = b\}$. *Define*

$$\widetilde{W_t} = \begin{cases} W_t & if \quad T_b > t \\ 2b - W_t & if \quad T_b \leq t \end{cases} \quad . \tag{3.1.7}$$

Then the process $\{\widetilde{W_t}\}_{0\leq t<\infty}$ *is also a Brownian motion that has the same distribution as* $\{W_t\}_{0\leq t<\infty}$.

This knowledge can be applied on a more general process, namely a Brownian motion with non-zero drift. The following results are going to be very useful for calculating the value of barrier options in section 4.2. The next paragraphs represent the result of Musiela and Rutkowski

[2005].

Let $X = \{X_t\}_{t \geq 0}$ be a process such that

$$X_t = vt + \sigma W_t \quad , \tag{3.1.8}$$

where $v \in \mathbb{R}$, $\sigma > 0$ and W_t is a standard Brownian motion under $\mathbb{P}$. For the maximum and the minimum of this process,

$$M_t^X := \max_{u \in [0,t]} X_u \,, \quad m_t^X := \min_{u \in [0,t]} X_u \quad . \tag{3.1.9}$$

According to Girsanov's theorem (see section 3.2.2), X is a Brownian motion under an equivalent probability measure and therefore, $\mathbb{P}[M_t^X > 0] = 1$ for every $t \geq 0$.

Lemma *For every $t > 0$, the joint distribution of X_t and M_t^X is given by the formula*

$$\mathbb{P}[X_t \leq x,\ M_t^X \geq y] = \exp\left[\frac{2vy}{\sigma^2}\right] \mathbb{P}[X_t \geq 2y - x + 2vt] \quad , \tag{3.1.10}$$

for every $x, y \in \mathbb{R}$ such that $y \geq 0$ and $x \leq y$.

In order to proof this lemma, the reflection principle and Girsanov's theorem are used (see Musiela and Rutkowski [2005]). As a consequence, the joint distribution of a Brownian motion with non-zero drift and its maximum value - (X_t, M_t^X) - is

$$\mathbb{P}[X_t \leq x,\ M_t^X \geq y] = \exp\left[\frac{2vy}{\sigma^2}\right] \Phi\left(\frac{x - 2y - vt}{\sigma\sqrt{t}}\right) \quad , \tag{3.1.11}$$

for every $x, y \in \mathbb{R}$ such that $y \geq 0$ and $x \leq y$.

For the minimum value of X and $y < 0$,

$$\begin{aligned} \mathbb{P}[\max_{u \in [0,t]} -X_u \geq -y] &= \mathbb{P}[\max_{u \in [0,t]} (-vu + \sigma W_u) \geq -y] = \mathbb{P}[\min_{u \in [0,t]} (-vu + \sigma W_u) \leq y] \\ &= \mathbb{P}[\min_{u \in [0,t]} X_u \leq y] \quad , \end{aligned}$$

because of the symmetry of the Brownian motion. Hence, $\mathbb{P}[m_t^X \leq y] = \mathbb{P}[M_t^{\hat{X}} \geq -y]$ for $y \leq 0$, $t > 0$ and process $\hat{X} = -vt + \sigma W_t$.

Consequently, the joint distribution of (X_t, m_t^X) satisfies

$$\mathbb{P}[X_t \geq x,\ m_t^X \geq y] = \Phi\left(\frac{-x + vt}{\sigma\sqrt{t}}\right) - \exp\left[\frac{2vy}{\sigma^2}\right] \Phi\left(\frac{2y - x + vt}{\sigma\sqrt{t}}\right) \tag{3.1.12}$$

for every $x, y \in \mathbb{R}$ such that $y \leq 0$ and $y \leq x$.

Now that the basic concept of Brownian motions is explained, the next step is to implement the random walk into a mathematical model that is able to describe risk factor developments and evaluate prices for assets and derivatives.

3.2 Black-Scholes Model

The Black-Scholes model was issued in 1973 and since then, has proofed to be a reliable framework to price options, future and other derivatives. It had a huge impact on the valuation and hedging of options and largely contributed to the growth and success of financial engendering. Ultimately, Robert Merton and Myron Scholes won the Nobel prize for economics for in 1997 for their achievements.
Its major assumption is that asset prices, or risk factors, are log-normal distributed, meaning that they follow a geometric Brownian motion.

3.2.1 General Assumptions on the Market

In order to apply the mathematical theory of the Black-Scholes model, several simplifying assumptions have to be made. On the one hand, they prevent the model to become too complex to be used in option pricing as well as risk management. On the other hand, these assumptions are needed to make use of the beneficial properties the model provides in order to analytically derive formulas for different scenarios in the financial market.
Hence, the following simplifications for the market are assumed:

- Continuous time trading possible
- No restrictions on trades (short-selling allowed)
- Deposit and lending rates are equal
- No transaction costs
- No taxes

The next restriction that comes along with the model is that it only provides two financial instruments - a risk-free and and a risky one.

Risk-free Financial Instrument

The risk-free security is represented by a bond or bank account, which are assumed to have no default risk. In a real market, there is no such thing as a risk-free financial instrument, because even banks, companies or governments with the best credit worthiness have strictly positive default probabilities.

Anyway, it is assumed that the interest rate $r > 0$ on this bank account is constant. Hence, the value of the risk-free bond at time $t \geq 0$ is

$$B_t = B_0 \exp[rt] \tag{3.2.1}$$

where $B_0 = 1$. Consequently, the derivative at time t is

$$B'_t = rB_t \quad .$$

Alternatively, the risk-free bond between $[0, T]$ can also be described as a zero-coupon bond at time $t \in [0, T]$ with maturity T and face value 1:

$$P(t, T) = \exp[-r(T - t)] \quad . \tag{3.2.2}$$

Obviously, $B_t = \exp[rT]P(t, T)$ and $P(t, T) = \frac{B_t}{B_T}$, $t \in [0, T]$.

Risky Financial Instrument

The risky financial instrument is the counterpart of the risk free one and represents a stock, swap rate or any other risk factor that underlies any random value fluctuation. It is none negative and driven by a standard Brownian motion.

Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtrated probability space, where $\mathcal{F} = \{\mathfrak{F}_t\}_{t\geq 0}$ is a filtration on $\mathfrak{F}$ which contains all null sets of $\mathfrak{F}$ and is continuous from the right. Further, let $\sigma > 0$ be the volatility, $\mu \in \mathbb{R}$ the appreciation rate, S_0 the price of the risky security at time 0 and $\{W_t\}_{t\geq 0}$ a one-dimensional, standard Brownian motion with respect to $\mathbb{F}$ and $\mathbb{P}$. Then

$$S_t := S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right] \tag{3.2.3}$$

is the stock price at time $t \geq 0$.

This kind of stochastic process is called a geometric Brownian motion (GBM) and follows a log-normal distribution. It is a very handy assumption, because many properties of the normal distribution can be applied. Furthermore, it provides the stock price to be non-negative all the time. However, it has to be said that is not the most realistic assumption, because it does not allow the stock price to perform jumps in prices at any time. If weekends are considered, the continuous path of a geometric Brownian motion might not seem to be an adequate approximation. In addition, price fluctuations will not be heavy tailed if they are assumed to be driven by a Brownian motion, hence, the probability of huge and drastic price changes is generally underestimated by this model.

Lemma *$\{S_t\}_{t\geq 0}$ is the unique strong solution of the stochastic differential equation (SDE)*

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \,, \quad t \geq 0 \tag{3.2.4}$$

with initial value S_0, respectively, the one of the stochastic integral equation

$$S_t = S_0 + \int_0^t \mu S_s \, ds + \int_0^t \sigma S_s \, dW_s \,, \quad t \geq 0 \quad . \tag{3.2.5}$$

For the proof of this lemma, which makes use of Ito's theorem with function $f(t,x) = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma x\right]$, $x \in \mathbb{R}$, $t > 0$, it is referred to Øksendal [2007].

Remark *For all $t \geq 0$, $f_t(x) := S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma x\right]$ is strictly monotonically non-decreasing and surjective, hence invertible. For all $n \in \mathbb{N}$ and $0 \leq t_1 < t_2 < \ldots < t_n$, $y_1, y_2, \ldots, y_n \in (0, \infty)$, the following holds: $\left\{\omega \in \Omega \mid S_{t_1} \leq y_1, S_{t_2} \leq y_2, \ldots, S_{t_n} \leq y_n\right\} = \left\{\omega \in \Omega \mid W_{t_1} \leq f^{-1}(y_1), W_{t_2} \leq f^{-1}(y_2), \ldots, W_{t_n} \leq f^{-1}(y_n)\right\}$. Hence, the σ-algebra generated by S_t equals the one generated by W_t, $t \in [0,t]$: $\sigma(S_t : t \in [0,T]) = \sigma(W_t : t \in [0,T])$. This means that the Brownian motion and the stock price contain the same information.*

The next step is to calculate its moments which are going to be very useful in later chapters. The expected value of (3.2.3) is obtained as follows:

$$\begin{aligned}
\mathbb{E}\left[S_t\right] &= \mathbb{E}\left[S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right]\right] = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t\right]\mathbb{E}\left[\exp\left[\sigma W_t\right]\right] \\
&= S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t\right] \int_{\mathbb{R}} \exp\left[\sigma x \sqrt{t}\right]\phi(x)\, dx \\
&= S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \frac{\sigma^2 t}{2}\right] \underbrace{\int_{\mathbb{R}} \phi\left(x - \sigma\sqrt{t}\right)\, dx}_{=1} \\
&= S_0 \exp[\mu t] \quad , \qquad (3.2.6)
\end{aligned}$$

where $\phi(x)$ is the probability density function of a standard normal distribution. For the variance, the second moment has to be calculated:

$$\begin{aligned}
\mathbb{E}\left[S_t^2\right] &= \mathbb{E}\left[S_0^2 \exp\left[\left(2\mu - \sigma^2\right)t + 2\sigma W_t\right]\right] = S_0^2 \exp\left[\left(2\mu - \sigma^2\right)t + 2\sigma^2 t\right] \int_{\mathbb{R}} \phi(x - 2\sigma\sqrt{t})\, dx \\
&= S_0^2 \exp\left[\left(2\mu + \sigma^2\right)t\right] \quad .
\end{aligned}$$

Thus, the variance of the stock price is

$$\mathbb{V}[S_t] = \mathbb{E}[S_t^2] - \mathbb{E}[S_t]^2 = S_0^2 \exp[2\mu]\left(\exp[\sigma^2 t] - 1\right) \quad . \qquad (3.2.7)$$

Lemma *$M_t = \exp\left[\sigma W_t - \frac{1}{2}\sigma^2 t\right]$, $t \geq 0$ is a $(\mathbb{F}, \mathbb{P})$ martingale.*

As a consequence, $S_t = \exp[\mu t] M_t$, $t \geq 0$, is generally not a $(\mathbb{F}, \mathbb{P})$-martingale, because for $t > s$,

$$\mathbb{E}[S_t \mid \mathfrak{F}_s] = \exp[\mu t]\; M_s = \underbrace{\exp[\mu(t-s)]}_{\neq 1 \text{ for } \mu \neq 0}\; S_s \quad \text{a.s.}$$

Finally, it can easily be observed that $\{S_t\}_{t\geq 0}$ is a $(\mathbb{F}, \mathbb{P})$-Markov process that is homogeneous regarding time.

3.2.2 Martingale Measures

The term martingale has previously been mentioned without being explained properly. However, the fact that a Brownian motion and, in certain circumstances, a geometric Brownian motion is a martingale is essential for the valuation of option prices. Øksendal [2007] is going to be taken as reference for the following paragraphs.

Definition *An n-dimensional stochastic process* $\{M_t\}_{t\geq 0}$ *on* $(\Omega, \mathfrak{F}, \mathbb{P})$ *is called a* *martingale* *with respect to a filtration* $\{\mathfrak{F}_t\}_{t\geq 0}$ *and* $\mathbb{P}$ *if*

a) M_t *is* $\mathfrak{F}_t$*-measurable for all* t*,*

b) $\mathbb{E}[|M_t|] < \infty$ *for all* t *and*

c) $\mathbb{E}[M_t \mid \mathfrak{F}_s] = M_s$ *for all* $t \geq s$ *.*

As mentioned before, a geometric Brownian motion is a martingale if the drift term $\mu = 0$. However, this is generally not the case in real world. Therefore it is always important to find an equivalent martingale measure. Brigo and Mercurio [2001] defines it as follows

Definition *An* *equivalent martingale measure* $\mathbb{Q}$ *is a probability measure on the space* $(\Omega, \mathfrak{F})$ *such that*

a) $\mathbb{P}$ *and* $\mathbb{Q}$ *are equivalent measures, that is* $\mathbb{P}(A) = 0 \Leftrightarrow \mathbb{Q}(A) = 0$*, for every* $A \in \mathfrak{F}$*;*

b) the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^2(\Omega, \mathfrak{F}, \mathbb{P})$*;*

c) the discounted asset price process $\frac{S_t}{B_t}$ *is an* $(\mathbb{F}, \mathbb{Q})$*-martingale, that is* $\mathbb{E}_{\mathbb{Q}}\left[\frac{S_t}{B_t} \middle| \mathfrak{F}_u\right] = \exp[-ru] S_u$*,* $0 \leq u \leq t \leq T$*.*

$$\tilde{S}_t = \frac{S_t}{B_t} = \exp[-rt] S_t = \overbrace{\tilde{S}_0}^{=S_0} \exp\left[\left(\mu - r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right] \tag{3.2.8}$$

is the discounted stock price, $t, r \geq 0$, and is the unique strong solution of the stochastic differential equation

$$d\tilde{S}_t = (\mu - r)\tilde{S}_t \, dt + \sigma \tilde{S}_t \, dW_t \ , \quad t, r \geq 0, \ \tilde{S}_0 = S_0 \quad .$$

Then, $\tilde{S}$ is a $\mathbb{P}$-martingale if $\mu = r$. The next step is to find an equivalent probability measure to $\mathbb{P}$ such that $\tilde{S}$ becomes a martingale for $\mu \neq r$.

The next theorem is probably the most fundamental one in the general theory of stochastic analysis and very important in many applications. It basically says that if the drift coefficient of a given Itô process is changed, then the law of the new process will be absolutely continuous with respect to the law of the original one and an explicit Radon-Nikodym derivative is given.

Theorem *(Girsanov's theorem)*
Let $\{Y_t\}_{t\geq 0} \in \mathbb{R}^n$, $n \in \mathbb{N}$ *be an Itô process of the form*

$$dY_t = a(t,\omega)\, dt + dW_t \,; \quad t \leq T,\ Y_0 = 0 \,,$$

where $T \leq \infty$ *is a given constant and* W_t *a n-dimensional Brownian motion. Put*

$$M_t = \exp\left[-\int_0^t a(s,\omega)\, dW_s - \frac{1}{2}\int_0^t a^2(s,\omega)\, ds\right] \,; \quad 0 \leq t \leq T \,. \tag{3.2.9}$$

Assume that M_t *is a martingale with respect to* $\mathfrak{F}_T$ *and* $\mathbb{P}$*. Define the measure* $\mathbb{Q}$ *on* $\mathfrak{F}_T$ *by*

$$d\mathbb{Q}(\omega) = M_T(\omega)\, d\mathbb{P}(\omega) \quad . \tag{3.2.10}$$

Then, $\mathbb{Q}$ *is a probability measure on* $\mathfrak{F}_T$*, and* Y_t *an n-dimensional Brownian motion with respect to* $\mathbb{Q}$*, for* $0 \leq t \leq T$*.*

Remark *The transformation* $\mathbb{P} \to \mathbb{Q}$ *given by (3.2.10) is called the Girsanov transformation of measures.*
The following condition is called Novikov condition and is sufficient to guarantee that $\{M_t\}_{t\leq T}$ *is a martingale with respect to* $\mathfrak{F}_T$ *and* $\mathbb{P}$*:*

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left[\frac{1}{2}\int_0^T a^2(s,\omega)\, ds\right]\right] < \infty \quad . \tag{3.2.11}$$

Girsanov also introduced a second theorem that works as a follow up of the one mentioned above. For an detailed explanation of both of them and proofs, the reader is encouraged to take a look into Øksendal [2007].

Girsanov's theorems motivates the following lemma that puts all the gained information into an applicable form. Let $F := \{\mathfrak{F}_t\}_{t\in[0,T^*]}$ be a complete, continuous from the right filtration that is generated by the one dimensional Brownian motion $\{W_t\}_{t\in[0,T^*]}$, $T^* < \infty$.

Lemma *Under the premise of Girsnaov's theorem, a unique martingale measure for* $\tilde{S}$ *exists with Radon-Nikodyn derivative*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left[\frac{r-\mu}{\sigma} W_{T^*} - \frac{1}{2}\frac{(r-\mu)^2}{\sigma^2} T^*\right] \quad .$$

Further, $d\tilde{S}_t = \sigma \tilde{S}_t\, dW_t^*$ *with* $W_t^* := W_t - \frac{r-\mu}{\sigma} t$*,* $t \in [0, T^*]$ *being a Brownian motion with respect to* $\mathcal{F} := \{\mathfrak{F}_t\}_{t\in[0,T^*]}$ *and* $\mathbb{Q}$*.*

Generally, the bank account $\{B_t\}_{t\geq 0}$ is taken as numeraire. However, more convenient numeraires can be introduced in order to calculate claims of financial derivatives. Brigo and Mercurio [2001] list the following fundamental tool for pricing financial instruments.

Proposition *Assume there exists a numeraire N and a probability measure $\mathbb{Q}^N$, equivalent to the initial $\mathbb{P}$, such that the price of any traded asset X relative to N is a martingale under $\mathbb{Q}^N$, i.e.,*

$$\frac{X_t}{N_t} = \mathbb{E}_{\mathbb{Q}^N}\left[\frac{X_T}{N_T}|\mathfrak{F}_t\right] \quad 0 \leq t \leq T\,. \tag{3.2.12}$$

Let U be an arbitrary numeraire. Then there exists a probability measure $\mathbb{Q}^U$, equivalent to the initial $\mathbb{P}$, such that the price of any attainable claim Y normalised by U is a martingale under $\mathbb{P}$, i.e.,

$$\frac{Y_t}{U_t} = \mathbb{E}_{\mathbb{Q}^U}\left[\frac{Y_T}{U_T}|\mathfrak{F}_t\right] \quad 0 \leq t \leq T\,. \tag{3.2.13}$$

Moreover, the Radon-Nikodym derivative defining the measure $\mathbb{Q}^U$ is given by

$$\frac{d\mathbb{Q}^U}{d\mathbb{Q}^N} = \frac{U_T N_0}{U_0 N_T} \quad . \tag{3.2.14}$$

So, for any asset price Z,

$$\mathbb{E}_{\mathbb{Q}^N}\left[\frac{Z_T}{N_T}\right] = \mathbb{E}_{\mathbb{Q}^U}\left[\frac{Z_T}{N_T}\frac{d\mathbb{Q}^N}{d\mathbb{Q}^U}\right] \quad . \tag{3.2.15}$$

3.2.3 Black-Scholes Formula

The purpose of the Black-Scholes model is to evaluate option prices based on a market with certain assumption mentioned in section 3.2.1.

Consider an European call option that gives the owner the right, not the obligation, to buy the underlying asset at a fixed price $K \geq 0$ (=strike) at maturity $T \leq T^* < \infty$. It is assumed that the underlying stock described by (3.2.3) pays a dividend rate $q \geq 0$, hence follow the price process

$$S_t = S_0 \exp\left[\left(\mu - q - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right]$$

Then, the arbitrage free price process of the European call option in the Black-Scholes Model $\forall t \in [0,T]$ is

$$
\begin{aligned}
c_t &= c_t(S_t, K, T-t, \sigma, r, q) = \mathrm{BS}(S_t, K, T-t, \sigma, r, q, 1) \\
&= S_t \exp[-q(T-t)]\, \Phi\big(d_1(S_t, T-t)\big) - K \underbrace{\exp[-r(T-t)]}_{=\frac{B_t}{B_T}}\, \Phi\big(d_2(S_t, T-t)\big) \quad \text{a.s.}\,,
\end{aligned}
\tag{3.2.16}
$$

with

$$
d_{1,2}(s,u) = \frac{\log\left[\frac{S}{K}\right] + \left(r - q \pm \frac{1}{2}\sigma^2\right) u}{\sigma\sqrt{u}}
\tag{3.2.17}
$$

and the distribution function of the normal distribution

$$
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left[-\frac{1}{2}y^2\right] dy\,,\ x \in \mathbb{R} \quad .
\tag{3.2.18}
$$

As usual, it is rather referred to the literature if anyone is interested in the proof of the formula, Hull [2009], for example, has listed one that uses the risk-neutral pricing method, though it can also be shown using stochastic differential equations.

In order to obtain the formula for an European put option, either the same derivation as for the call can be used, or the call-put-parity can be applied. It says that in a non-arbitrary world,

$$
c_t - p_t = S_t - K\frac{B_t}{B_T} \quad \text{a.s.}\,,\ t \in [0,T]\,,
\tag{3.2.19}
$$

where c_t and p_t are the corresponding prices for call and put with similar strike and maturity. Thus, (3.2.16) and (3.2.19) result in the arbitrage free price process of an European put option with strike $K \geq 0$ and maturity $T \leq T^* < \infty$ in the Black-Scholes Model: $\forall t \in [0,T]$,

$$
\begin{aligned}
p_t &= p_t(S_t, K, T-t, \sigma, r, q) = \mathrm{BS}(S_t, K, T-t, \sigma, r, q, -1) \\
&= S_t\ \exp[-q(T-t)]\, \Phi\big(d_1(S_t, T-t)\big) - K \exp[-r(T-t)]\, \Phi\big(d_2(S_t, T-t)\big) \\
&\quad - \big(S_t - K\exp[-r(T-t)]\big) \\
&= K\exp[-r(T-t)]\Phi\big(-d_2(S_t, T-t)\big) - S_t \exp[-q(T-t)]\Phi\big(-d_1(S_t, T-t)\big) \quad \text{a.s.} \quad .
\end{aligned}
\tag{3.2.20}
$$

The Black-Scholes model can be applied to many different derivatives, resulting in similar formulas than those mentioned above. If the underlying instrument can be described as a geometric Brownian motion, the model is a very powerful and solid tool to draw price processes for financial derivatives. Therefore, the Black-Scholes model is going to be used throughout the whole thesis in order to simulate exposure profiles over the course of time.

4 Lomibao and Zhu Model

The goal of this thesis is to quantify the counterparty credit risk of specific, path depending OTC derivatives. To do so, the risk factor evolution is described by a geometric Brownian motion, which provides the advantage of using results from the Black-Scholes framework (see figure 4.1). The basic motivation comes from Lomibao and Zhu [2005] introducing a model and algorithm based on the conditional valuation of MtF values. Therefore, this section is closely tied to this paper, where most of the following ideas originate. However, calculations are going to be done in more detail and results compared to other approaches later on.

The finesse of the approach proposed by Lomibao and Zhu [2005] is to simulate a single underlying S_{t_k} at time t_k and use it to calculate the mark-to-market value of a path-dependent derivative, although none of the risk factor's path is known. For example, an up-and-out barrier option gives the owner the right to buy a stock for a pre-defined price at exercise date T unless the stock price crosses an upper, fixed barrier on its way. If only knowing the stock price S_{t_k} at t_k, it is uncertain whether the underlying stock price has previously exceeded this barrier or not. In order to overcome this issue, the Brownian bridge introduced in section 3.1.1 is used to derive the mark-to-future values of path-dependent financial derivatives. The parameters of the stochastic process are calibrated for each derivative by using historical data. However, some instruments require the calculation of implied volatilities that are taken from at-the-money options, representing the market's perception on the evolution.

Before potential future exposures can be calculated, closed formulas of the future values have to be derived analytically. Thereby, the value-at-future is described by expected values conditioned on the simulated future scenario. Thereby, the properties of the Brownian bridge are used when calculating the expected values of the pay-off functions conditioned on S_{t_k}, which occupies the biggest part of the analytical calculation.

4.1 Scenario Generation

Having finished the analytical part, the simulation of the MtF values for each of the thousands of scenarios has to be done via R, a language and environment for statistical computing and graphics. The basic problem of evaluating path-dependent instruments in this framework is that future scenarios con only be simulated at discrete set of dates $\{t_1, t_2, \dots, t_N\}$, while the value of the actual instrument may depend on the full continuous path prior to the simulation date. Thereby, an equidistant time grid , with step-size $\Delta t = t_{l+1} - t_l$, $l \in \{0, 1, \ldots, N-1\}$, is assumed for all simulations.

In order to generate market scenarios like FX rates, stock prices and interest rates at certain future points in time, a reliable framework has to be found that is build on a real measure based on historical data. Therefore, the use of a log-normal model is proposed that can easily be embedded into the Black-Scholes framework. The underlying prices are described by a geometric Brownian motion:

$$S_t = S_0 \exp\left[(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right] \quad , \tag{4.1.1}$$

where S_0 is the initial price at t_0, μ the appreciation rate, σ its volatility and W_t a standard Brownian motion. In case of option pricing, the volatility $\sigma = \sigma_{iv}$, which is the implied volatility driven by the market perception of the risk of the instrument. Under the real measure, the two parameters are estimated from historical data as follows:

$$\sigma_h = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\log\left[\frac{S_t}{S_{t-1}}\right] - \mu_h\right)^2} \quad , \quad \mu_h = \frac{1}{T}\sum_{t=1}^{T}\log\left[\frac{S_t}{S_{t-1}}\right] . \tag{4.1.2}$$

In order to derive the parameters of (4.1.1), the drift has to be adjusted to $\mu = \mu_h + \frac{1}{2}\sigma^2$.

The next subsection deals with the different approaches to simulate possible future values. The first one is to simulate the whole path from t_0, describing the whole trajectory. The other method is to directly simulate to the time t_k. The first approach is called Path-Depending Simulation (PDS), the second one Direct Jump to Simulation Date (DJS).

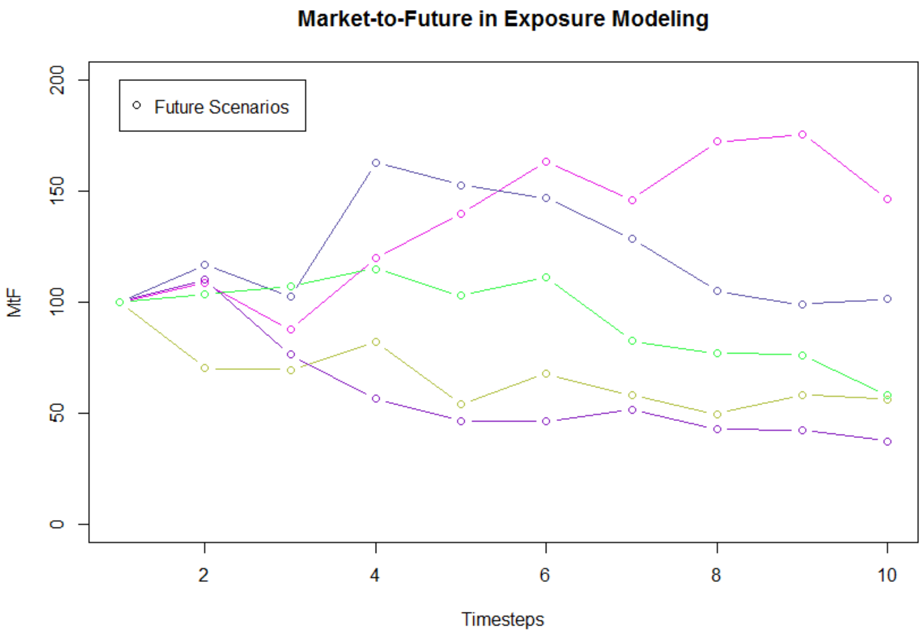

Figure 4.1: Illustration of generated GBMs for different future scenarios

4.1.1 Path-Depending Simulation

Path-depending simulation (PDS) captures the whole evolution through discrete time intervals, which means that each value depends on the previous values simulated. It is a step by step generation of the risk factor over time and each simulation is expressed in term of the previous simulated values and the differences in times.

Taking a look at the log-normal evolution, this means that a risk factor $X(t_{i+1})$ at time t_{i+1} is written as

$$\begin{aligned} X(t_{i+1}) &= X(t_i)\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}\,W_1\right] \\ &= X(t_i)\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)(t_{i+1} - t_i) + \sigma\,W_{t_{i+1}-t_i}\right], \end{aligned} \tag{4.1.3}$$

where W_t is a standard Brownian motion.

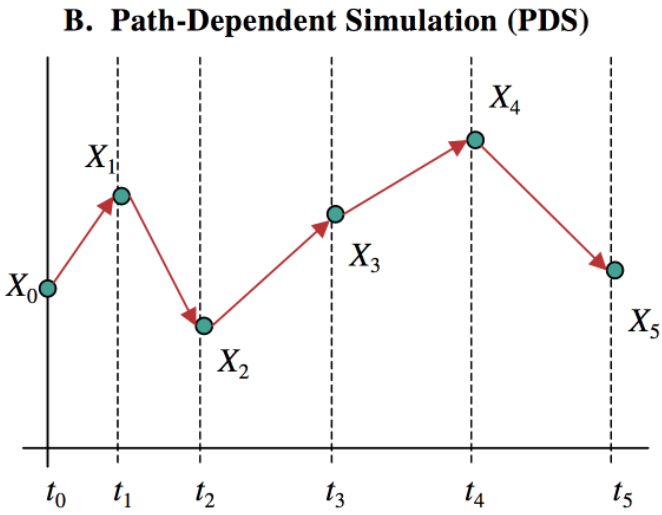

Figure 4.2: Path-Dependent Simulation (PDS)

4.1.2 Direct-Jump to Simulation Date

In contrast to PDS, the direct-jump to simulation date method (DJS) describes an evolution process that merely depends on the initial value, $X(t_0)$, and its distance to the future date. In case of the lognormal evolution function, the risk factor process is

$$\begin{aligned} X(t) &= X(0)\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}\,W_1\right] \\ &= X(0)\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right], \end{aligned} \tag{4.1.4}$$

where, again, W_t is the standard Brownian motion at t.

In order to illustrate the two different methods, it is referred to figure 4.2 and 4.3 .

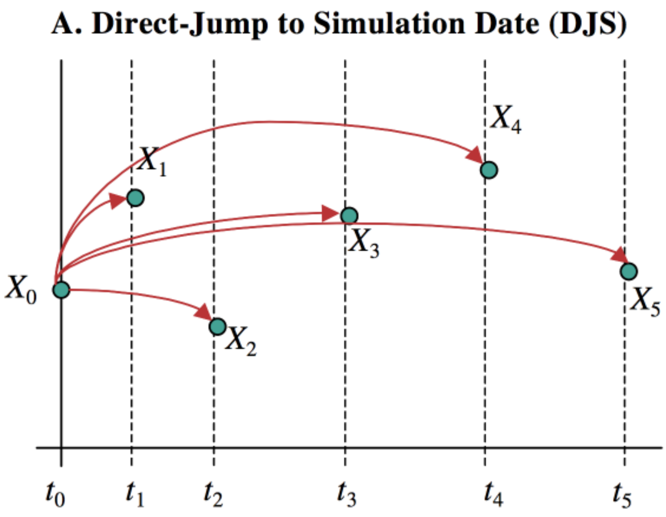

Figure 4.3: Direct-Jump to Simulation Date (DJS)

The concept of conditional valuation is all about the distribution of the market factor scenarios rather than the trajectory.

4.1.3 Conditional Valuation

In order to evaluate future mark-to-market prices of path-dependent derivatives, the conditional valuation is a reliable technique it is applicable to any instrument across the range of derivatives.

The limitation of computational resources merely allows the valuation at a discrete set of times, however, the MtF might depend on the full path over the continuum of dates prior to the simulation date. Therefore, this chapter deals with establishing a general formulation of the conditional valuation approach that can be adapted for all financial instruments. Thereby, either the DJS approach from section 4.1.2 or the PDS approach form section 4.1.1 can be used for generating the market scenarios. First of all, the notation taken from Lomibao and Zhu [2005] is introduced:

$\{t_k = t_1, t_2, \dots, t_N\}$	...	discrete simulation dates
$\{X(t_k) = X(t_1), X(t_2), \dots, X(t_N)\}$	...	market risk factor scenarios
$\{V(t_k) = V(t_1), V(t_2), \dots, V(t_N)\}$	...	mark-to-future value / future value of the transaction

As mentioned before, certain instruments can not be determined strictly by the state of the underlying at simulation the date. Therefore, those path-dependent derivatives have to be evaluated considering not only the scenario at time t_k, but also the path leading to the simulation date. Hence, using the expected value of the pay-off function of the financial derivative, conditioned on the underlying market risk factor at time t_k, the future value of instrument i and its position p_i can generally be formulated as

$$V(j, t_k, x) = \mathbb{E}\left[f\left(t_k, \{X(t)\}_{0 \le t < t_k}\right) | X_j(t_k) = x\right] , \tag{4.1.5}$$

where the expected value is conditioned on the simulated risk factor $X_j(t_k) = x$ at t_k for scenario $j \in \{1, 2, \ldots, M\}$. Furthermore, $M > 0$ is the number of scenarios/simulations, $\{X(t)\}_{0 \le t < t_k}$ is the path of the risk factor evolution and f the pay-off function of the financial derivative. If the instrument was not path-dependent, the expected value at (4.1.5) would degenerate to its MtM value at t_k:

$$V(j, t_k, x) = f\left(t_k, X_j(t_k) = x\right)$$

It is important to be aware of the difference between the future MtM value and the MtF value introduced by Demob et al. [2000], which essentially equals the value-at-future (VaF) notation used by Lomibao and Zhu [2005]. Last one therefore shows an example that illustrated the difference between the two terms, considering two cases where the valuation function is separable:

$$f\left(t_k, \{X(t)\}_{0 \le t < t_k}\right) = g\left(t_k, X(t_k)\right) \cdot h\left(t_k, \{X(t)\}_{0 \le t < t_k}\right) \tag{4.1.6}$$

$$f\left(t_k, \{X(t)\}_{0 \le t < t_k}\right) = g\left(t_k, X(t_k)\right) + h\left(t_k, \{X(t)\}_{0 \le t < t_k}\right) . \tag{4.1.7}$$

Hence, the expected valuation conditioned on the simulated risk factor at t_k is

$$V\left(t_k, x\right) = g\left(t_k, x\right) \cdot \mathbb{E}\left[h\left(t_k, \{X(t)\}_{0 \le t < t_k}\right) | X\left(t_k\right) = x\right] \tag{4.1.8}$$

$$V\left(t_k, x\right) = g\left(t_k, x\right) + \mathbb{E}\left[h\left(t_k, \{X(t)\}_{0 \le t < t_k}\right) | X\left(t_k\right) = x\right] , \tag{4.1.9}$$

where $g(t_k, x)$ determines the MtM value of the transaction and $h\left(t_k, \{X(t)\}_{0 \le t < t_k}\right)$ a payoff function for a path-dependent instrument that strictly depends on the evolution of the risk factor up to t_k. For instance, the up-and-out barrier option is an example for case (**??**) with

$$h\left(t_k, \{X(t)\}_{0 \le t < t_k}\right) = \mathbb{1}_{(X(t) < H \,:\, 0 \le t < t_k)} ,$$

whereas the average option is an example of (**??**) with

$$h\left(t_k, \{X(t)\}_{0\leq t<t_k}\right) = \frac{1}{t_k}\sum_{j=0}^{k} X(t_j)\ .$$

4.2 Barrier Option

Barrier options are options whose payments strictly depend on whether the price of the underlying passes a certain barrier within a given time period. The event when the underlying crosses the barrier level is called barrier event. They are traded OTC and multiple variations exists, mainly divided into knock-out options and knock-in options. The first ones expire if the price of the underlying crosses a certain level, called barrier, whereas second ones become valid after breaching the barrier. They are always cheaper than their vanilla counterpart as there is always a strictly positive probability of the stock price to be above or below the barrier level and thus, making the option extinguish. For example, if an investor believes that a certain stock will go up within a certain period of time, but will not reach a certain height H, then he will buy a knock-out barrier option with upper level H and pay less premium than for the vanilla call option. As the pay-off at maturity completely depending whether the stock price reaches a certain level, barrier options are perfect examples for path-dependent derivatives. Figure 4.4 shows two trajectories of stock prices modeled by geometric Brownian motions. The first one, the solid black line, breaches the barrier, which is represented by the dotted green line, on its way to maturity 1, but still remains above the strike (solid blue line). However, the owner would not be allowed to exercise the option, because the upper barrier was crossed at least once. On the other hand, the second path, which is illustrated by the black dashed line, never hits the barrier, but still remains above the strike. Therefore, the option would remain intact and the owner would exercise the call.

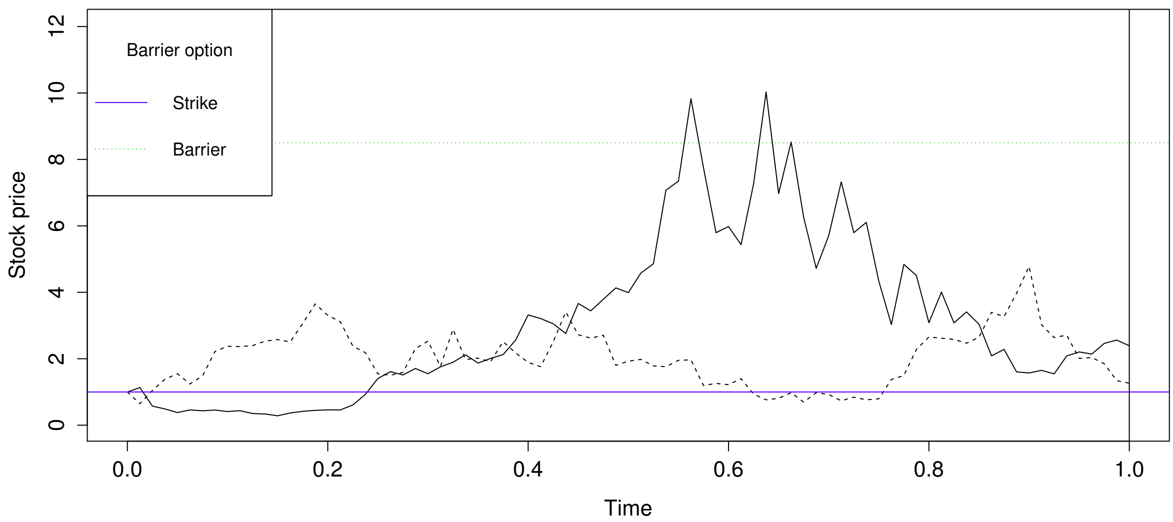

Figure 4.4: Illustration of Up-and-Out Call Option

4.2.1 Pricing of Barrier Options

As shown in chapter 3.2, the value of a plain vanilla call or put at time t is

$$c(t) = S_t \exp\big[-q(T-t)\big]\Phi(d_1) - K\exp\big[-r(T-t)\big]\Phi(d_2) \tag{4.2.1}$$

$$p(t) = K\exp\big[-r(T-t)\big]\Phi(-d_2) - S_t \exp\big[-q(T-t)\big]\Phi(-d_1)\ , \tag{4.2.2}$$

with

$$d_1 = \frac{\log\frac{S_t}{K} + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = \frac{\log\frac{S_t}{K} + (r - q - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}\ ,$$

where S_t is the price of the underlying stock at time t, K the strike of the option at maturity T, r the risk-free interest rate p.a., q the dividend rate of the stock p.a., σ the volatility of the stock and Φ the cumulative distribution function of the normal distribution (see (3.2.18)).
An example of a knock-out option is the down-and-out call option. It is basically a regular call option with the condition that the stock price must not reach a certain barrier H - otherwise the option vanishes. Obviously, the barrier H has to be below the level of the current stock price. The corresponding knock-in option is the down-and-in call option, which will spring

into existence if the stock price passes barrier H. According to Hull [2009], the value of a down-and-in call for $H > K$ at time t is

$$c_{di}(t) = S_t \exp\big[-q(T-t)\big]\left(\frac{H}{S_t}\right)^{2\lambda}\Phi(y)$$
$$-K\exp\big[-r(T-t)\big]\left(\frac{H}{S_t}\right)^{2\lambda-2}\Phi(y-\sigma\sqrt{T-t}), \qquad (4.2.3)$$

where

$$\lambda = \frac{r-q+\frac{\sigma^2}{2}}{\sigma^2}$$
$$y = \frac{\log\left[\frac{H^2}{S_t K}\right]}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t},$$

and zero if $H \leq K$.

Looking at a vanilla call option, one can easily observe that its value has to equal the value of a down-and-out and a down-and-in call since the payoffs are the same. Hence, for $H \geq K$

$$\begin{aligned} c_{do}(t) &= c(t) - c_{di}(t) \\ &= S_t\Phi(x_1)\exp\big[-q(T-t)\big] \\ &\quad - K\exp\big[-r(t-t)\big]\Phi(x_1-\sigma\sqrt{T-t}) \\ &\quad - S_t\exp\big[-q(T-t)\big]\left(\frac{H}{S_t}\right)^{2\lambda}\Phi(y_1) \\ &\quad + K\exp\big[-r(T-t)\big]\left(\frac{H}{S_t}\right)^{2\lambda-2}\Phi(y_1-\sigma\sqrt{(T-t)}) \quad , \end{aligned} \qquad (4.2.4)$$

where

$$x_1 = \frac{\log\left[\frac{S_t}{H}\right]}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t}$$
$$y_1 = \frac{\log\left[\frac{H}{S_t}\right]}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t} \quad .$$

An up-and-out call option is a plain vanilla call option that expires if the stock price hits a barrier above the current level. Its counterpart consequently is the up-and-in call option, analogue to the down-and-in and down-and-out calls above.

For $H \leq K$, the value of an up-and-out call $c_{uo}(t) = 0$ and that of an up-and-in call $c_{ui}(t) = c(t)$. If $H > k$,

$$\begin{aligned} c_{ui}(t) &= S_t\Phi(x_1)\exp\big[-q(T-t)\big] \\ &\quad - K\exp\big[-r(t-t)\big]\Phi(x_1-\sigma\sqrt{T-t}) \end{aligned}$$

$$
\begin{aligned}
&- S_t \exp\big[-q(T-t)\big]\left(\frac{H}{S_t}\right)^{2\lambda}\big[\Phi(-y)-\Phi(-y_1)\big] \\
&+ K \exp\big[-r(T-t)\big]\left(\frac{H}{S_t}\right)^{2\lambda-2} \\
&\times\big[\Phi(-y+\sigma\sqrt{T-t})-\Phi(-y_1+\sigma\sqrt{T-t})\big] \quad ,
\end{aligned} \tag{4.2.5}
$$

and

$$
c_{uo} = c - c_{ui} \quad . \tag{4.2.6}
$$

For information on barrier put options as well as all the derivations of the valuation formulas above, it is referred to Hull [2009]. For the purpose of this thesis, call options are chosen in order to calculate their future exposures of which the calculations are similar to those of put options.

Obviously, the evolution of the underlying is very important in order to evaluate the MtF values of barrier options, because it is uncertain whether the option is still valid at any given time t_k. Anyway, let $\big(\Omega, \mathfrak{F}, \{\mathfrak{F}\}_{t\geq 0}, \mathbb{P}\big)$ be a filtered measure space and $\mathbb{P}^*$ an equivalent martingale measure to $\mathbb{P}$ for the discounted evolution process $\tilde{S}_t = S_t \exp[-rt]$. Then, the MtM values of a up-and-out call and a down-and-out put under the risk neutral measure $\mathbb{P}^*$ are given by

$$
\mathrm{MtM}_{uo}(t) = B_t\, \mathbb{E}_{\mathbb{P}^*}\left[\frac{1}{B_T}\max\{0, S_T - K\}\, \mathbb{1}_{(M^S_{t,T}<H)}\right] = c_{uo}(t) \tag{4.2.7}
$$

$$
\mathrm{MtM}_{do}(t) = B_t\, \mathbb{E}_{\mathbb{P}^*}\left[\frac{1}{B_T}\max\{0, K - S_T\}\, \mathbb{1}_{(m^S_{t_k,T}>L)}\right] = c_{do}(t) \quad , \tag{4.2.8}
$$

where $M^S_{t,T} = \max\{S_\tau | t < \tau \leq T\}$, $m^S_{t_k,T} = \min\{S_\tau | t < \tau \leq T\}$, H is the up barrier, L the down barrier and B_t the value of the bond under the assumptions of the Black-Scholes framework at time $t \geq 0$ (see section 3.2.1). In contrast to the calculations of Lomibao and Zhu [2005], the discount factor is also considered here in order to go along with the Black-Scholes risk-neutral valuation. This also leads to the first flaw of the paper this thesis as it does not clearly state how to handle the risk-free interest rate. Unfortunately, Lomibao and Zhu [2005] do not write their mark-to-market valuation formula of the corresponding barrier option down, which should include the risk-free rate r if the Black-Scholes framework was used (see beginning of this section). Hence, it is assumed that they probably used $r = 0.$. Furthermore, they do not clarify under which measure the valuation is performed - the calculation above use the risk-neutral measure as intended for the Black-Scholes framework.

Basically, formulas (4.2.7) and (4.2.8) describe the valuation formulas of plain vanilla call options conditioned on the barrier event not occurring, which equals the condition that the maximum of the stock has to be below H or the minimum above L. The evolution of the stock price is assumed to follow a GBM, which under the actual measure $\mathbb{P}$ looks like

$$S_t = S_0 \exp\left[(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right] \quad , \tag{4.2.9}$$

whose drift can be estimated by using historical data. Under the risk neutral measure $\mathbb{P}^*$

$$S_t = S_0 \exp\left[-\frac{1}{2}\sigma^2 t + \sigma W^*(t)\right] \quad , \tag{4.2.10}$$

where W_t^* is a Brownian motion under $\mathbb{P}^*$. For more information about the change of measure and the application of the Girsanov theorem, it is referred to section 3.2.2.
For the purpose of illustrating the different possible paths the risk factor may pursue for a given scenario $S_{t_k} = x$, Lomibao and Zhu [2005] describe the following example:

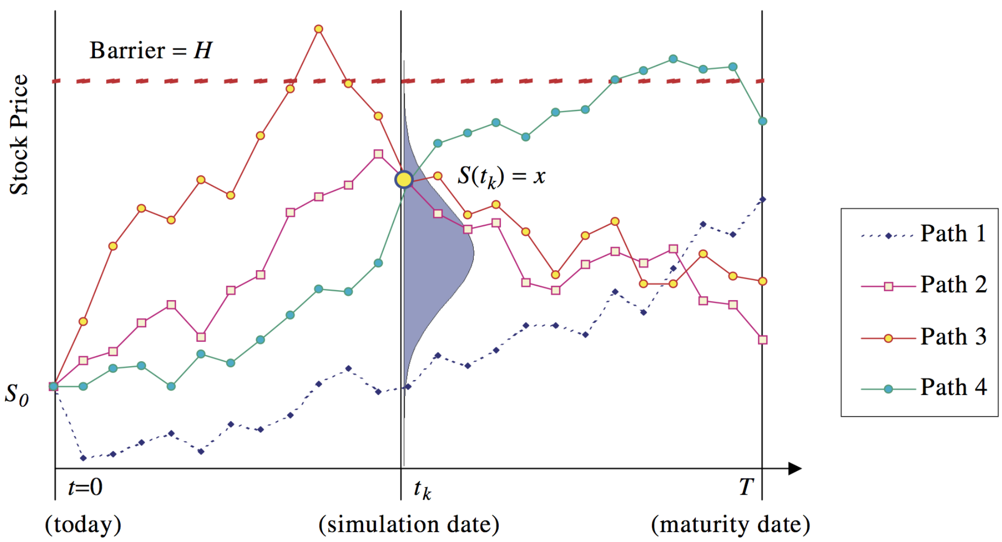

Figure 4.5: Example of different scenario paths (Source: Lomibao and Zhu [2005])

Figure 4.5 shows four different scenario paths for a given scenario $S_{t_k} = x$ at simulation date t_k. The first one shows the evolution of the stock price that did not hit S_{t_k} at all. The second trajectory does hit $S_{t_k} = x$ at simulation date and stays below H for the whole tenor. In case of the third path, the barrier is reached before the simulation date, and in the last one, H is crossed before maturity but after the simulation date t_k. The third path is an interesting one as it will affect the calculation of exposure at t_k, because it hits the barrier before simulation date, which determines the existence or extinction of the underlying option.

For further calculations, the focus is going to be on down-and-out call and the up-and-out call as representatives of all the other barrier option variations. Applying the conditional expectation on formula (4.2.7) and (4.2.8) and using the independence of $\mathbb{1}_{\{M^S_{0,t_k}<H\}}$ and $\mathbb{1}_{\{M^S_{t_k,T}<H\}}$, as well as $W_{t_k} \perp\!\!\!\perp W_T - W_{t_k}$, the values-at-future are

$$\begin{aligned}
\mathrm{VaF}_{uo}(t_k,x) &= \mathbb{E}_{\mathbb{P}^*}\left[\frac{B_{t_k}}{B_T}\max\{0,S_T-K\}\,\mathbb{1}_{\{M^S_{t_k,T}<H\}}\,\mathbb{1}_{\{M^S_{0,t_k}<H\}}\mid S_{t_k}=x\right] \\
&= \mathbb{E}_{\mathbb{P}^*}\left[\frac{B_{t_k}}{B_T}\max\{0,S_T-K\}\,\mathbb{1}_{\{M^S_{t_k,T}<H\}}\mid S_{t_k}=x\right]\mathbb{E}_{\mathbb{P}^*}\left[\mathbb{1}_{\{M^S_{0,t_k}<H\}}\mid S_{t_k}=x\right] \\
&= \mathrm{MtM}_{uo}(t_k;x)\times\mathbb{P}^*\left[M^S_{0,t_k}<H\mid S_{t_k}=x\right] \qquad (4.2.11)
\end{aligned}$$

$$\begin{aligned}
\mathrm{VaF}_{do}(t_k,x) &= \mathbb{E}_{\mathbb{P}^*}\left[\frac{B_{t_k}}{B_T}\max\{0,S_T-K\}\,\mathbb{1}_{\{m^S_{0,t_k}>L\}}\,\mathbb{1}_{\{m^S_{0,t_k}>L\}}\mid S_{t_k}=x\right] \\
&= \mathbb{E}_{\mathbb{P}^*}\left[\frac{B_{t_k}}{B_T}\max\{0,S_T-K\}\,\mathbb{1}_{\{m^S_{0,t_k}>L\}}\mid S_{t_k}=x\right]\mathbb{E}_{\mathbb{P}^*}\left[\mathbb{1}_{\{m^S_{0,t_k}>L\}}\mid S_{t_k}=x\right] \\
&= \mathrm{MtM}_{do}(t_k;x)\times\mathbb{P}^*\left[m^S_{0,t_k}>L\mid S_{t_k}=x\right] \qquad (4.2.12)
\end{aligned}$$

So, the value-at-future of both the down-and-out and up-and-out call can be separated into its mark-to-market value at t_k, which can easily be calculated by using formula (4.2.6) and (4.2.4) and the probability of the stock price to stay within the according borders. Note that Lomibao and Zhu [2005] wrongly neglected the condition $S_{t_k} = x$ of the first expected value in each of the formulas above. However, they end up having the correct results at the end.

In order to derive the probability to stay either below or upon the barrier, the geometric Brownian motion

$$S_t = S_0 \exp\left[(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right]$$

has to be rewritten to a drifted Brownian motion

$$X_t := \log\left[\frac{S_t}{S_0}\right] = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t \quad ,$$

with drift term $c := \left(\mu - \frac{1}{2}\sigma^2\right)$. In order to simplify the following calculations and ensure the comparability with the results of Lomibao and Zhu [2005], $r = 0$ is assumed. The reason for the transformation above is to use the results for the distribution of the maximum of a Brownian motion with non-zero drift discussed in section 3.1.2. Furthermore, it is referred to Karatzas and Shreve [1988], Shepp [1979], Buffet [2003] and Musiela and Rutkowski [2005], who provide a deep insight into the subject of maxima and mimima of Brownian motions.

In case of the up-and-out call option, the barrier H and the simulated scenario $S_{t_k} = x$ have to be transformed to $H' = \log\left[\frac{H}{S_0}\right]$ and $x' = \log\left[\frac{x}{S_0}\right]$ respectively. For $x > H$, the barrier option is obviously out-of-the money. Therefore, $\log\left[\frac{x}{H}\right] < 0$ is assumed and the probability from (4.2.11) is

$$\mathbb{P}^*\left[M^S_{0,t_k} < H \mid S_{t_k} = x\right] = \begin{cases} 1 - \mathbb{P}^*\left[M^S_{0,t_k} \geq H \mid S_{t_k} = x\right] & \text{if } x < H \\ 0 & \text{otherwise} \end{cases} \tag{4.2.13}$$

Transforming the process into a Brownian motion with drift, (4.2.13) turns into

$$\mathbb{P}^*\left[M^S_{0,t_k} < H \mid S_{t_k} = x\right] = \begin{cases} 1 - \mathbb{P}^*\left[M^X_{0,t_k} \geq H' \mid X_{t_k} = x'\right] & \text{if } x' < H' \\ 0 & \text{otherwise} \end{cases} \tag{4.2.14}$$

For $x' < H'$, the theory of section 3.1.2 can be applied, however, the density of (3.1.11) with respect to x is needed. Thus, partially differentiating with respect to x and using the fundamental theorem of calculus leads to

$$\begin{aligned}\mathbb{P}^*[M^X_{0,t_k} \geq H', X_{t_k} = x] = \frac{\partial}{\partial x}\left[\mathbb{P}^*[M^X_{0,t_k} \geq H', X_{t_k} \leq x]\right] &= \frac{\partial}{\partial x}\left[\exp\left[\frac{2vy}{\sigma^2}\right]\Phi\left(\frac{x - 2y - vt_k}{\sigma\sqrt{t_k}}\right)\right] \\ &= \exp\left[\frac{2vy}{\sigma^2}\right]\phi\left(\frac{x - 2y - vt_k}{\sigma\sqrt{t_k}}\right)\end{aligned}$$

This result is eventually used to calculate (4.2.14) for $x' < H'$:

$$\begin{aligned}\mathbb{P}^*\left[M^S_{0,t_k} < H \mid S_{t_k} = x\right] &= 1 - \frac{\mathbb{P}^*\left[M^X_{0,t_k} \geq H', X_{t_k} = x'\right]}{\mathbb{P}^*\left[X_{t_k} = x'\right]} \\ &= 1 - \frac{\exp\left[\frac{2(\mu - \frac{1}{2}\sigma^2)H'}{\sigma^2}\right]\phi\left(\frac{x' - 2H' - (\mu - \frac{1}{2}\sigma^2)t_k}{\sigma\sqrt{t_k}}\right)}{\phi\left(\frac{x' - (\mu - \frac{1}{2}\sigma^2)t_k}{\sigma\sqrt{t_k}}\right)} \\ &= 1 - \exp\left[\frac{2(\mu - \frac{1}{2}\sigma^2)H'}{\sigma^2} - \frac{-4H'\left(x' - (\mu - \frac{1}{2}\sigma^2)t_k\right) + 4H'^2)}{2\sigma^2 t_k}\right] \\ &= 1 - \exp\left[\frac{2H'(x' - H')}{\sigma^2 t_k}\right]\end{aligned} \tag{4.2.15}$$

So, using formula (4.2.15), the valuation formula (4.2.11) of the value-at-future of the up-and-out barrier call option

$$\mathrm{VaF}_{uo}(t_k, x) = \begin{cases} \mathrm{MtM}_{uo}(t_k; x) \times \left(1 - \exp\left[\frac{2H'(x'-H')}{\sigma^2 t}\right]\right) & \text{if } x < H \\ 0 & \text{otherwise} \end{cases} \tag{4.2.16}$$

for $H' = \log\left[\frac{H}{S_0}\right]$ and $x' = \log\left[\frac{x}{S_0}\right]$.

For the down-and-out barrier option whose value-at-future is (4.2.12), the probability for the stock to stay above the barrier $L < (x \wedge S_0)$ can be derived in the same way as for up-and-out barrier options in (4.2.15). With the help of (3.1.12) and the symmetry of the density of the normal distribution about its mean,

$$\begin{aligned}\mathbb{P}^*\left[m^S_{0,t_k} > L \mid S_{t_k} = x\right] = \mathbb{P}^*\left[m^X_{0,t_k} > L' \mid X_{t_k} = x'\right] &= \frac{\mathbb{P}^*\left[m^X_{0,t_k} \geq L',\ X_{t_k} = x'\right]}{\mathbb{P}[X_{t_k} = x']} \\ &= \frac{\phi\left(\frac{-x'+(\mu-\frac{1}{2}\sigma^2)t_k}{\sigma\sqrt{t_k}}\right) - \exp\left[\frac{2(\mu-\frac{1}{2}\sigma^2)L'}{\sigma^2}\right]\phi\left(\frac{2L'-x'+(\mu-\frac{1}{2}\sigma^2)t_k}{\sigma\sqrt{t_k}}\right)}{\phi\left(\frac{x'-(\mu-\frac{1}{2}\sigma^2)t_k}{\sigma\sqrt{t_k}}\right)} \\ &= 1 - \exp\left[\frac{2L'(x'-L')}{\sigma^2 t_k}\right]\end{aligned}$$

for $L' = \log\left[\frac{L}{S_0}\right]$ and $x' = \log\left[\frac{x}{S_0}\right]$.

So, combining (4.2.17) and (4.2.12) gives the value-at-future of the down-and-out call:

$$\mathrm{VaF}_{do}(t_k, x) = \begin{cases} \mathrm{MtM}_{do}(t_k; x) \times \left(1 - \exp\left[\frac{2L'(x'-L')}{\sigma^2 t_k}\right]\right) & \text{if } x > L \\ 0 & \text{otherwise} \end{cases} \tag{4.2.17}$$

for $L' = \log\left[\frac{L}{S_0}\right]$ and $x' = \log\left[\frac{x}{S_0}\right]$.

In the next section, the implementation and calculation of the potential future exposure of the up-and-out call option is shown by generating scenarios with R.

4.2.2 Implementation of Barrier Options

For the purpose of implementation, the statistic programming software R is used, which can be downloaded for free for any given operating system. For all the path depending derivatives listed and implemented in this thesis, the package SDE is used, which provides simulations for geometric Brownian motion and Brownian bridge:

```
library(sde)    # load sde package
```

Using the model introduced in section 4.1, the first step is to set the number M of scenarios generated and the number of time intervals N used in the process of discretising the tenor up to maturity T - $M = 10000$ scenarios and $N = 100$ time intervals are going to be generated.
Generally, parameters for the risk factor - the stock price in this case - are estimated based on (4.1.2). However, for the demonstration of the implementation, a simple example is used in which the parameters are predefined instead:

Let S be the stock price that follows a geometric Brownian motion with appreciation rate $\mu = 0.05$, volatility $\sigma = 0.1$ and initial value $S_0 = 100\$$ at $t_0 = 0$. The up-and-out call option has strike $K = 100$ at maturity $T = 1$ with an upper barrier $H = 110\$$. For starters, the risk free interested rate r used to discount the price process is assumed to be zero like intended by Lomibao and Zhu [2005]. However, is changed to $r > 0$ for showcasing the difference later on .
The next step is to generate the stock price process up to T for every time node t_k, $k \in \{0, 1, 2, \dots, N\}$. As the indexing in R starts at one, $t_0 \hat{=}$ `t[1]` and $t_N = T \hat{=}$ `t[N+1]`. These parameters are implemented as follows:

```
mu<-0.05      # appreciation rate
r<-0        # risk-free rate
sigma<-0.1  # volatility
T<-1        # maturity
S0<-100     # initial value
K<-100      # strike
H<-110      # upper barrier
```

The function GBM of the SDE package generates a geometric Brownian motion for the given parameters and is used to generate the path of the stock price evolution:

```
S<-GBM(S0,mu,sigma,T,N)      # stock price
```

In order to calculate the value-at-future

$$\mathrm{VaF}_{do}(t_k,x) = \begin{cases} \mathrm{MtM}_{do}(t_k;x) \times \left(1 - \exp\left[\frac{2L'(x'-L')}{\sigma^2 t_k}\right]\right) & \text{if } x > L \\ 0 & \text{otherwise} \end{cases} \tag{4.2.18}$$

the probability of the stock price to stay below barrier H has to be calculated first:

```
P<-rep(0,N+1)              # probability to stay below H
P[(which(S<H))]<- 1-(S[(which(S<H))]/H)^(2*log(H/S0)/(sigma^2*(t
   [(which(S<H))])))
```

Afterwards, the mark-to-market values $\mathrm{MtM}_{uo}(t_k;x)$, $k \in \{0,1,2,\ldots,N\}$, have to be calculated, which can easily be implemented according to chapter 4.2 and the results of Hull [2009]:

```
d1 <- (log(S/K)+(r+sigma^2/2)*(T-t))/(sigma*sqrt(T-t))
lambda <- (r+1/2*sigma^2)/sigma^2
x1 <- log(S/H)/(sigma*sqrt(T-t)) + lambda*sigma*sqrt(T-t)
y  <- log(H^2/(S*K))/(sigma*sqrt(T-t)) + lambda*sigma*sqrt(T-t)
c_uo<-S*pnorm(d1)-K*exp(-r*(T-t))*pnorm(d1 - sigma * sqrt(T-t))
    - (S*pnorm(x1)-K*exp(-r*(T-t))*pnorm(x1-sigma*sqrt(T-t))-S*
   (H/S)^(2*lambda)*(pnorm(-y)-pnorm(x1-sigma*sqrt(T-t)))+K*exp
   (-r*(T-t))*(H/S)^(2*lambda-2)*(pnorm(-y+sigma*sqrt(T-t))-
   pnorm(x1)))   # MtM of up-an-out barrier call option
```

Putting the pieces together, the value-at-future for the up-and-out call option is calculated by multiplying the probability from above with the mark-to-market value of the barrier option. The results for each scenario j, $j \in \{1,2,\ldots,M\}$ and each time-step t_k, $k \in \{0,1,2,\ldots,N\}$ are saved in a matrix:

```
VaF[j,]<-P*c_uo                    # value-at-future
```

```
VaF[j,(which(VaF[j,]<0))]<-0        # exposure
```

The second line ensures that each negative value-at-risk equals zero in order to transfer the values into exposures.

Finally, the metrics introduced in 2.1.1 have to be applied, merely concentrating on potential future exposures. Thereby, the quantiles for the PFEs are set to 5%, 50% and 95% for all examples in order to make the results comparable. Hence,

```
q<-apply(VaF, 2, quantile, probs = c(0.05, 0.5, 0.95),  na.rm
   = TRUE)              # PFEs quantiles
```

Figure 4.6 shows the PFEs of the up-and-out call option with confidence levels of 5%, 50% and 95 %. The x-axis represents the tenor up to maturity $T = 1$. The PFE can be interpreted as a value-at-risk from the investor's profit perspective. One can easily recognize the difference between the exposure profile of the barrier option and that of a vanilla call option with the same parameters shown in figure 4.7. Taking a closer look at the peak exposure profile, the convex shape of the barrier option's PFEs stands in contrast with the concave profile of the quantiles of the exposure of a plain vanilla option, which has to do with the likelihood of the barrier option to be knocked out during the tenor. Furthermore, the fact that at any t_k, $k \in \{0, 1, 2, \ldots, N\}$, the path up that point in time as well as the development up to maturity T are unknown has a huge effect on the option price in general, as a significant discrepancy between the exposure of the vanilla option and that of an up-and-out barrier option can be seen in figure 4.6 and . Even at its peak, the potential future exposures of the barrier option are far lower than those of the vanilla option, providing that they use an equal strike and underlying asset.

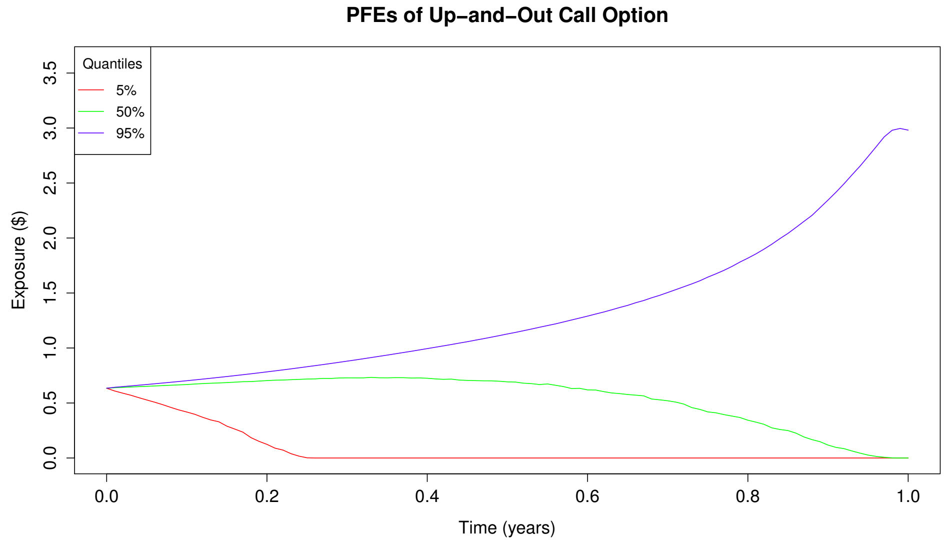

Figure 4.6: PFEs of up-and-out call option with $\mu = 0.05$

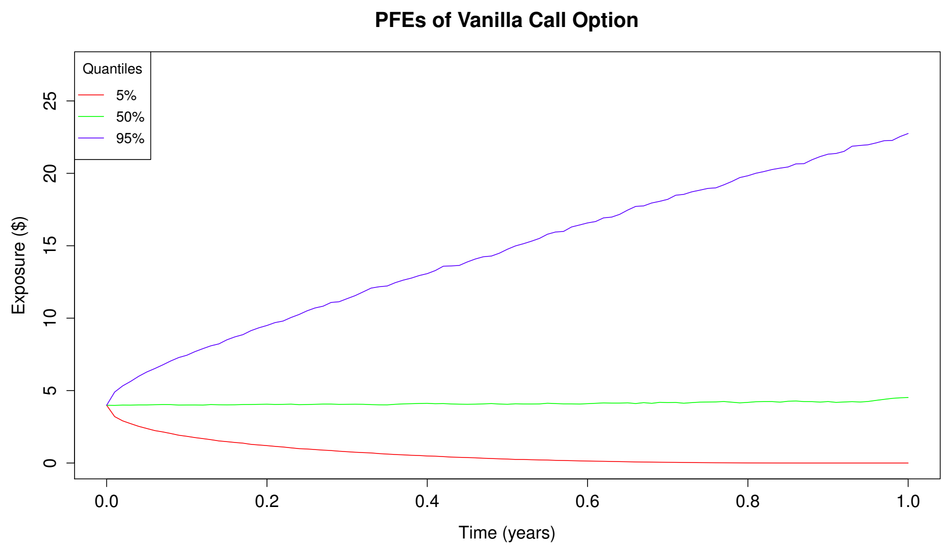

Figure 4.7: PFEs of vanilla call option with $\mu = 0.05$

Furthermore, note that the 50%-quantile of the up-and-out barrier call option converges 0 zero, whereas that of the vanilla option does not. That has to do with the fact that under these parameters, the real probability of the vanilla option to be "out-of-the-money" is lower than 50% - hence it stays above zero. On the other hand, if $\mu = 0$ is assumed - like intended by Lomibao and Zhu [2005] - the probability to be "out-of-the-money" is greater or equal 50%, which makes the corresponding quantile converging to zero (see figure 4.8. Since the barrier option requires the stock price to stay within certain borders to have a positive payoff, its 50% quantile converges to zero regardless.

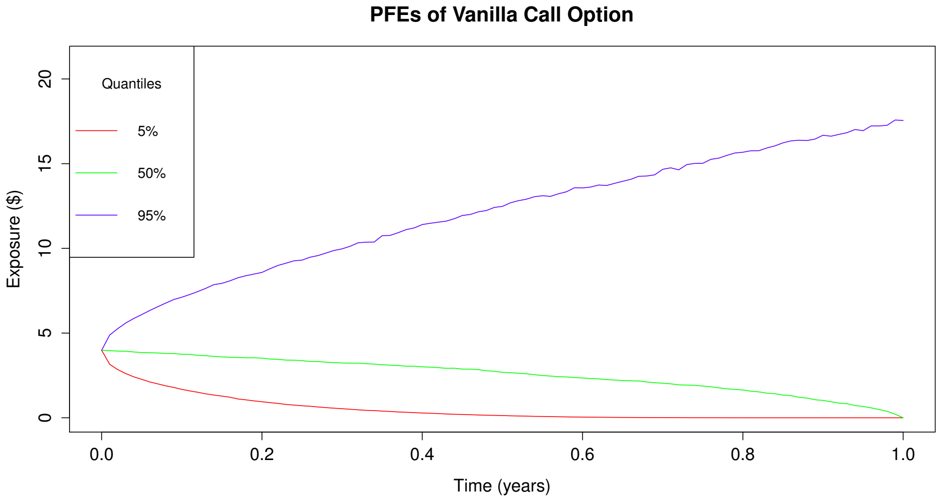

Figure 4.8: PFEs of vanilla call option

The next plot shows the expected exposure of the barrier option, though it has to be noted that the plot is going to become smoother when altering the risk-free interest rate to any value greater zero (figure 4.9).

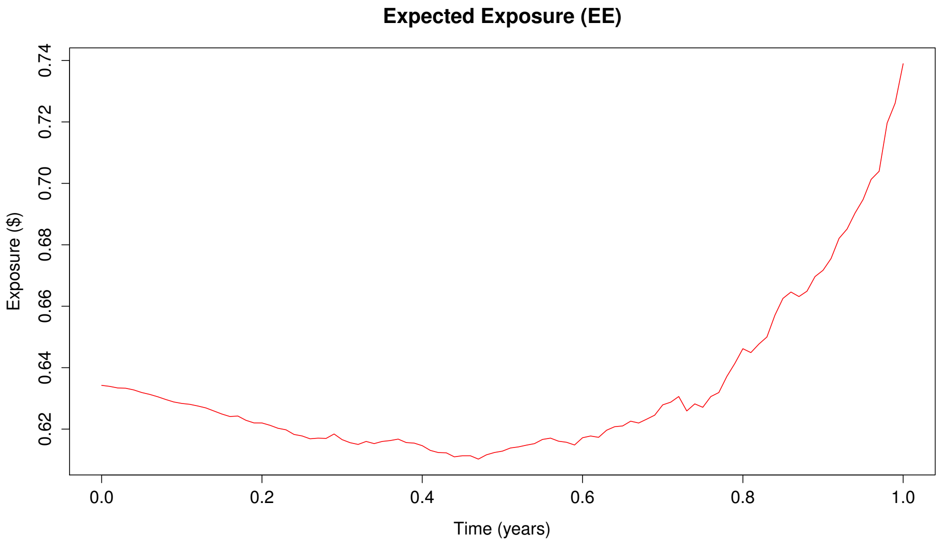

Figure 4.9: EE of up-and-out barrier option

Setting the interest rate to $r = 0.02$ and leaving the appreciation rate at $\mu = 0.05$, the value-at-future turns negative for the first period of time, afterwards however, the exposures show the same behavior as those that neglect the discount factor. Figure 4.10 and 4.11 show the different exposure profiles taking a discount rate into account.

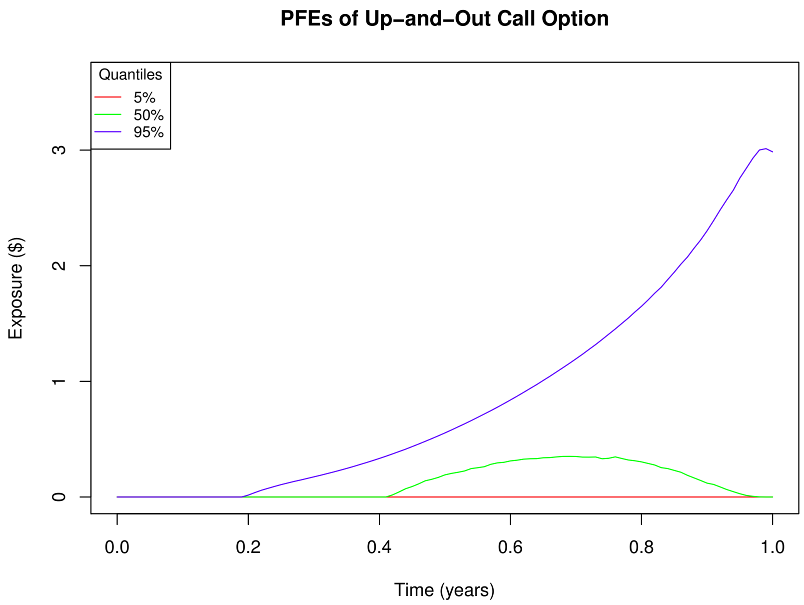

Figure 4.10: PFEs of up-and-out call option with $r = 0.02$ and $\mu = 0.05$

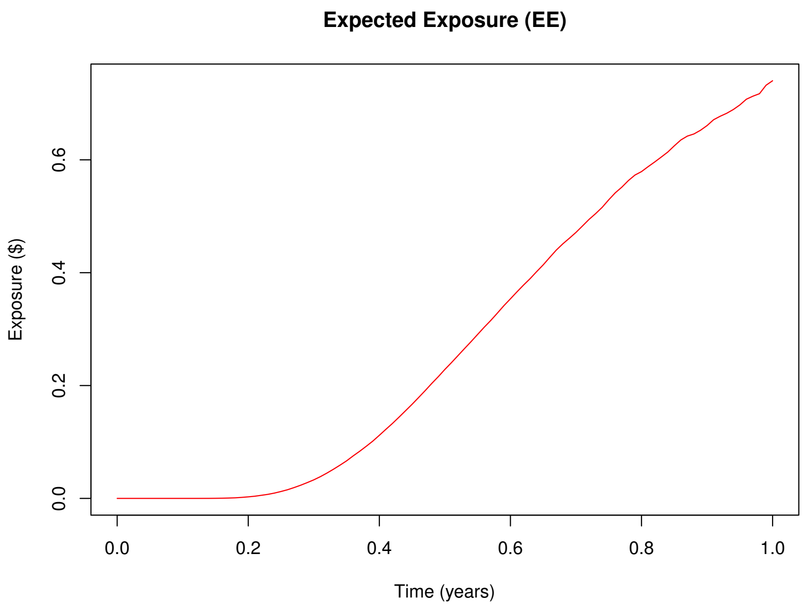

Figure 4.11: EE of up-and-out call option with $r = 0.02$ and $\mu = 0.05$

4.3 Swap-Settled Swaption

The next path-dependent derivative of which the exposure profiles are going to be calculated is the swap-settled swaption. The difference between this swaption and a regular one is the settlement: Where the regular swaption is cash-settled at expiry date, a swap-settled swaption settles into the underlying swap if the option is exercised at maturity. The crucial part of a swap-settled swaption in the topic of credit exposure is the fact that its future credit exposure does not stop at the expiry date, but will continue well beyond to the maturity of the swap if the option is exercised. However, to understand the peculiarities of this derivative, the regular swaption has to be explained first, referring to the work of Hull [2009].

The European swaption is an option on entering an interest-rate swap, which is a financial instrument that obliges the owner to pay or receive cash flows in the amount of a predefined interest rate (swap rate) on a fixed notional over a certain period. In exchange, he receives or pays a floating interest rate (f.e. LIBOR, EURIBOR, ...) on the same notional over the same timespan. These derivatives are called fixed-for-floating swaps and are basically an exchange of different cash flows.
There are two possible positions:

- Payer swap: The party that has to pay the fixed interest rate calls the swap a payer swap. It receives a floating rate in exchange.

- Receiver swap: The party that receives the fixed interest rate calls the swap a receiver swap. It has to pay a floating rate in exchange.

In most cases, the London interbank offered rate (LIBOR) is taken for the floating interest rate. It is the average rate estimated by leading banks in London at which banks can borrow funds from other banks in London's interbank market. It is fixed on a daily basis and is usually offered as one, three or six months LIBOR, which refers to the maturity of the funds. In order to understand swaps, assume a six months swap between company A and company B starting at 5^{th} March, 2012. Further, assume that A is obliged to pay a swap rate of 5% on a notional of 100 million $ to company B, which itself has to pay six months LIBOR on the same notional to company A in return. The exchange takes place every six months and to keep the example simple, the swap rate of 5% is semi-annual.

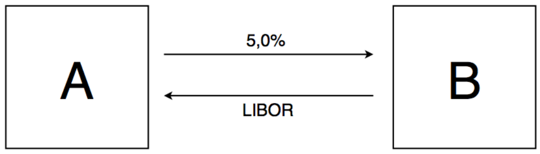

Figure 4.12: Illustration of interest rate swap between company A and B

Consequently, the first exchange of cash flows will occur on 5^{th} September, 2012, exactly six months after the swap started. Company A pays $100 \times 0.05 = 2.5$ million \$, and company B, assuming that the six months LIBOR is 4.2% at 5^{th} March, 2012, $0.042 \times 0.5 \times 100 = 2.1$ million \$. The floating interest rate for a swap period is always fixed at the end of the last period. Hence, the first exchange of cash flows does not underlie any uncertainty since the first so-called fixing date matches the date the contract is signed. The second exchange is going to take place at 5^{th} March, 2013 - this pattern is going to continue until expiry date. It has to be noted though that only the margin of the two cash flows is transferred from one party to the other. Relating to the example, company A has to pay $2.5 - 2.1 = 0.4$ million \$ to company B at 5^{th} September, 2012.

Hence, the value of a payer swap is

$$V_{\text{swap}} = B_{\text{fl}} - B_{\text{fix}} \tag{4.3.1}$$

where B_{fl} is the value of the floating Bond and B_{fix} that of the fixed one. For more information on swaps, it is referred to Hull [2009].
Anyway, now that a swap has been established, the motivation behind an option on a swap is much clearer. So, the owner of an European swaption has the right, but not the obligation, to enter an interest rate swap at a specific date. Similar to swaps, it is distinguished between a payer swaption and a receiver swaption which determines the underlying swap.

- Payer swaption: The right, but not the obligation, to pay fixed rate and receive floating rate throughout the tenor of the underlying swap.
- Receiver swaption: The right, but not the obligation, to receive fixed rate and pay floating rate throughout the tenor of the underlying swap.

Several factors and parameters have to be fixed in such a contract:

- the premium of the swaption (price)
- the strike of the swaption which equals the swap rate of the underlying swap
- the exercise/expiry date of the option
- the tenor of the swap
- the swap periods and their frequency
- the notional amount

However, when does an investor want to buy a swaption and if so, when does he want to exercise his option?
Swaptions provide their owner the security that the fixed rate, which he has to pay for future funds, does not exceed a certain level. It is an alternative to a forward swap, which, in contrast to a swaption, obliges the owner to enter the swap at a future time - not giving the owner any choice. However, the option makes the swaption also more expensive than a simple forward swap, which does not need a premium to be paid upfront, because the forward does not protect the owner from unfavorable interest rate developments. The owner of payer swaption will exercise his option at exercise date, if the floating rate, f.e. LIBOR, is higher than the predefined swap rate at that date. If LIBOR is below the fixed swap rate, the investor has no interest in entering the underlying swap at expiry date, since a regular swap will be cheaper. Therefore, he will buy the regular one which has a present value of zero (no-arbitrage) in contrast to the negative value of underlying swap (see formula (4.3.1)).

4.3.1 Pricing of Swap-Settled Swaptions

In order to evaluate a swaption, it first has to be constructed from more basic instruments, whereby following definitions and notations are closely tied to Brigo and Mercurio [2001]. The forward rate agreement (FRA) is a contract that gives its owner an interest-rate payment for the period between T_α and T_β, where T_α is the expiry time and $T_\beta > T_\alpha$ the maturity. At maturity T_β, a fixed rate K is exchanged against a floating payment based on the spot rate $L(T_\alpha, T_\beta)$ resetting in T_α with maturity T_β, which is defined as

$$L(t,T) := \frac{1 - P(t,T)}{\tau(t,T)\, P(t,T)} \quad , \tag{4.3.2}$$

where $P(t,T)$ is the value of a zero-coupon bond at t defined in (3.2.2), and $\tau(t,T)$ the year fraction between t and T. For a notional $\hat{L}$, the value of the contract at maturity is therefore

$$\hat{L}\tau(T_\alpha,T_\beta)(K - L(T_\alpha,T_\beta)) \quad .$$

Hence, the value of the forward rate agreement at time $t < T_\alpha < T_\beta$ is

$$\mathrm{FRA}(t,T_\alpha,T_\beta,\tau(T_\alpha,T_\beta),\hat{L},K) = \hat{L}P(t,T)\tau(T_\alpha,T_\beta)(K - F(t;T_\alpha,T_\beta)) \quad , \tag{4.3.3}$$

with $F(t;T_\alpha,T_\beta) := \frac{1}{\tau(T_\alpha,T_\beta)}\left(\frac{P(t,T_\alpha)}{P(t;T)} - 1\right)$ being the forward interest rate, which is the value of the fixed rate in the FRA that renders it a fair contract at time t.

The next step is to calculate an interest-rate swap starting from T_α and ending at T_β by using the results above. At every T_i in a prespecified set of dates $T_{\alpha+1}, T_{\alpha+2}, \ldots T_\beta$, with T_α and T_β, the fixed leg pays out

$$\hat{L}\tau_i K$$

and the floating one

$$\hat{L}\tau_i L(T_{i-1},T_i) \quad ,$$

with τ_i being the year fraction between T_{i-1} and T_i and $L(T_{i-1},T_i)$ resetting at dates $T_\alpha, T_{\alpha+1}, \ldots T_{\beta-1}$.

Consequently, the value of the payer interest rate swap (PFS) at time t is the discounted sum of the values of all the cash-flow exchanges, hence

$$\begin{aligned}
\mathrm{PFS}(t,\mathcal{T},\tau,\hat{L},K) &= \sum_{i=\alpha+1}^{\beta} -\mathrm{FRA}(t,T_{i-1},T_i,\tau_i,\hat{L},K) \\
&= \hat{L}\sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)\big(F(t;T_{i-1},T_i) - K\big) \quad ,
\end{aligned} \tag{4.3.4}$$

where $\mathcal{T} := \{T_\alpha, T_{\alpha+1}, \ldots, T_\beta\}$ and $\tau = \{\tau_{\alpha+1}, \tau_{\alpha+2}, \ldots, \tau_\beta\}$.

Analogously, for an according receiver interest rate swap (RFS),

$$\begin{aligned}
\mathrm{RFS}(t,\mathcal{T},\tau,\hat{L},K) &= \sum_{i=\alpha+1}^{\beta} \mathrm{FRA}(t,T_{i-1},T_i,\tau_i,\hat{L},K) \\
&= \hat{L}\sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)\big(K - F(t;T_{i-1},T_i)\big) \quad .
\end{aligned} \tag{4.3.5}$$

The forward swap rate $S\alpha, \beta_t$ at time t for the sets of times $\mathcal{T}$ and year fractions τ is defined as the rate in the fixed leg of the IRS that makes it a fair contract at t. Basically, it is the fixed rate K from above that provides that $\mathrm{RFS}(t, \mathcal{T}, \tau, \hat{L}, K) = 0 = \mathrm{PFS}(t, \mathcal{T}, \tau, \hat{L}, K)$. Hence,

$$S_t^{\alpha,\beta} = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)} \quad . \tag{4.3.6}$$

In order to lead over to swaptions, the discounted pay-off of a payer swaption with maturity T_α and Tenor $T_\alpha - T_\beta$, can be written by considering the value of the underlying payer IRS at the swaptions maturity T_α. This value is

$$\hat{L} \sum_{i=\alpha+1}^{\beta} \tau_i P(T\alpha, T_i)\big(F(T_\alpha; T_{i-1}, T_i) - K\big) \quad .$$

Obviously, the option will only be exercised when upper value is positive at T_α. Hence, the payer swaption pay-off, discounted from T_α to the current time t equals to

$$\mathrm{MtM}_{\mathrm{ps}}(t) = \hat{L}\ D(t,T_\alpha)\ \max\Big(0, \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i)\big(F(T_\alpha; T_{i-1}, T_i) - K\big)\Big) \quad . \tag{4.3.7}$$

Thereby

$$D(t,T) = \frac{B_t}{B_T} = \exp\Big[-\int_t^T r_s\, ds\Big]$$

is the (stochastic) discount factor between t and T.

In order to evaluate a swaption analytically from formula (4.3.7), the Black-Scholes framework is chosen again. However, the forward swap measure has to be defined, under which the swap forward swap rate is a martingale.

Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P})$ be introduced like in section 3. The forward swap measure $\mathbb{P}^{\alpha,\beta}$ is defined on $\mathfrak{F}_{\alpha+1}$ as the equivalent martingale measure associated to the annuity numeraire

$$A_t^{\alpha,\beta} := \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i) \tag{4.3.8}$$

under which the forward swap rate $S_t^{\alpha,\beta}$ from (4.3.6) is a martingale.

By assuming a log-normal dynamic,

$$dS_t^{\alpha,\beta} = \sigma S_t^{\alpha,\beta} dW_t^{\alpha,\beta} \quad , \tag{4.3.9}$$

where $\sigma > 0$ and $W_t^{\alpha,\beta}$ is a standard Brownian motion under $\mathbb{P}^{\alpha,\beta}$. Consequently,

$$S_t^{\alpha,\beta} = S_0^{\alpha,\beta} \exp\big[-\frac{1}{2}\sigma^2 t + \sigma W_t^{\alpha,\beta}\big] \quad ,$$

Under this assumption, the results of the Black-Scholes model from section 3.2.3 can be applied on (4.3.7). A change of numeraire from bond B under the equivalent martingale measure $\mathbb{P}^*$ to the annuity $A^{\alpha,\beta}$ under $\mathbb{P}^{\alpha,\beta}$ (see (3.2.13)) leads to a mark-to-market value of the payer swaption of

$$
\begin{aligned}
\mathrm{MtM}_{\mathrm{ps}}(t;K,\mathcal{T},\tau) &= \mathbb{E}_{\mathbb{P}^*}\left[\hat{L}\, D(t,T_\alpha)\ \max\left(0, A^{\alpha,\beta}_{T_\alpha}\left(S^{\alpha,\beta}_{T_\alpha} - K\right)\right) \,\Big|\, S^{\alpha,\beta}_t\right] \\
&= \hat{L}\ \mathbb{E}_{\mathbb{P}^*}\left[\frac{B_t}{B_{T_\alpha}}\ A^{\alpha,\beta}_{T_\alpha}\ \max\left(0,\ S^{\alpha,\beta}_{T_\alpha} - K\right) \,\Big|\, S^{\alpha,\beta}_t\right] \\
&= \hat{L}\ \mathbb{E}_{\mathbb{P}^{\alpha,\beta}}\left[\frac{A^{\alpha,\beta}_t}{A^{\alpha,\beta}_{T_\alpha}}\ A^{\alpha,\beta}_{T_\alpha}\ \max\left(0,\ S^{\alpha,\beta}_{T_\alpha} - K\right) \,\Big|\, S^{\alpha,\beta}_t\right] \\
&= \hat{L}\ A^{\alpha,\beta}_t \mathrm{BS}(S^{\alpha,\beta}_t, K, T_\alpha - t, \sigma, 0, 0, 1) \quad \text{a.s.}\,,
\end{aligned}
\tag{4.3.10}
$$

where $\mathrm{BS}(S^{\alpha,\beta}_t, K, T_\alpha - t, \sigma, 0, 0, 1)$ is the value of the Black-Scholes formula defined in section 3.2.3).

For a receiver-swaption, the mark-to-market value is calculated similarly, hence

$$
\begin{aligned}
\mathrm{MtM}_{\mathrm{rs}}(t;K,\mathcal{T},\tau) &= \mathbb{E}_{\mathbb{P}^*}\left[\hat{L}\, D(t,T_\alpha)\ \max\left(0, A^{\alpha,\beta}_{T_\alpha}\left(K - S^{\alpha,\beta}_{T_\alpha}\right)\right) \,\Big|\, S^{\alpha,\beta}_t\right] \\
&= \hat{L}\ A^{\alpha,\beta}_t \mathrm{BS}(S^{\alpha,\beta}_t, K, T_\alpha - t, \sigma, 0, 0, -1) \quad \text{a.s.}\,.
\end{aligned}
\tag{4.3.11}
$$

These formulas hold for every $t_k < T_\alpha$, where $S^{\alpha,\beta}_{t_k}$ is going to be the simulated swap rate in the MtF model. Comparing them with those given by Lomibao and Zhu [2005], it is clear that they did not take the notional into account, hence assuming $\hat{L} = 1$. Furthermore, they use the Black-Scholes formula with parameter S_0 for the valuation at time t, although the S_t has to be used since the stock price at time t is known.

For any $t > T_\alpha$, Lomibao and Zhu [2005] proposed to continue calculating the potential future exposures, because if the swaption was exercised and the owner settled into the underlying swap, the exposure would have to be calculated continuously up to T_β. They argued that for the simulated scenario $S^{\alpha,\beta}_t$, $T_\alpha < t < T_\beta$, there are two possibilities for the trajectory of the swap rate:

1. The swap rate, starting at $S^{\alpha,\beta}_0$, follows a path that hits $\underline{S^{\alpha,\beta}_{T_\alpha} > K > 0}$ at exercise date T_α, and proceeds until reaching $S^{\alpha,\beta}_{t_k}$, $T_\alpha < t_k < T_\beta$.
2. The swap rate, starting at $S^{\alpha,\beta}_0$, follows a path that hits $\underline{S^{\alpha,\beta}_{T_\alpha} < K}$ at exercise date T_α, and proceeds until reaching $S^{\alpha,\beta}_{t_k}$, $T_\alpha < t_k < T_\beta$.

In case of the first path, the swaption would be exercised, the owner would then enter the swap and exposure calculations would have to be continued up to maturity T_β. However, if the simulated swap rate takes the second path, the swaption would not be exercised and therefore, no exposure would have to be calculated for any $t_k > T_\alpha$.

However, they do not realize that for any $T_\alpha < t \leq T_\beta$, the forward swap rate $S_t^{\alpha,\beta}$ does not exist. Therefore, trying to model any swap rates above exercise date T_α does not make any sense, especially if the forward swap rate is considered. Thus, the potential future exposures can only be calculated up to the first settlement date of the underlying swap, which equals the exercise date of the swaption. That also means that the exposure profile of a swap-settled swaption in the framework of Black-Scholes equals that of a cash-settled one. Thereby, Brigo and Mercurio [2001] gives a good insight into the so-called log-normal forward-swap model (LSM).

4.3.2 Implementation of Swap-Settled-Swaptions

The implementation of a swap-settled swaption is going to be shown on a simple example of an European payer swaption introduced in the section before. Again, a geometric Brownian motion is used to model the risk factor, which, in this case, is the forward swap rate $S_t^{\alpha,\beta}$. The sde package provides all the simulation that are needed for this model.

```
library(sde)          # load sde package
```

The following Monte-Carlo simulation contains $M = 10000$ scenarios. Let S be the swap rate that follows a geometric Brownian motion with appreciation rate $\mu = 0.05$ volatility $\sigma = 0.1$ and initial value $S_0 = 0.05$ at $t_0 = 0$. The swaption with fixed rate $K = 0.05$ has an exercise date of $T_\alpha = 1$ and if exercised, lets the owner enter the underlying swap at T_α with tenor $T = 5$. Therefore, underlying swap with notional $L = 100\$$ starts at T_α and will end at $T_alpha + T = T_\beta$. Finally, the risk-free interest rate is $r = 0.02$.

```
mu<-0.05     # appreciation rate
sigma<-0.1   # volatility
T<-5         # tenor of swap
T_a<-1       # exercise date of swaption
S0<-0.05     # initial value
```

```
K<-0.05      # fixed swap rate (=strike)
L<-100       # Nominale of swap in $
r<-0.02      # risk-free rate
```

The next step is to generate the time grid, which again is equidistant. Thereby, the tenor of the option on the swap is divided into $n = 100$ intervals and it is assumed that the swap pays quarterly, hence four swap periods per year.

```
M<-10000            # number of generated scenarios
n<-100             # number of time intervals up to exercise date
m<-4                # number of swap periods per year
N<-floor(m*T)       # total number of swap periods
t<-c(seq(0,T_a-1/n, by=1/n) , seq(T_a,(T_a+T),by=1/m))        #
   time nodes
A<-rep(0,(T_a)*n+1)
for (k in 1:((T_a)*n+1)) {
 A[k] <- sum(1/m*exp( -r*(t[(T_a*n+2):(T_a*n+T*m+1)] - t[k])))
    # annuity
}
```

For every scenario j out of the 10000 simulations, the risk factor is generated and both the value-at-future (4.3.10) for time nodes before T_α.

```
S<-GBM(S0,mu,sigma,T_a,(T_a)*n)      # swap rate
d1 <- (log(S[(1:(T_a*n+1))]/K)+(sigma^2/2)*(T_a-t[(1:(T_a*n+1))])
   )/(sigma*sqrt(T_a-t[(1:(T_a*n+1))]))
MtM[j,]<- L*A*(S[(1:(T_a*n+1))]*pnorm(d1)-K*pnorm(d1-sigma*sqrt(T
   _a-t[(1:(T_a*n+1))])))              # mark-to-market value
ifelse(MtM<0, 0, MtM)                  # exposure
```

After the computer-intense simulation is done, the exposure measures have to be applied on the results of the VaF table. For several different input parameters, the expected exposure (EE) and the quantiles of the potential future exposures are calculated and plotted.

```
ExpExposure<-apply(MtM[,(1:(T_a*n+1))],2,mean)    # expected
  exposure
Quantile_vanilla<-apply(MtM,2, quantile, probs = c(0.05, 0.5,
  0.95),  na.rm = TRUE)   # PFEs
```

Figure 4.13 shows the potential future exposures for the quantiles 0.05, 0.5 and 0.95. Unsurprisingly, the profile of the exposures in figure 4.13 is similar to that of a vanilla call option on a stock - see figure 4.9. That is due to the fact the swaption simply is a vanilla option on a swap, where the forward swap rate under the forward swap measure is modeled similar to the stock price under the risk-neutral measure with the bond as numeraire.

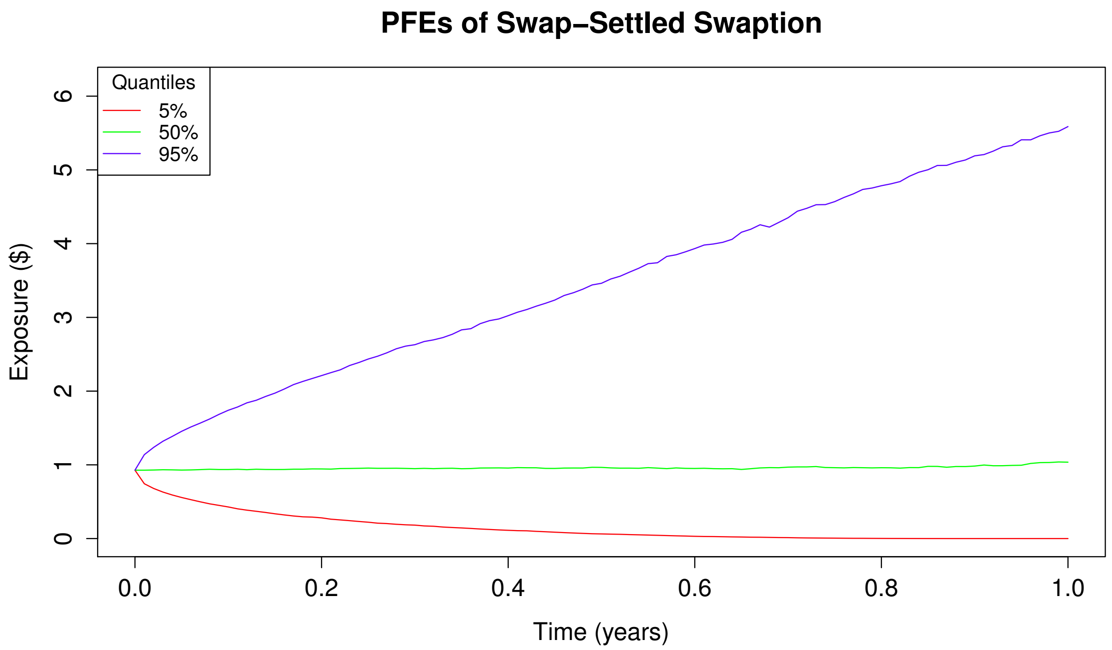

Figure 4.13: PFEs of Swap-Settled-Swaption (Payer) with $\mu = 0.05$ and $r = 0.02$

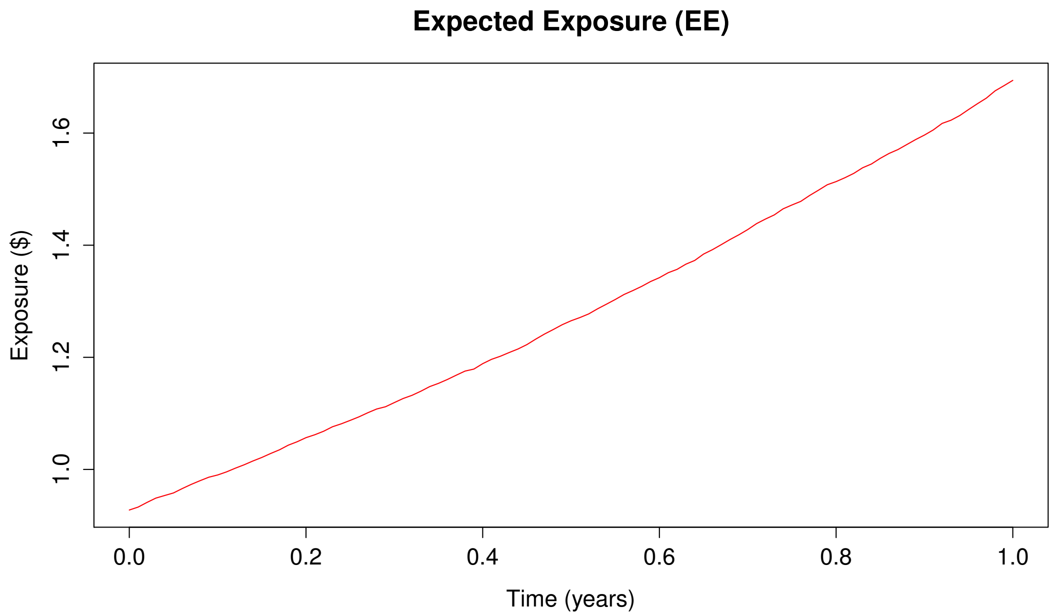

Figure 4.14: EE of Swap-Settled-Swaption (Payer) with $\mu = 0.05$ and $r = 0,02$

Again, if $\mu = 0$ is chosen, as apparently done by Lomibao and Zhu [2005], the 50%-quantile will converge to zero like its vanilla counterpart. This behavior has already been discussed in section 4.2.2 and can be seen in figure 4.15.

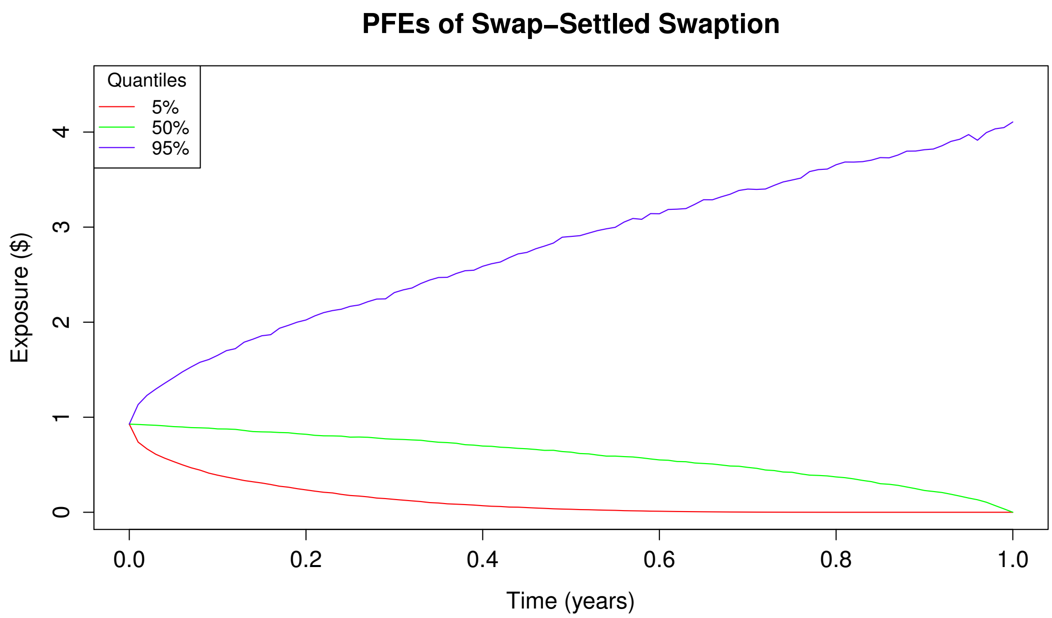

Figure 4.15: PFEs of Swap-Settled-Swaption (Payer) with $\mu = 0$ and $r = 0.02$

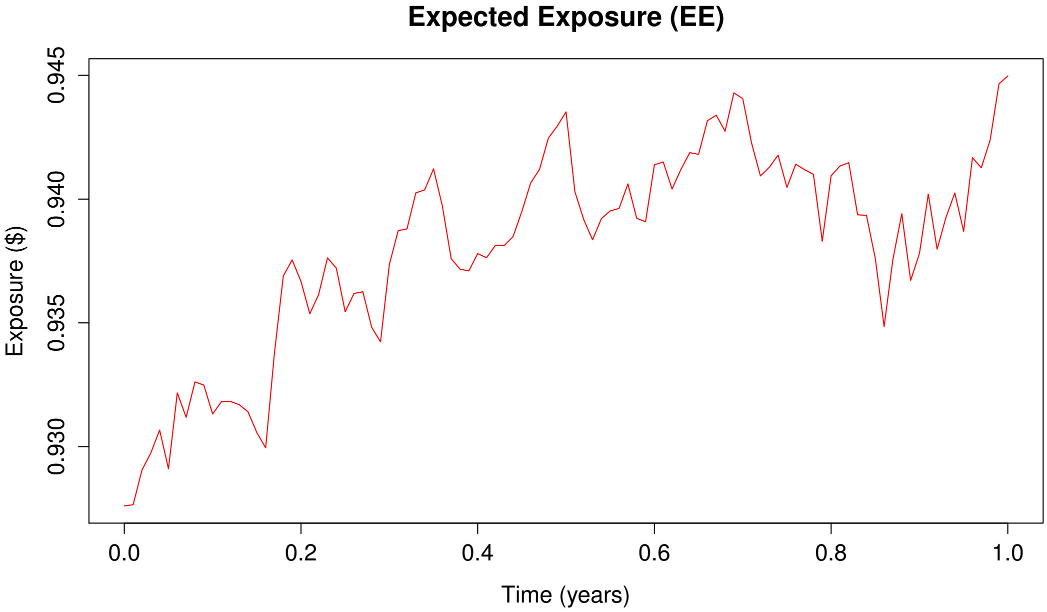

Figure 4.16: EE of Swap-Settled-Swaption (Payer) with $\mu = 0$ $r = 0,02$

4.4 Asian Option

Asian options are options where the payoff depends on the average price of the underlying within the tenor of the option. Thereby, daily, weekly or even monthly stock prices are taken into account for calculating either an arithmetic average, or a geometric one. Considering the stock prices of the discrete set of which the average is taken, time of a discrete set of time, the two averages are defined as

$$A_n = \sum_{i=0}^{n} \frac{S_{t_i}}{n} \tag{4.4.1}$$

and

$$G_n = \prod_{i=0}^{n} S_{t_i}^{\frac{1}{n}} \tag{4.4.2}$$

where n is the number of stock prices S_{t_i}, $i \in \{0, \ldots, n\}$, taken into account.
For the purpose of this paper, it is focused on the arithmetic average only, as the geometric one can simply be calculated by using the convolution of normal distributed random variables.
The payoff of an average price call is

$$\max(0, A_n - K) \ ,$$

and that of an according put option

$$\max(0, K - A_n) \ .$$

Both of them have in common that they are cheaper to buy than their plain vanilla counterparts, which makes them more appealing in some situations. As Hull [2009] points out, a manager that expects certain, evenly distributed cash flows over the course of a year, is surely interested in an option that provides the security of the average cash flows to be higher than a certain limit (strike). Thus, an average option becomes handy. Alternatively, an average strike option with payoff

$$\max(0, S_T - A_n)$$

for the call, and

$$\max(0, A_n - S_T)$$

for the put option can also be obtained. They provide that the average price of a heavily traded asset stays above or below the price at maturity.

To trace an arc to the methods used in this thesis, it is assumed that the price S of the underlying follows a geometric Brownian motion

$$S_t = S_0 \exp\left[(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right]$$

with A_n being its arithmetic average.
Unfortunately, the distribution of the arithmetic average of log-normal distributed random variables does not have any properties allowing to derive the valuation analytically, hence, there is no closed form of it. However, Fenton [1960] proposed to approximate the sum of log-normal distributed random variables with another log-normal distributed random variable using moment matching. This method accordingly is called Fenton-Wilkinson method. Additionally, Schwartz and Yeh [1981] provide a good approach to evaluate the moments for the resulting random variate. These observation are going to be very useful when pricing an Asian option in the following section.

4.4.1 Pricing of Asian Options

For the purpose of calculating the exposure profiles of an Asian option, the value-at-future has to be derived first, similar to the other derivatives that are focused on in this thesis. For simulation, a discrete time grid $\{t_0, t_1, \ldots, t_N\}$ is used. Note that t_N does not have to equal the maturity T, since the last date the average is taken from does not need to fall on the date of maturity. For example, if a weekly basis is considered, an Asian option with the maturity of one year á 360 days does meet this requirement, because $7 \times 51 = 357$. Hence, $t_N = \frac{\lfloor 360/7 \rfloor \times 7}{360} \neq 1 = T$, which shows that the day count conversion is crucial.

As the concept of the Brownian bridge is going to be applied, it is crucial to divide the arithmetic average into one part that deals with the average from $[t_0, t_k]$ and another one for $[t_{k+1}, t_N]$ - with simulation date t_k,. That is important as for any $t \in \{t_0, t_1, \ldots t_k\}$, the concept of the Brownian Bridge applies, and for $t \in \{t_{k+1}, \ldots, t_N\}$, the common valuation formulas conditioned on the information at t_k is used. Therefore, the following notation for the discrete case is introduced:

$$A(t_m, t_l) = \frac{1}{l - m + 1} \sum_{i=m}^{l} S_{t_i} \quad , \quad 0 \leq m < l \leq n$$

It has to be mentioned though that it is merely concentrated on the discrete case, totally abandoning the continuous arithmetic average. As the model is based on a discrete set of time and Monte-Carlo simulation, it is quite plausible to do so. As a consequence, the value-at-future of the Asian option with arithmetic average at time $t_k \in \{t_0, t_1, \dots, t_{N-1}\}$ is given by

$$\begin{aligned}\mathrm{VaF}(t_k, S_{t_k}; \psi) &= \mathbb{E}\left[\frac{B_{t_k}}{B_T}\max\left(0, \psi\left(A(t_0, t_N) - K\right)\right) \mid S_{t_k}\right] \\ &= \frac{B_{t_k}}{B_T}\mathbb{E}\left[\max\left(0, \psi\left(\frac{k+1}{N+1}A(t_0, t_k) + \frac{N-k}{N+1}A(t_{k+1}, t_N) - K\right)\right) \,\middle|\, W_{t_k}\right] \end{aligned} \tag{4.4.3}$$

where $W_{t_k} = \frac{\log\left[\frac{S_{t_k}}{S_0}\right] - \left(\mu - \frac{1}{2}\sigma^2\right)t_k}{\sigma}$ is generated accordingly and B_t is the value of a risk-free bond at time t introduced in section 3.2.1. For $t_k = t_N$, $A(t_{k+1}, t_N) := 0$ such that $\frac{N-k}{N+1}A(t_{k+1}, t_N) = 0$. The next step is to split up the joint conditional expectation, whereby $A(t_0, t_k)$ is independent from $A(t_{k+1}, t_N)$ under the condition that W_{t_k} is known. Be aware that Lomibao and Zhu [2005] have falsely rewritten the expected value above, because they made a false assumption on the independence of $A(t_0, t_k)$, $A(t_{k+1}, t_N)$ and W_{t_k}. Unfortunately, it seems to be impossible to find an analytical solution for upper expected value for arithmetic average option, which requires a proper semi-analytical approach. However, as mentioned before, the distribution of the sum of log-normal distributed random variables comes fairly close to be log-normal distributed again. Therefore, Lomibao and Zhu [2005] introduced an approximation where the averages are assumed to be log-normal distributed to reach accord with previously used Black-Scholes model. However, they assumed that both approximations are correlated, which is not a plausible assumption on the averages. Since the stock prices, and consequently, their averages are solely driven by a Brownian motion, which has independent increments, the following random variables are chosen to be independent from each other:

$$\tilde{A}(t_0, t_k; Z) = \exp\left[M(t_0, t_k) + \sqrt{V(t_0, t_k)}\, Z\right] \tag{4.4.4}$$

$$\tilde{A}(t_{k+1}, t_N; U) = \exp\left[M(t_{k+1}, t_N) + \sqrt{V(t_{k+1}, t_N)}\, U\right] \quad , \tag{4.4.5}$$

where $U, Z \sim N(0,1)$ i.i.d., and $\big(M(t_0, t_k), V(t_0, t_k)\big)$, $\big(M(t_{k+1}, t_N), V(t_{k+1}, t_N)\big)$ are the parameters of the log-normal distributed approximation such that

$$A(t_0, t_k) \mid W_{t_k} \approx \tilde{A}(t_0, t_k; Z)$$

$$A(t_{k+1}, t_N) \mid W_{t_k} \approx \tilde{A}(t_{k+1}, t_N; U)$$

with

$$\mathbb{E}[\tilde{A}(t_0, t_k; Z)^i] = \mathbb{E}[A(t_0, t_k)^i \mid W_{t_k}]$$

$$\mathbb{E}[\tilde{A}(t_{k+1}, t_N; U)^i] = \mathbb{E}[A(t_{k+1}, t_N)^i \mid W_{t_k}], \quad i \in \{1, 2\} \quad .$$

The parameters of the approximated arithmetic averages can be calculated by deriving their first and second moments for both time intervals, $[t_0, t_k]$ and $[t_{k+1}, t_N]$ - t_k being fixed. However, note that with time interval, the discrete set of time within the limits is meant. Thereby, the common formulas for the calculations of the parameters for a log-normal distributed random variable are used:

$$\mathbb{E}[\tilde{A}(t_{k+1}, t_N; U) \mid W_{t_k}] = \exp\left[M(t_{k+1}, t_N) + \frac{1}{2}V(t_{k+1}, t_N)\right] \quad \Leftrightarrow$$
$$V(t_{k+1}, t_N) = 2\left(\log\left[\mathbb{E}[\tilde{A}(t_{k+1}, t_N; U) \mid W_{t_k}]\right] - M(t_{k+1}, t_N)\right)$$

$$\mathbb{V}[\tilde{A}(t_{k+1}, t_N; U) \mid W_{t_k}] = \Big(\exp\left[V(t_{k+1}, t_N)\right] - 1\Big) \exp\left[2M(t_{k+1}, t_N) + V(t_{k+1}, t_N)\right] \quad .$$

Hence, the parameters for $\tilde{A}(t_{k+1}, t_N; U)$ are

$$M(t_{k+1}, t_N) = 2\log\left[\mathbb{E}[\tilde{A}(t_{k+1}, t_N; U) \mid W_{t_k}]\right] - \frac{1}{2}\log\left[\mathbb{E}[\tilde{A}(t_{k+1}, t_N; U)^2 \mid W_{t_k}]\right]$$
$$V(t_{k+1}, t_N) = \log\left[\mathbb{E}[\tilde{A}(t_{k+1}, t_N; U)^2 \mid W_{t_k}]\right] - 2\log\left[\mathbb{E}[\tilde{A}(t_{k+1}, t_N; U) \mid W_{t_k}]\right] \quad ,$$

and for $\tilde{A}(t_1, t_k; Z)$ - analogically -

$$M(t_0, t_k) = 2\log\left[\mathbb{E}[\tilde{A}(t_0, t_k; Z) \mid W_{t_k}]\right] - \frac{1}{2}\log\left[\mathbb{E}[\tilde{A}(t_0, t_k; Z)^2 \mid W_{t_k}]\right]$$
$$V(t_0, t_k) = \log\left[\mathbb{E}[\tilde{A}(t_0, t_k; Z)^2 \mid W_{t_k}]\right] - 2\log\left[\mathbb{E}[\tilde{A}(t_0, t_k; Z) \mid W_{t_k}]\right] \quad .$$

Therefore, the next steps are to calculate the moments of $A(t_{k+1}, t_N)$ and $A(t_1, t_k)$, match them with those of the approximations $\tilde{A}(t_1, t_k)$ and $\tilde{A}(t_{k+1}, t_N)$, and eventually calculate the parameters according to the formulas above. Splitting the average up into two time intervals is crucial, however, as the knowledge of W_{t_k} has a great impact on the conditional valuation. For time interval $[t_{k+1}, t_N]$ and $\Delta t = t_i - t_{i-1}$, $i \in \{1, \ldots, N\}$, both moments of $A(t_{k+1}, t_N)$ can be calculated using familiar calculations seen in section 3.2.1:

$$\mathbb{E}[A(t_{k+1}, t_N) \mid W_{t_k}] = \mathbb{E}\left[\frac{1}{N-k}\sum_{j=1}^{N-k} S_{t_{k+j}} \;\middle|\; W_{t_k}\right] = \frac{1}{N-k}\sum_{j=1}^{N-k} \mathbb{E}\left[S_{t_{k+j}} \mid W_{t_k}\right]$$

$$
\begin{aligned}
&= \frac{S_{t_k}}{N-k}\sum_{j=1}^{N-k}\exp\left[\left(\mu-\frac{1}{2}\sigma^2\right)(t_{k+j}-t_k)\right]\mathbb{E}\left[\exp\left[\sigma W_{t_{k+j}-t_k}\right]\right]\\
&= \frac{S_{t_k}}{N-k}\sum_{j=1}^{N-k}\exp\left[\left(\mu-\frac{1}{2}\sigma^2\right)(t_{k+j}-t_k)\right]\exp\left[\frac{\sigma^2}{2}(t_{k+j}-t_k)\right]\\
&\quad\times\int_{\mathbb{R}}\phi\big(x-\sigma\sqrt{t_{k+j}-t_k}\big)\\
&= \frac{S_{t_k}}{N-k}\sum_{j=1}^{N-k}\exp\left[\mu(t_{k+j}-t_k)\right]\quad\text{a.s.}
\end{aligned}
\tag{4.4.6}
$$

$$
\begin{aligned}
\mathbb{E}[A(t_{k+1},t_N)^2\mid W_{t_k}] &= \mathbb{E}\left[\left(\frac{1}{N-k}\sum_{j=1}^{N-k}S_{t_{k+j}}\right)^2\,\middle|\, W_{t_k}\right]\\
&= \frac{1}{(N-k)^2}\sum_{j=1}^{N-k}\left(\mathbb{E}\left[S_{t_{k+j}}^2\,\middle|\,W_{t_k}\right]+2\sum_{i=1}^{j-1}\mathbb{E}\left[S_{t_{k+j}}S_{t_{k+i}}\mid W_{t_k}\right]\right)\\
&= \frac{1}{(N-k)^2}\sum_{j=1}^{N-k}\left\{\mathbb{E}\left[S_{t_k}^2\exp\left[2\left(\mu-\frac{1}{2}\sigma^2\right)(t_{k+j}-t_k)+2\sigma W_{t_{k+j}-t_k}\right]\right]\right.\\
&\quad+2\sum_{i=1}^{j-1}\mathbb{E}\Big[S_{t_k}^2\exp\Big[\big(\mu-\frac{1}{2}\sigma^2\big)(t_{k+j}-t_k+t_{k+i}-t_k)\\
&\quad\left.+\sigma(W_{t_{k+j}-t_k}+W_{t_{k+i}-t_k})\Big]\Big]\right\}\\
&= \left(\frac{S_{t_k}}{N-k}\right)^2\sum_{j=1}^{N-k}\left\{\exp\left[2\left(\mu-\frac{1}{2}\sigma^2\right)(t_{k+j}-t_k)\right]\mathbb{E}\left[\exp\left[2\sigma W_{t_{k+j}-t_k}\right]\right]\right.\\
&\quad+2\sum_{i=1}^{j-1}\exp\Big[\big(\mu-\frac{1}{2}\sigma^2\big)(t_{k+j}+t_{k+i}-2t_k)\Big]\\
&\quad\left.\times\mathbb{E}\Big[\exp\Big[\sigma(W_{t_{k+j}-t_k}+W_{t_{k+i}-t_k})\Big]\Big]\right\}\\
&= \left(\frac{S_{t_k}}{N-k}\right)^2\sum_{j=1}^{N-k}\left\{\exp\left[2\left(\mu-\frac{1}{2}\sigma^2\right)(t_{k+j}-t_k)+2\sigma^2(t_{k+j}-t_k)\right]\right.\\
&\quad+2\sum_{i=1}^{j-1}\exp\Big[\big(\mu-\frac{1}{2}\sigma^2\big)(t_{k+j}+t_{k+i}-2t_k)\Big]\\
&\quad\left.\times\mathbb{E}\Big[\exp\Big[\sigma(W_{t_{k+j}-t_{k+i}}+2W_{t_{k+i}-t_k})\Big]\Big]\right\}\\
&= \left(\frac{S_{t_k}}{N-k}\right)^2\sum_{j=1}^{N-k}\left\{\exp\left[\left(2\mu+\sigma^2\right)(t_{k+j}-t_k)\right]\right.\\
&\quad+2\sum_{i=1}^{j-1}\exp\Big[\big(\mu-\frac{1}{2}\sigma^2\big)(t_{k+j}+t_{k+i}-2t_k)\\
&\quad\left.+\frac{1}{2}\sigma^2(t_{k+j}-t_{k+i})+2\sigma^2(t_{k+i}-t_k)\Big]\right\}\\
&= \left(\frac{S_{t_k}}{N-k}\right)^2\sum_{j=1}^{N-k}\Big\{\exp\left[\left(2\mu+\sigma^2\right)(t_{k+j}-t_k)\right]
\end{aligned}
$$

$$+ 2\sum_{i=1}^{j-1}\exp\left[\mu(t_{k+j}+t_{k+i}-2t_k)+\sigma^2(t_{k+i}-t_k)\right]\Bigg\} \quad \text{a.s.} \; , \tag{4.4.7}$$

where the independence of W_s and W_{t-s} provides

$$W_s + W_t \sim W_s + [W_s + W_{t-s}] = 2W_s + W_{t-s} \quad , \; t > s \quad .$$

However, the moments for $[t_0, t_k]$ are a little bit more difficult to derive as a repeated use of the Brownian Bridge is essential. In the following calculations, the result of (3.1.2) is used multiple times:

$$\begin{aligned}
\mathbb{E}[A(t_0,t_k) \mid W_{t_k}] &= \mathbb{E}\left[\frac{1}{k+1}\sum_{j=0}^{k} S_{t_j} \;\middle|\; W_{t_k}\right] \\
&= \frac{S_0}{k+1}\sum_{j=0}^{k}\exp\left[\left(\mu-\frac{1}{2}\sigma^2\right)t_j\right]\,\mathbb{E}\left[\exp\left[\sigma W_{t_j}\right] \mid W_{t_k}\right] \\
&= \frac{S_0}{k+1}\sum_{j=0}^{k}\exp\left[\left(\mu-\frac{1}{2}\sigma^2\right)t_j\right]\,\exp\left[\frac{\sigma^2 t_j}{2}\left(1-\frac{t_j}{t_k}\right)+\frac{\sigma t_j W_{t_k}}{t_k}\right] \\
&= \frac{S_0}{k+1}\sum_{j=0}^{k}\exp\left[\mu t_j - \frac{\sigma^2 t_j^2}{2t_k}+\frac{\sigma t_j W_{t_k}}{t_k}\right]
\end{aligned} \tag{4.4.8}$$

and

$$\begin{aligned}
\mathbb{E}[A(t_0,t_k)^2 \mid W_{t_k}] &= \mathbb{E}\left[\left(\frac{1}{k+1}\sum_{j=0}^{k} S_{t_j}\right)^2 \;\middle|\; W_{t_k}\right] \\
&= \left(\frac{S_0}{k+1}\right)^2\sum_{j=0}^{k}\Bigg\{\exp\left[\left(\mu-\frac{1}{2}\sigma^2\right)2t_j\right]\,\mathbb{E}\big[\exp\left[2\sigma W_{t_j}\right] \mid W_{t_k}\big] \\
&\quad + 2\sum_{i=0}^{j-1}\exp\left[\left(\mu-\frac{1}{2}\sigma^2\right)(t_i+t_j)\right]\,\mathbb{E}\big[\exp\left[\sigma\left(W_{t_i}+W_{t_j}\right)\right] \mid W_{t_k}\big]\Bigg\} \\
&= \left(\frac{S_0}{k+1}\right)^2\sum_{j=0}^{k}\Bigg\{\exp\left[\left(\mu-\frac{1}{2}\sigma^2\right)2t_j\right]\,\exp\left[2\sigma^2 t_j\left(1-\frac{t_j}{t_k}\right)+\frac{2\sigma t_j W_{t_k}}{t_k}\right] \\
&\quad + 2\sum_{i=0}^{j-1}\exp\left[\left(\mu-\frac{1}{2}\sigma^2\right)(t_i+t_j)\right]\,\mathbb{E}\big[\exp\left[\sigma\left(2W_{t_i}+W_{t_j-t_i}\right)\right] \mid W_{t_k}\big]\Bigg\} \\
&= \left(\frac{S_0}{k+1}\right)^2\sum_{j=0}^{k}\Bigg\{\exp\left[\left(\mu-\frac{1}{2}\sigma^2\right)2t_j\right]\,\exp\left[2\sigma^2 t_j\left(1-\frac{t_j}{t_k}\right)+2\sigma\frac{t_j W_{t_k}}{t_k}\right] \\
&\quad + 2\sum_{i=0}^{j-1}\exp\left[\left(\mu-\frac{1}{2}\sigma^2\right)(t_i+t_j)\right]\,\mathbb{E}\big[\exp\left[2\sigma W_{t_i}\right] \mid W_{t_k}\big] \\
&\quad \times \mathbb{E}\big[\exp\left[\sigma W_{t_j-t_i}\right] \mid W_{t_k}\big]\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{S_0}{k+1}\right)^2 \sum_{j=0}^{k} \Bigg\{ \exp\left[\left(2\mu+\sigma^2\right) t_j + 2\sigma^2 \frac{t_j^2}{t_k} + 2\sigma \frac{t_j W_{t_k}}{t_k}\right] \\
&\quad + 2 \sum_{i=0}^{j-1} \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)(t_i + t_j)\right] \exp\left[2\sigma^2\left(1 - \frac{t_i}{t_k}\right) - 2\sigma \frac{W_{t_k} t_i}{t_k}\right. \\
&\quad \left. + \sigma^2 \frac{t_j - t_i}{2}\left(1 - \frac{t_j - t_i}{t_k}\right) + \sigma W_{t_k} \frac{t_j - t_i}{t_k}\right]\Bigg\} \\
&= \left(\frac{S_0}{k+1}\right)^2 \sum_{j=0}^{k} \Bigg\{ \exp\left[\left(2\mu+\sigma^2\right) t_j + 2\sigma \frac{t_j}{t_k}\left(\sigma t_j + W_{t_k}\right)\right] \\
&\quad + 2 \sum_{i=0}^{j-1} \exp\left[\mu(t_i + t_j) + \sigma^2 t_i - \frac{\sigma^2}{t_k}\left(2t_i^2 + \frac{(t_j - t_i)^2}{2}\right) + \sigma \frac{W_{t_k}}{t_k}(t_i + t_j)\right]\Bigg\} \quad \text{.a.s.}
\end{aligned} \tag{4.4.9}
$$

Let $\psi = 1$, meaning that a call option on the average is considered. As $U \perp\!\!\!\perp Z$,

$$
\begin{aligned}
\mathrm{VaF}(t_k, S_{t_k}; 1) &= \frac{B_{t_k}}{B_T}\, \mathbb{E}\left[\max\left(0, \frac{k+1}{N+1} A(t_0, t_k) + \frac{N-k}{N+1} A(t_{k+1}, t_N) - K\right) \,\middle|\, W_{t_k}\right] \\
&\approx \frac{B_{t_k}}{B_T}\, \mathbb{E}\left[\max\left(0, \frac{k+1}{N+1} \tilde{A}(t_0, t_k; Z) + \frac{N-k}{N+1} \tilde{A}(t_{k+1}, t_N; U) - K\right)\right] \\
&= \exp\left[-r(T - t_k)\right] \int_{\mathbb{R}} \int_{\mathbb{R}} \max\left(0, \frac{k+1}{N+1} \exp\left[M(t_0, t_k) + \sqrt{V(t_0, t_k)}\, z\right]\right. \\
&\quad \left. - \left(K - \frac{N-k}{N+1} \exp\left[M(t_{k+1}, t_N) + \sqrt{V(t_{k+1}, t_N)}\, u\right]\right)\right) \phi(z)\phi(u)\, dz\, du
\end{aligned}
$$

In order to get rid of the maximum function, the integral is split up at the point where

$$
\begin{aligned}
&K - \frac{N-k}{N+1} \exp\left[M(t_{k+1}, t_N) + \sqrt{V(t_{k+1}, t_N)}\, u\right] = 0 \Leftrightarrow \\
&u = \frac{\log\left[\frac{K(N+1)}{N-k}\right] - M(t_{k+1}, t_N)}{\sqrt{V(t_{k+1}, t_N)}} =: d_u
\end{aligned}
$$

and restricted with an indicator function on the set

$$
\begin{aligned}
&\left\{\frac{k+1}{N+1} \exp\left[M(t_0, t_k) + \sqrt{V(t_0, t_k)}\, z\right] > K - \frac{N-k}{N+1} \exp\left[M(t_{k+1}, t_N) + \sqrt{V(t_{k+1}, t_N)}\, u\right]\right\} \Leftrightarrow \\
&\left\{ z \geq \frac{\log\left[\frac{(N+1)\left(K - \frac{N-k}{N+1}\exp\left[M(t_{k+1}, t_N) + \sqrt{V(t_{k+1}, t_N)}\, u\right]\right)}{k+1}\right] - M(t_0, t_k)}{\sqrt{V(t_0, t_k)}} =: -d_z(u) \right\}
\end{aligned}
$$

Thus,

$$
\mathrm{VaF}(t_k, S_{t_k}; 1) \approx \exp\left[-r(T - t_k)\right] \Bigg\{ \int_{-\infty}^{d_u} \int_{-\infty}^{\infty} \max\left(0, \frac{k+1}{N+1} \exp\left[M(t_0, t_k) + \sqrt{V(t_0, t_k)}\, z\right]\right.
$$

$$- \Big(K - \frac{N-k}{N+1} \exp\Big[M(t_{k+1}, t_N) + \sqrt{V(t_{k+1}, t_N)}\ u\Big]\Big)\Big) \phi(z)\phi(u)\ dz\ du$$
$$+ \int_{d_u}^{\infty} \int_{-\infty}^{\infty} \Big(\frac{k+1}{N+1} \exp\Big[M(t_0, t_k) + \sqrt{V(t_0, t_k)}\ z\Big]$$
$$- \Big(K - \frac{N-k}{N+1} \exp\Big[M(t_{k+1}, t_N) + \sqrt{V(t_{k+1}, t_N)}\ u\Big]\Big)\Big) \phi(z)\phi(u)\ dz\ du\Big\}$$

The first double integral is calculated as follows:

$$\int_{-\infty}^{d_u} \int_{-\infty}^{\infty} \max\Big(0, \frac{k+1}{N+1} \exp\Big[M(t_0, t_k) + \sqrt{V(t_0, t_k)}\ z\Big]$$
$$- \Big(K - \frac{N-k}{N+1} \exp\Big[M(t_{k+1}, t_N) + \sqrt{V(t_{k+1}, t_N)}\ u\Big]\Big)\Big) \phi(z)\phi(u)\ dz\ du =$$
$$\int_{-\infty}^{d_u} \int_{-\infty}^{\infty} \Big(\frac{k+1}{N+1} \exp\Big[M(t_0, t_k) + \sqrt{V(t_0, t_k)}\ z\Big]$$
$$- \Big(K - \frac{N-k}{N+1} \exp\Big[M(t_{k+1}, t_N) + \sqrt{V(t_{k+1}, t_N)}\ u\Big]\Big)\Big) \mathbb{1}_{\{-z \le d_z(u)\}}\ \phi(z)\phi(u)\ dz\ du =$$
$$\int_{-\infty}^{d_u} \int_{-\infty}^{d_z(u)} \Big(\frac{k+1}{N+1} \exp\Big[M(t_0, t_k) - \sqrt{V(t_0, t_k)}\ z\Big]$$
$$- \Big(K - \frac{N-k}{N+1} \exp\Big[M(t_{k+1}, t_N) + \sqrt{V(t_{k+1}, t_N)}\ u\Big]\Big)\Big) \phi(z)\phi(u)\ dz\ du =$$
$$\int_{-\infty}^{d_u} \Big(\frac{k+1}{N+1} \exp\Big[M(t_0, t_k) + \frac{V(t_0, t_k)}{2}\Big] \int_{-\infty}^{d_z(u)} \phi\big(z + \sqrt{V(t_0, t_k)}\big)\ dz$$
$$- \Big(K - \frac{N-k}{N+1} \exp\Big[M(t_{k+1}, t_N) + \sqrt{V(t_{k+1}, t_N)}\ u\Big]\Big) \Phi\big(d_z(u)\big)\Big) \phi(u)\ du =$$
$$\int_{-\infty}^{d_u} \Big(\frac{k+1}{N+1} \exp\Big[M(t_0, t_k) + \frac{V(t_0, t_k)}{2}\Big] \Phi\big(d_z(u) + \sqrt{V(t_0, t_k)}\big)$$
$$- \Big(K - \frac{N-k}{N+1} \exp\Big[M(t_{k+1}, t_N) + \sqrt{V(t_{k+1}, t_N)}\ u\Big]\Big) \Phi\big(d_z(u)\big)\Big) \phi(u)\ du \quad .$$

Apparently, the integral does not have any closed form, thus has to be calculated numerically with R. The second integral, however, has the following explicit form:

$$\int_{d_u}^{\infty} \int_{-\infty}^{\infty} \Big(\frac{k+1}{N+1} \exp\Big[M(t_0, t_k) + \sqrt{V(t_0, t_k)}\ z\Big]$$
$$- \Big(K - \frac{N-k}{N+1} \exp\Big[M(t_{k+1}, t_N) + \sqrt{V(t_{k+1}, t_N)}\ u\Big]\Big)\Big) \phi(z)\phi(u)\ dz\ du =$$
$$\int_{d_u}^{\infty} \Big(\frac{k+1}{N+1} \exp\Big[M(t_0, t_k) + \frac{V(t_0, t_k)}{2}\Big]$$
$$- \Big(K - \frac{N-k}{N+1} \exp\Big[M(t_{k+1}, t_N) + \sqrt{V(t_{k+1}, t_N)}\ u\Big]\Big)\Big) \phi(u)\ du =$$

$$\frac{k+1}{N+1}\exp\left[M(t_0,t_k)+\frac{V(t_0,t_k)}{2}\right]\Phi(-d_u)-K\Phi(-d_u)$$
$$+\frac{N-k}{N+1}\exp\left[M(t_{k+1},t_N)+\frac{V(t_{k+1},t_N)}{2}\right]\Phi\big(-d_u+\sqrt{V(t_{k+1},t_N)}\big)$$

Hence, value-at-future of the approximated arithmetic average call option is

$$\begin{aligned}
\mathrm{VaF}(t_k,S_{t_k};1)\approx\exp\left[-r(T-t_k)\right]\Bigg\{ & \\
&\int_{-\infty}^{d_u}\Bigg(\frac{k+1}{N+1}\exp\left[M(t_0,t_k)+\frac{V(t_0,t_k)}{2}\right]\Phi\big(d_z(u)+\sqrt{V(t_0,t_k)}\big)\\
&\quad-\Big(K-\frac{N-k}{N+1}\exp\Big[M(t_{k+1},t_N)+\sqrt{V(t_{k+1},t_N)}\,u\Big]\Big)\,\Phi\big(d_z(u)\big)\Bigg)\,\phi(u)\,du\\
&+\Bigg(\frac{k+1}{N+1}\exp\left[M(t_0,t_k)+\frac{V(t_0,t_k)}{2}\right]-K\Bigg)\,\Phi(-d_u)\\
&+\frac{N-k}{N+1}\exp\left[M(t_{k+1},t_N)+\frac{V(t_{k+1},t_N)}{2}\right]\Phi\big(-d_u+\sqrt{V(t_{k+1},t_N)}\big)\Bigg\}
\end{aligned}\tag{4.4.10}$$

For an according put option, meaning that $\psi=-1$, similar steps and calculations can be applied, resulting in

$$\begin{aligned}
\mathrm{VaF}(t_k,S_{t_k};-1)\approx\exp\left[-r(T-t_k)\right]\Bigg\{ & \\
&\int_{-\infty}^{d_u}\int_{-\infty}^{\infty}\max\Bigg(0,\Big(K-\frac{N-k}{N+1}\exp\Big[M(t_{k+1},t_N)+\sqrt{V(t_{k+1},t_N)}\,u\Big]\Big)\\
&\quad-\frac{k+1}{N+1}\exp\Big[M(t_0,t_k)+\sqrt{V(t_0,t_k)}\,z\Big]\Bigg)\,\phi(z)\phi(u)\,dz\,du\Bigg\}\\
=\exp\left[-r(T-t_k)\right]\Bigg\{ & \\
&\int_{-\infty}^{d_u}\int_{d_z(u)}^{\infty}\Bigg(\Big(K-\frac{N-k}{N+1}\exp\Big[M(t_{k+1},t_N)-\sqrt{V(t_{k+1},t_N)}\,u\Big]\Big)\\
&\quad-\frac{k+1}{N+1}\exp\Big[M(t_0,t_k)+\sqrt{V(t_0,t_k)}\,z\Big]\Bigg)\,\phi(z)\phi(u)\,dz\,du\Bigg\}\\
=\exp\left[-r(T-t_k)\right]\Bigg\{ & \\
&\int_{-\infty}^{d_u}\Bigg(\Big(K-\frac{N-k}{N+1}\exp\Big[M(t_{k+1},t_N)+\sqrt{V(t_{k+1},t_N)}\,u\Big]\Big)\,\Phi\big(-d_z(u)\big)\\
&\quad-\frac{k+1}{N+1}\exp\left[M(t_0,t_k)+\frac{V(t_0,t_k)}{2}\right]\,\Phi\big(-d_z(u)-\sqrt{V(t_0,t_k)}\big)\Bigg)\,\phi(u)\,du\Bigg\}\quad.
\end{aligned}\tag{4.4.11}$$

Again, a closed form of which the values can be calculated analytically does not exist. Therefore, the value-at-future and hence, the exposure profiles again have to be calculated numerically.

The next section deals with a small, simplified example of the application of these formulas. It is going to show the steps in order to implement an Asian call option into R and calculating its potential future exposures.

4.4.2 Implementation of Asian Options

Now that the valuation formulas are obtained, a small example of an Asian option with arithmetic average should illustrate the exposure evaluation over time. Package wise, the sde package is used again in order to generate the geometric Brownian motion

```
library(sde)     # load sde package
```

As it is assumed that Consider an Asian call option on a stock with initial price S_0 = 100\$, parameters μ = 0.05 and σ = 0.1, strike K = 100\$, maturity T = 1 and a weekly averaging frequency. Besides, the risk-free interest rate is considered to be r = 0.02. In R, the variables are defined as follows:

```
mu<-0.05          # drift term
r<-0.02           # risk-free interest rate
sigma<-0.1        # volatility
T<-1              # maturity
S0<-100           # initial stock value in dollars
K<-100            # strike in dollars
```

Like simulating the barrier option in section 4.2.2, the the first step is to set the number M of scenarios generated and the number of time intervals N used in the process of discretizing the time to maturity T. In case of the Asian option, M = 10000 scenarios with a 7/360 day count conversion are generated.

The discrete time grid is generally not required to be equidistant, however, it is assumed for this example. Hence, calculating the arithmetic average on a weekly basis,

```
N<-floor(360/7) *T                  # number of time intervals
t<-seq(0, N*7/360,by=7/360)         # day count conversion = 7/360
```

The next step is to generate the stock price process up to T at every time node t_k, $k \in \{1, 2, \ldots, N\}$. The function `GBM` of the sde package, which generates the path of a geometric Brownian motion with the given input parameters, is used in order to do so. Since the indexing in R starts at 1, $0 = t_0 \mathrel{\hat{=}}$ `t[1]`, $t_1 \mathrel{\hat{=}}$ `t[2]`, … $t_N \mathrel{\hat{=}}$ `t[N+1]`. The same is true for the stock price process generated by the `GBM` function, starting with $S_0 \mathrel{\hat{=}}$ `S[1]` and ending with $S_{t_n} \mathrel{\hat{=}}$ `S[N+1]`:

```
S<-GBM(S0,mu,sigma,1,N)                       # stock price
X<-(log(S/S0)-(mu-1/2*sigma^2)*t)/sigma      # according BM
```

The next step is to calculated the parameters of approximations (4.4.4) and (4.4.4). Therefore, the first two moments conditioned on the simulated stock price have to be calculated, which are implemented according to the results of section 4.4.1:

```
for (k in (1:(N+1))) {

  # 1st moment of A(t_1,t_k) | W_{t_k}
  if (k==1){ M1[1,k] <- S0/k*exp(sigma*X[k])  } else {
  M1[1,k] <- S0/(k)* (sum(exp(mu*t[1:(k)] - sigma^2*t[1:(k)]^2/
     (2*t[k]) + sigma * t[1:(k)]*X[k]/t[k])))
  }

  # 1st moment of A(t_{k+1},t_N) | W_{t_k}
  if (k==(N+1)) { M1[2,k] <- 0 }
  else {
    M1[2,k] <- S[k]/(N-k+1) * sum(exp(mu*(t[(k+1):(N+1)] - t[k
       ])))
  }
```

```r
# 2st moment of A(t_1,t_k) | W_{t_k}
if (k==1) {M2[1,k]<- (S0/(k))^2 }
# else if (k==2) {
#   M2[1,k] <- (S0/(k))^2*(1+(exp((2*mu + sigma^2)*t[1] + 2 *
    sigma*t[1]/t[k] * (sigma*t[1] + X[k])))+2*exp(mu*t[k] -
   sigma^2*t[k]/2+sigma*X[k])  )}
   else {
  #j=1
  M2[1,k] <- (S0/(k))^2*(1+(exp((2*mu + sigma^2)*t[1] + 2 *
     sigma*t[1]/t[k] * (sigma*t[1] + X[k]))+2*exp(mu*t[1] -
     sigma^2*t[1]/t[k]/2+sigma*X[k]/t[k]*t[1])  ) )
  for (j in (2:k)){
    M2[1,k]<- M2[1,k] + (S0/(k))^2*(exp((2*mu + sigma^2)*t[j]
        + 2*sigma*t[j]/t[k] * (sigma*t[j] + X[k])) + 2*sum(
       exp(mu*(t[1:(j-1)] + t[j]) + sigma^2 * t[1:(j-1)] -
       sigma^2/t[k]*(2*t[1:(j-1)]^2 + (t[j]-t[1:(j-1)])^2/2 )
        + sigma*X[k]/t[k]*(t[1:(j-1)]+t[j]) )) )
  }
}

# 2st moment of A(t_{k+1},t_N) | W_{t_k}
if (k==(N)){
  M2[2,k]<-(S[k]/(N-k+1))^2 * (exp((2*mu + sigma^2)*(t[k+1]-t
     [k])))}
   # else if (k==N){M2[2,k]<-(S[k]/(N-k+1))^2   }
   else if (k==(N+1)) {}
else {
  #j=1
  M2[2,k]<-(S[k]/(N-k+1))^2 * (exp((2*mu + sigma^2)*(t[k+1]-t
     [k])))
  for (j in (2:(N-k+1))){
```

```
        M2[2,k]<-M2[2,k] + (S[k]/(N-k+1))^2 * (exp((2*mu + sigma^
            2)*(t[k+j]-t[k])) + 2*sum(exp(mu*(t[(k+1):(k+j-1) ] +
            t[k+j] -2*t[k]) + sigma^2*(t[(k+1):(k+j-1)] - t[k])))
          )
      }
    }
  }
```

These moments are eventually applied on the formulas for calculating the parameters of the approximations, which were derived in the last subsection.

```
# parameters of approximations
Mz <- 2*log(M1)-1/2*log(M2)
Vz <- log(M2)-2*log(M1)
```

Finally, the value-at-future of the Asian call option can be calculated for each t_k, $k \in \{0,1,2,\ldots,N\}$. As discussed in the last section, (4.4.10) can merely be derived numerically, because no closed form of the integral of the cumulative normal distribution function exists. Additionally, the value-at-future is transformed to exposures by setting its negative part to zero.

```
for (k in (1:(N))){
    VaF[m,k]<-exp(-r*(T-t[k]))*(integrate(function(u) {(((k)/(N
       +1))*exp(Mz[1,k] + Vz[1,k]/2)* pnorm(-1*(log((N+1)*(K-(N
       -k+1)/(N+1)*exp(Mz[2,k]+ sqrt(Vz[2,k])*u ))/k)-Mz[1,k] )
       /sqrt(Vz[1,k]) + sqrt(Vz[1,k])) - (K-(N-k+1)/(N+1) * exp
       (Mz[2,k] + sqrt(Vz[2,k]) *u ) )*pnorm(-1*(log((N+1)*(K-(
       N-k+1)/(N+1)*exp(Mz[2,k]+ sqrt(Vz[2,k])*u ))/k)-Mz[1,k]
       )/sqrt(Vz[1,k]))) * dnorm(u)  },lower=-10, upper=(log(K*
       (N+1)/(N-k+1) )-Mz[2,k] )/sqrt(Vz[2,k]))$val + ((k)/(N
       +1)*exp(Mz[1,k]+ Vz[1,k]/2)-K) *pnorm(-1*(log(K*(N+1)/(N
       -k+1) )-Mz[2,k] )/sqrt(Vz[2,k])) + (N-k+1)/(N+1)*exp(Mz
       [2,k] + Vz[2,k]/2)*pnorm(-1*(log(K*(N+1)/(N-k+1) )-Mz[2,
       k] )/sqrt(Vz[2,k]) +sqrt(Vz[2,k])))
```

```
}
VaF[m,N+1] <- exp(Mz[1,N+1] + (Vz[1,N+1])/2 )* pnorm( -1*((log(
   K)-Mz[1,N+1])/sqrt(Vz[1,N+1]) ) + sqrt(Vz[1,N+1]) )-K *
   pnorm(-1*((log(K)-Mz[1,N+1])/sqrt(Vz[1,N+1]) ))
  VaF[j,(which(VaF[j,]<0))]  <-0     # replacing negative
     values with zeros
```

Obviously, this has to be done for all $min\{1, 2, \ldots, 10000$. Finally, the exposure profiles are calculated. Again, the quantiles for the PFEs are set to 5%, 50% and 95%.

```
q<-apply(VaF, 2, quantile, probs = c(0.05, 0.5, 0.95),  na.rm
    = TRUE)              # PFEs
```

Plot 4.17 shows the results of this implementation with $\mu = 0.05$ and $r = 0.02$. Comparing its exposure profiles with those of a vanilla call option with the same parameters and same maturity $T = 1$, which can be seen in figure 4.18, the exposure profile of the Asian option apparently has a lower peak exposure, hitting its highest points before maturity, whereas a vanilla option peaks way at the end of the tenor. This concave profile is the result of the gaining knowledge over the course of time that effects the value of an Asian option more than that of a vanilla one. They start out with the same uncertainty of the payoff at maturity, however, whereas the vanilla option solely rely on the last value at maturity T and therefore is very vulnerable for "surprising" changes within a short period of time, it is harder for an Asian option to make big changes the more the stock price progressed up to maturity. Assuming that the Asian option is in-the-money well within its tenor, it will be more likely for it to stay in-the-money due to its nature of taking the average over the whole path of the stock price evolution than for a plain vanilla option. The more is known about the underlying over time, the lower the uncertainty of the payoff of the path-dependent Asian option is and hence, its exposure.

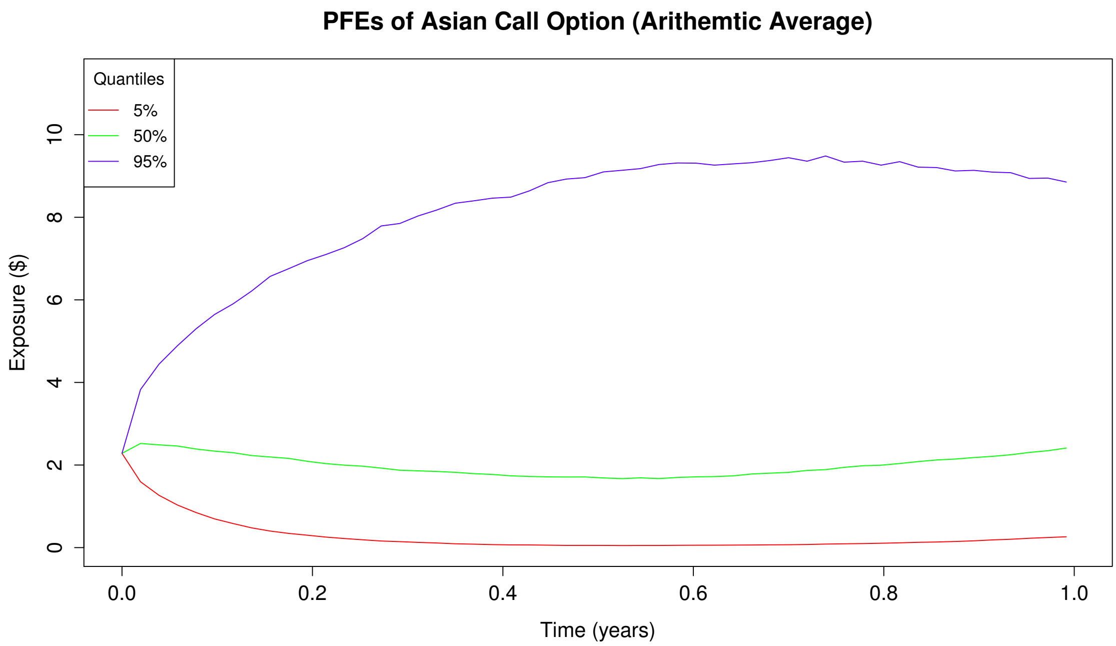

Figure 4.17: PFEs of Asian call option with $T = 1$

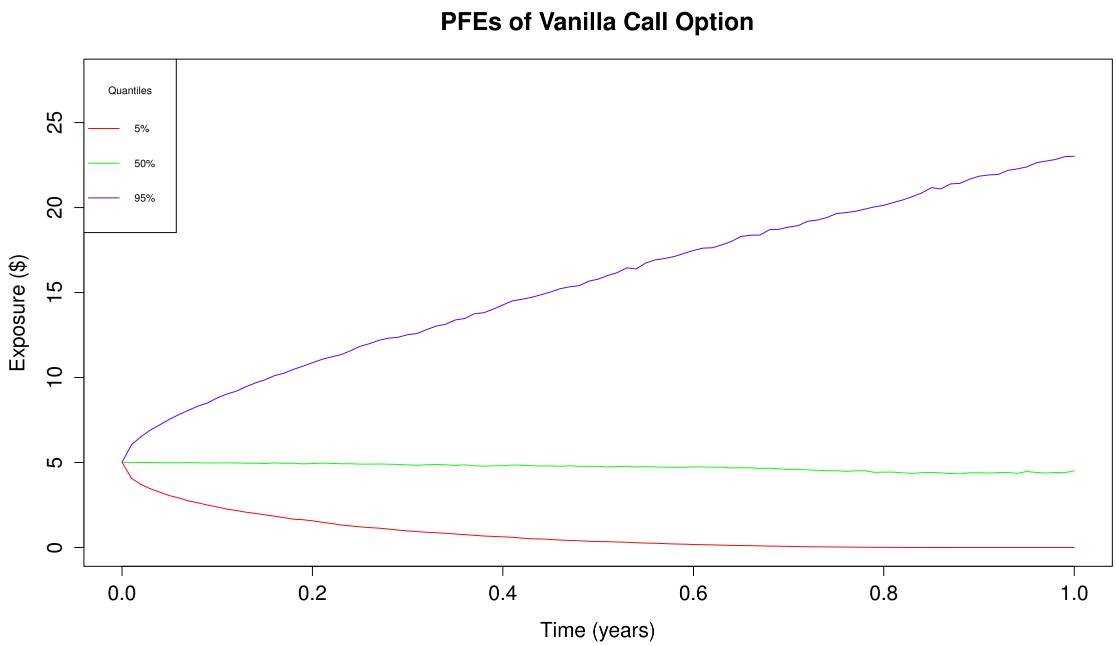

Figure 4.18: PFEs of vanilla call option with $T = 1$

5 Conclusion

This thesis started out to take the work of Lomibao and Zhu [2005] as a basis for a gradual enhancement of the existing knowledge in order to both list more precise and slicker calculations and apply the model on other path-dependent derivatives. However, during the analysis of the paper and consulting its references, more and more flaws of the basic idea, as well as its application on the chosen financial derivatives have emerged.

Generally, the paper appears to be considerable inaccurate in its assumptions, nomenclature and implementation. Often the authors take conditions and assumptions for granted without explaining or even mentioning, which, combined with their often misleading naming of variables and functions, and the lack of references, leads to a very vague picture of their intentions. A deeper examination has often revealed severe flaws of their ideas which makes further calculations obsolete.

Two out of three derivatives that are discussed in detail are simply incorrect, even though simplifying assumptions have already been taken. Where the barrier option lacks any precise listing of the steps of calculation, which are necessary to understand the model, the idea behind the swap-settled swaption to calculate exposures after the exercise date is plainly wrong. If there was a way to derive those additional exposures, it would certainly not work with the use of the forward swap rate and the log-normal forward-swap model. Therefore, the concept of the Brownian bridge cannot be applied on this financial instrument and the calculated exposure profiles equal those of a cash-settled swaption, unless a more refined approach to derive the exposures connected to the underlying swap is developed.

Finally, the concept is applied on the Asian option: Unfortunately, the readers of Lomibao and Zhu [2005] are firstly confused by their so-called $\sqrt{3}$-rule, which does not seem to be from any importance for the rest of the calculations. Additionally, they often forget to take "todays" stock price into account which results in a wrong weighting of the split arithmetic averages and often do not define their functions and variables correctly or intuitively. Although it is plausible to approximate the sum of log-normal distributed random variables with another log-normal distributed random variable (see Fenton [1960]), their approach to assume that the approximation of the average before simulation date t_k and that of the average after t_k are correlated is not adequate. That is the main mistake they make in their section about Asian options, hence, their results greatly differ from those of section 4.4.2 in this thesis.

Another inaccuracy within the paper is the inconsistence regarding discounting and choosing the probability measure. On the one hand they properly use the right numeraire when calculating the swap-settled swaption under the forward swap measure. On other occasions, they do not integrate the risk-free bonds for discounting while calculating mark-to-market values, but eventually use Black-Scholes formulas that strictly rely on risk-neutral valuation. In general, they never state which measure they actually use for calculating the expected values, which is crucial for the reliability of the results. That and the fact that Lomibao and Zhu [2005] never declares the parameter they use for plotting their potential future exposure quantiles makes the results difficult to compare and forces anyone that tries to mirror their results to guess which parameters they could have used.

Although the idea of Lomibao and Zhu [2005] to use the Brownian bridge in oder to capture the uncertainty of a risk factor evolution over a certain time is a highly interesting one, all in all, their execution lacks a little bit of preciseness and sophistication. As the main goal of this thesis is to prove the correctness of the model and provide additional mathematical background, the results listed in the sections above hopefully give a better understanding of the topic of evaluating exposures of path-dependent derivatives via Monte-Carlo simulation. Obviously, no one is freed from any mistake and this thesis is not be an exception. It should also be stated that the work of Lomibao and Zhu [2005], though having some flaws, is highly appreciated and greatly helped to calculate the exposure profiles listed the last chapters. The reader should use this thesis to make himself aware of the difficulty and problems that occur when trying to quantify probabilities of certain trajectories of general Brownian motions while only knowing a single future value.
Since OTC-derivatives are going to continue to play an important role in the financial market, the models to capture their risk are going to become more sophisticated during the following years. The more exotic the derivatives are, the more difficult are they to evaluate, which can easily be seen in this thesis. Although the approach of Lomibao and Zhu is limited in its way to capture certain developments - take a look at swap-settled swaptions for instance - it certainly is a good starting point to develop a procedure that can calculate future exposures more reliably. Hopefully, this thesis has helped to give a good presentation of these problems in order to find a way to apply the concept on other financial derivatives as well.

References

[1] Basel Committee on Banking Supervision [2005]: *The Application of Basel II to Trading Activities and Treatment of Double Default Effects.* Basel Committee on Banking Supervision; www.bis.org

[2] Dimitri P. Bertsekas and John N. Tsitsiklis [2002]: *Introduction to Probability.* Athena Scientific

[3] Damiano Brigo and Fabio Mercurio [2001]: *Interest Rate Models - Theory and Practice.* 2nd Edition (2006), Springer

[4] Emannuel Buffet [2003]: *On the Time of the Maximum of Brownian Motion with Drift.* Journal of Applied Mathematics and Stochastic Analysis, 16:3 (2003)

[5] Chang-Chih Chen, Chou-Wen Wang and David Shyu [2010]: *Pricing Double Barrier Options Using Reflection Principle.* International Research Journal of Finance and Economics; EuroJournals Publishing

[6] Ron S. Demob, Andrew R. Aziz, Dan Rosen and Michael Zerbs [2000]: *Mark to Future: A framework for measuring risk and reward.* Algorithmics Publications

[7] L. F. Fenton [1960]: *The Sum of Lognormal Probability Distributions in Scatter Transmission Systems.* IRE Trans. Commun. Syst. 8

[8] Jon Gregory [2010]: *Counterparty Credit Risk.* John Wiley & Sons Ltd.

[9] John C. Hull [2009]: *Options, Futures and Other Derivatives.* Pearson Studium

[10] Stefano M. Iacus [2008]: Simulation and Inference for Stochastic Differential Equations: With R Examples. Springer

[11] Ioannis Karatzas and Steven E. Shreve [1988]: *Brownian Motion and Stochastic Calculus.* Springer-Verlag

[12] Paul Pierre Lévy [1948]: *Processus Stochastiques et Mouvement Brownien.* Gauthier-Villars, Paris

[13] Dante Lomibao and Steven Zhu [2005]: *A Conditional Valuation Approach for Path-Dependent Instruments.* Wilmott Magazine, 11/04/2012

[14] David Meintrup and Stefan Schäffler [2005]: *Stochastic: Theory und Anwendung.* Springer

[15] Marek Musiela and Marek Rutkowski [2005]: *Martingale Methods in Financial Modelling.* Springer

[16] Berndt Øksendal [2007]: *Stochastic Differential Equations.* Springer

[17] Ben De Prisco and Dan Rosen [2005]: *Modelling Stochastic Counterparty Credit Exposure for Derivatives Portfolios.* In Counterparty Credit Risk Modelling, edited by Michael Pykhtin, Risk Books

[18] S. Schwartz and Y. Yeh [1981]: *On the Distribution Function and Moments of Power Sum with Lognormal Components.* Bell Syst. Tech. J., vol. 61

[19] Larry Shepp [1979]: *The Joint Densitiy of the Maximum and Its Location for a Wiener Process with Drift.* Journal of Applied Probability, Vol. 16, No. 2 (Jun., 1979)

[20] Damiano Brigo and Fabio Mercurio [2001]: *Interest Rate Models - Theory and Practise.* 2nd edition (2006),Springer