DIPLOMARBEIT

Barrier options and their Application to Structure Floors

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Statutory Declaration

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Abstract

We determine the price of digital double barrier options with an arbitrary number of barrier periods in the Black-Scholes model. This means that the barriers are active during some time intervals, but are switched off in between.

As an application, we calculate the value of a structure floor for structured notes whose individual coupons are digital double barrier options. This value can also be approximated by the price of a corridor put. We also address the issue which arises when using Monte Carlo simulation to price a barrier option, namely the discretization bias inherent when using a discrete setting in a continuously monitored model.

This work is largely based on the paper *Digital Double Barrier Options: Several Barrier Periods and Structure Floors* of Altay, Gerhold, Haidinger and Hirager, published in the *International Journal of Theoretical and Applied Finance*, Volume 16, 2013.[1] which the author has co-authored.

keywords: Double barrier option, digital option, binary option, structure oor, occupation time, corridor option

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1 Preface

Options with some kind of barrier feature have become quite popular in recent years, especially in foreign exchange (FX) markets. Barrier options are path-dependent, their price depends on whether the underlying has touched a barrier during the life of the option.
They are usually structured as a modification of standard European options. For example a common option is a front end single barrier up-and-out call option. Up to time t in the life of the option there exists a barrier and if the underlying crosses this barrier the option expires worthless, after time t the option becomes a standard call option. Due to the additional feature the premium is smaller compared to a standard European option. This is why those options have become so popular, they offer the same level of protection when used as a hedge but are considerably cheaper.
The value of such options can be obtaind by solving the Black-Scholes partial differential equation with appropriate boundary conditions, for example Hui [11].

We consider digital double barrier options with an arbitrary number of barrier periods. This means that the holder receives the payoff only if the underlying stays between the two barriers in certain specified time intervals. While such contracts might make sense by themselves (e.g. as a weather or energy derivative with seasonal barriers), the motivation is to use them for the pricing of certain structured notes with several coupons. Such trades often feature an aggregate floor at the final coupon date, which increases the total payoff to a guaranteed amount if the sum of the coupons is less than this amount. Pricing this terminal premium requires the law of the sum of the coupons, which can be recovered from its moments. If the individual coupons of the note are digital barrier options, then these moments can be computed from the prices of options of the kind described above, where the sets of barrier periods are subsets of the coupon periods of the note.

Recall that Monte Carlo pricing of barrier contracts is tricky, because the discretization produces a downward bias for the barrier hitting probability. For single barrier options, this difficulty can be overcome using the explicit law of the maximum of the Brownian bridge [2, 4]. For double barrier options, the exit probability of the Brownian bridge is not known; see Baldi et al. [3] for an approximate approach using sample path large deviations. These numerical challenges led us to investigate exact valuation formulas.

The paper is structured as follows. In Section 2 we present the mathematical theory behind option pricing, some results from stochastic calculus and large deviations theory. In Section 3 we define barrier options and barrier digital for which we define the payoffs we are interested in and price them for a single barrier period. Then we extend the result to arbitrarily many periods of active barriers. The pricing of structure floors, is also presented in Section 3. Numerical results showing an application of the derived theoretical pricing functions can be found in Section 4. Finally we present the Mathematica code used in this paper as well as the exact expression for a corridor option in the Appendix .

2 Mathematical Theory

In this part we state some basic results necessary for the later chapters. Throughout this chapter we present only some fundamental proofs vital for the purpose of this paper.

The results presented herein refer loosely to the works and books from [16],[17] and [19].

2.1 Preliminaries

2.1.1 Stochastic Calculus

To derive the Black-Scholes equation in the next chapter we need a mathematical framework in which we can work.

The next definitions state the probability space with which we work throughout the paper.

Definition 2.1. *The triple* $(\Omega, \mathfrak{A}, \mathbb{P})$ *is called a probability space, where* Ω *is a non-empty set,* $\mathfrak{A}$ *a* σ*-algebra and* $\mathbb{P}$ *a probability measure on* $(\Omega, \mathfrak{A})$*.*

The information up until a specific time is modelled by a filtration.

Definition 2.2. *A filtration* $(\mathcal{F}_t)_{t\geq 0}$ *is a non-decreasing sequence of* σ*-algebras, where* $\forall s,t \geq 0, s < t$*, it holds that* $\mathcal{F}_s \subset \mathcal{F}_t$*.* $\left(\Omega, \mathfrak{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}\right)$ *is usually called a filtered probability space.*

If $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous (i.e. $\mathcal{F}_t = \bigcap\limits_{s<t} \mathcal{F}_s,\ \forall t$) and complete (i.e. $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets) it is said that it satisfies the usual conditions.

Definition 2.3. *A real valued stochastic Process* $(B_t)_{t\geq 0}$ *on a probability space* $\left(\Omega, \mathfrak{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}\right)$ *is called a standard Brownian motion if*

- $B_0(\omega) = 0$ *a.s.*
- *The map* $t \mapsto B_t,\ t \in [0, \infty)$*, is a.s. a continuous function, i.e.* $\exists A \subset \mathfrak{A}, \mathbb{P}[A] = 1$ *and* $A \subset \{\omega \in \Omega : t \mapsto B_t(\omega)$ *is continuous*$\}$*.*
- *For all* $t \geq 0$ *and* $h > 0$ *the increments* $B_{t+h} - B_t$ *are normally distributed with mean* 0 *and variance* h*.*

- *The increments are independent, i.e. for all* $0 \leq t_t \leq \ldots \leq t_n$ *the random variables* $B_{t_n} - B_{t_{n-1}}, B_{t_{n-1}} - B_{t_{n-2}}, \ldots, B_{t_2} - B_{t_1}$ *are independent.*

For a proof of the existence of the Brownian motion see for example [17].

Definition 2.4. *A Brownian motion* $(B_t)_{t\geq 0}$, *or more generally a stochastic process* $X = (X_t)_{t\geq 0}$ *is adapted to the filtration* $(\mathcal{F}_t)_{t\geq 0}$ *if each random variable* X_t *is* $\mathcal{F}_t$*-measurable.*

Now we want to give meaning to the following integral

$$\int_0^T f(t)\; dB_t, \tag{1}$$

where $f(t)$ is a stochastic process and B_t a Brownian motion.

Since the paths of the Brownian motion can not be differentiated with respect to time and have infinite variation over every interval[1],expression (1) can not be defined as an Riemann-Stieltjes integral.

Definition 2.5. *Let* $\mathcal{V} = \mathcal{V}\,(S,T)$ *be the set of all functions* $f(t,\omega):\; I \times \Omega \to \mathbb{R}$ *such that*

- f *is progressively measurable, i.e.* $(t,\omega) \to f(t,\omega)$ *is* $\mathcal{B}\times\mathcal{F}_t$*-measurable, where* $\mathcal{B}$ *is the Borel* σ*-algebra on* $\mathbb{R}^+$.
- $f(t,\omega)$ *is* $\mathcal{F}_t$*-adapted.*
- $\mathbb{E}\left[\int_S^T f(t,\omega)^2 dt\right] < \infty.$

The next definition shows how for functions $f \in \mathcal{V}\,(S,T)$ expression (1) can be defined.

Definition 2.6. *Let* $f \in \mathcal{V}\,(S,T)$. *The Itō integral of* f *(with respect to the Brownian motion) is defined as*

$$\int_S^T f(t) dB_t = \lim_{n\to\infty} \int_S^T \phi_n(t) dB_t, \tag{2}$$

[1] c.f. [15]

where $(\phi_n)_{n\in\mathbb{N}}$ is a sequence of elementary functions satisfying

$$\mathbb{E}\left[\int_S^T (f(t)-\phi_n(t))^2\ dt\right] \to 0, \quad \text{as } n\to\infty, \tag{3}$$

and the limit in (2) in $L^2(\mathbb{P})$.

The following lemma shows the most important properties of the Itō integral, namely the Itō isometry and the martingale property. For a detailed proof the interested reader should refer to [17].

Lemma 2.1. *It holds that*

$$\mathbb{E}\left[\left(\int_S^T f(t)dB_t\right)^2\right] = \mathbb{E}\left[\int_S^T f(t)^2 dt\right] \quad \forall f\in\mathcal{V}(S,T). \tag{4}$$

further an Itō Integral w.r.t. a Brownian motion is a martingale:

A stochastic process $X=(X_t)_{t\geq 0}$ is called a martingale w.r.t. to a filtration $\mathcal{F}_t$ if

- *X_t is $\mathcal{F}_t$-measurable for all t,*
- *$\mathbb{E}[|X_t|]<\infty$ for all t,*
- *$\mathbb{E}[X_s|\mathcal{F}_t]=X_t$ for all $s>t$.*

Most of the time we are not only interested in an integral w.r.t. to the Brownian motion, it is also interesting to find an expression for functions of Brownian motions, i.e. $f(B_t)$, here $f(\cdot)$ refers to a differentiable funtion. It is worth noting that the following theorem not only applies to one-dimensional processes but can be stated for higher dimensions, too.[2]

Theorem 2.1. *Let $g(t,x)\in C^{1,2}(I\times\mathbb{R})$ and denote with g_x, g_t and g_xx the partial derivatives. Then for every $t\geq 0$,*

$$g(t,B_t) = g(0,B_0) + \int_0^t g_t(t,B_t)dt + \int_0^t g_x(t,B_t)dB_t + \frac{1}{2}\int_0^t g_{xx}(t,B_t)dt. \tag{5}$$

[2] cf [17].

Usually the following short-hand notation is used

$$dg(t, B_t) = g_t(t, B_t)dt + g_x(t, B_t)dB_t + \frac{1}{2}g_{xx}(t, B_t)dt.$$

Now we turn to the discussion of stochastic differential equations which are interesting to study from a mathematical view and necessary to describe the evolution of asset prices in mathematical finance.

2.1.2 Stochastic differential equations

A short introduction to stochastic differential equations (SDEs) which we will need later on is presented in this chapter. Additionally, we show a rather striking result that relates SDEs and partial differential equations.

Definition 2.7. *A stochastic differential equation is an equation of the following form*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t. \tag{6}$$

The coefficients $b(t, x)$ and $\sigma(t, x)$ are called the drift coefficient and the diffusion coefficient of the SDE. Similar to ordinary differential equations (ODEs) an initial condition of the form $X_0 = x_0,\ t \leq 0,\ x_0 \in \mathbb{R}$ is needed.

The challenge is to find a solution for (6), which is unique, if possible. Most SDEs are difficult to solve and do not admit an explicit solution. If the drift and diffusion coefficients of (6) satisfy certain growth and continuous restrictions it can be shown that there exists a unique solution which is adapted to the filtration generated by the Brownian motion.[3]

The following example shows an SDE where a unique explicit solution is known.

Example 2.1. *Let's take a look at the following SDE:*

$$dS_t = \mu S_t dt + S_t \sigma dB_t,\ t \geq 0 \tag{7}$$

Now we want to show that the unique strong solution to the SDE (given initial condition $S_0 > 0$) is given by

[3] cf [17] Theorem 5.2.1

$$S_t = S_0 \exp\left(\sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right), \ t \geq 0. \tag{8}$$

The solution is known as Geometric Brownian motion.

Proof. First we show that S_t is, indeed, a solution to the given SDE. Consider the function $g(x) = \ln(x)$ and apply (2.1):

$$\begin{aligned}
\ln(S_t) - \ln(S_0) &= \int_0^t \frac{1}{S_u} dS_u - \frac{1}{2} \int_0^t \frac{1}{S_u^2}(S_u^2 \sigma^2 du) \\
&= \int_0^t \frac{1}{S_u} d(S_u \mu du + S_u \sigma dB_u) - \frac{1}{2} \int_0^t \sigma^2 du \\
&= \int_0^t \sigma dB_u + \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) du \\
&= \sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right) t \\
\Rightarrow S_t &= S_0 \exp\left(\sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right)\right)
\end{aligned}$$

The solution is strong, because it is adapted.

To prove uniqueness assume that there exists another process R_t which fulfills the SDE. Then consider the function $f(x, y) = \frac{y}{x}$ the rest follows from Itōs fomula and the properties of the quadratic covariation. □

The geometric Brownian motion has a central role in mathematical finance as we will see later.

Before we state an important result which connects stochastic differential equations with partial differential equations we need the following two definitions.

Definition 2.8. *A stochastic process X_t is called an Itō diffusion if it satisfies a stochastic differential equation of the form*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \ t \geq s, \ X_0 = x_0 \tag{9}$$

and the coefficients are Lipschitz continuous, i.e.

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq D\,|x - y|\,,\; x, y \in \mathbb{R}. \tag{10}$$

Definition 2.9. *Let X_t be a Itō diffusion. The generator A of X_t is defined by*

$$Af(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}\left[f(X_t)\right] - f(x)}{t},\; x \in \mathbb{R}. \tag{11}$$

Theorem 2.2. *Let X_t be an multidimensional Itō diffusion*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t. \tag{12}$$

If $f \in C_0^2(\mathbb{R}^n)$ then

$$Af(x) = \sum_i b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}. \tag{13}$$

With this result we can derive the generator of the Brownian motion which solves the stochastic differential equation

$$dX_t = dB_t, \tag{14}$$

therefore we have $b = 0$ and $\sigma = I_n$, the n-dimensional identity matrix. The generator of the Brownian motion B_t is

$$Af = \frac{1}{2}\Delta, \tag{15}$$

where Δ denotes the Laplace operator.

The next theorem is known as the Feynmann-Kac formula, after the world-renowned American physicist Richard Feynmann (1918-1988) and the Polish mathematician Mark Kac (1914-1984).

Theorem 2.3. *Let $f \in C_0^2(\mathbb{R}^n)$ and $q \in C(\mathbb{R}^n)$. Assume that q is bounded from below. Let*

$$v(t,x) = \mathbb{E}\left[\exp\left(-\int_0^t q(X_s)ds\right) f(X_t)\right], \tag{16}$$

then

$$\begin{aligned} \frac{\partial v}{\partial t} &= Av - qv,\ t > 0, x \in \mathbb{R}^n, \\ v(0,x) &= f(x),\ x \in \mathbb{R}^n. \end{aligned} \tag{17}$$

Furthermore, if $w(t,x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ is bounded on $K \times \mathbb{R}^n$ for each compact $K \subset \mathbb{R}$ and w solves (17), then $w(t,x) = v(t,x)$.

2.1.3 Partial Differential Equations

Essential tools for the analysis of partial differential equation (PDE) are worked out here. We can only skim the surface of the theory of PDEs and present the connection to SDEs, a thorough analysis is out of the scope of this paper.

As we will see the so-called Black-Scholes partial differential equation is a parabolic equation with a second derivative with respect to one variable and a first derivative to the other. Equations of this type are usually called heat equations, because they are used to model the flow of heat from one part of an object to another.

The simplest heat equation for the temperature $u = u(x,\tau)$ over the domain $\mathbb{R} \times (0,\infty)$ is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(\cdot,0) = u_0 \tag{18}$$

where x is a spatial coordinate and t the time. How do solutions of this equation look like? The next theorem answers this question.

Theorem 2.4. *Let u be a solution of the heat equation (18) with $u(\cdot, t) \in L^1(\mathbb{R})$ for all $t \geq 0$. Then*

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int\limits_{\mathbb{R}} \exp\left\{ -\frac{|x-y|^2}{4t} \right\} u_0(y)\ dy. \tag{19}$$

Proof. This proof is based on Fourier transform methods. It holds that

$$0 = \widehat{u_t - u_{xx}}(k) = \widehat{u}_t \left| k^2 \right| \widehat{u}.$$

The second equation follows from the properties of the Fourier transform. Now we have an ordinary differential equation with initial condition $\widehat{u}(k,0) = \widehat{u_0}$. Integrating this equation and applying the inverse Fourier theorem (c.f. [15] (15.2)) gives us

$$u(x,t) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} e^{-|k|^2 t}\ \widehat{u}_0\ e^{ikx}\ dk.$$

sNow define $w := \widehat{e^{-|k|^2 t}}^{-1}$, then it follows easily using the fact that $\widehat{f * g} = \widehat{f}\widehat{g}$ for functions in $\mathrm{L}^1(\mathbb{R})$ that

$$u(x,t) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} \widehat{w}(k,t)\ \widehat{u}_0\ e^{ikx}\ dk = (w * u_0)(x,t).$$

Finally, the last step is to calculate $w(x,t)$

$$w(x,t) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} e^{-|k|^2 t} e^{ikx}\ dk = \frac{1}{\sqrt{4\pi t}} e^{-|x|^2/4t}.$$

□

For the rather technical proof that for t $\to 0$ *(19) one gets the initial condition refer to [8].*

Two interesting properties of the heat-equation are worth mentioning. Even for intial conditions which are not differentiable the solution has derivatives of all orders for all positive times. Furthermore the solution has infinte speed of propagation, even for small times t. The solution is positive even though at time $t = 0$ it is only non negative.

Now we have all necessary tools to derive the classical Black-Scholes equation for valuing option contracts.

2.2 Black-Scholes equation

In order to derive the valuation formula we need to make some assumptions about the market in which we will trade and value our option, i.e. we assume "ideal conditions".

- Trading is possible at every instant, i.e. continuously.
- The interest rate r is known, constant and equal for borrowing and lending.
- The volatility of the stock is constant.
- There are no transaction costs in buying and selling assets.
- It is possible to trade any fraction of an asset.

There are several ways to derive the Black-Scholes (BS) equation. We show two ways. The first is based on the Feynman-Kac theorem and the second one on the heat equation. Before doing this, we describe the market and the assets we consider.

2.2.1 Black-Scholes economy

Let's look at a market where two assets can be traded, a risky stock S and a riskless bond B. The dynamics of these assets are governed by the following SDEs

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t, \\ dB_t &= r_t B_t dt. \end{aligned} \tag{20}$$

The inital values are $B_0 = 1$ and S_0 of the bond and the stock respectivly.

Remarks: Remember that the above equations are just an informal way of expressing the integral equation $S_t = S_0 + \int_0^t \mu S_t \ dt + \int_0^t \sigma S_t \ dW_t$. The second point here is that we put W_t for the Brownian motion to distinguish it from the bond process.

Now we introduce a derivative on the stock whose value V_t can be found by applying Itō's Lemma.

$$
\begin{aligned}
dV_t &= \frac{\partial V}{\partial t}\ dt + \frac{\partial V}{\partial S}\ dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\ dt \\
&= \frac{\partial V}{\partial t}\ dt + \frac{\partial V}{\partial S}(\mu S_t\ dt + \sigma S_t\ dW_t) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\ dt \\
&= \left(\frac{\partial V}{\partial t}\ dt + \mu S_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)\ dt + \left(\sigma S_t \frac{\partial V}{\partial S}\right)\ dW_t.
\end{aligned}
$$

When forming a portfolio of a combination of these instruments we assume that it is self-financing, which means that the portfolio values are due only to changes in the value of the covered instruments, mathematically this is called self-financing.

Definition 2.10. *A trading strategy is a pair $\phi = (\phi^1, \phi^2)$ of progressively measurable stochastic processes. A trading strategy on the interval $[0, T]$ is self-financing if its wealth process $\Pi_t(\phi) = \phi^1_t S_t + \phi^2_t B_t$ satisfies*

$$
\Pi_t(\phi) = \Pi_0(0) + \int_0^t \phi^1_u\ dS_u + \int_0^t \phi^2_u\ dB_t \tag{21}
$$

At this point we have established all necessary conditions to derive the Black-Scholes PDE.

2.2.2 Black-Scholes PDE

We derive the PDE in the same way Fischer Black and Myron Scholes did in their seminal paper [5].

They looked at a portfolio composed of one share of the underlying stock and $\delta = 1/\Delta$ shares of an option V written on that stock. The value of that portfolio at time t is $\Pi_t = \theta V_t + S_t$. Now, since we work only with self-financing portfolios we can write

$$
\begin{aligned}
d\Pi &= \theta dV + dS \\
&= \left(\theta \frac{\partial V}{\partial t}\ dt + \theta \mu S_t \frac{\partial V}{\partial S} + \frac{1}{2}\theta \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S_t\right)\ dt + \left(\theta \sigma S_t \frac{\partial V}{\partial S} + \sigma S_t\right)\ dW_t
\end{aligned}
$$

This particular portfolio must be riskless and it must earn the risk free rate r. To be riskless the second term involving the Brownian motion dW must be zero, therefore $\theta = -\left(\frac{\partial V}{\partial S}\right)^{-1}$. To earn the risk free rate the dynamics of the portfolio has to be $d\Pi = r\Pi dt = r(\theta V + S)dt$.

Considering this we arrive at the following equation

$$\left(\theta\frac{\partial V}{\partial t} + \frac{1}{2}\theta\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right) dt = r(\theta V + S)dt.$$

The only step left to arrive at the Black-Scholes PDE is to drop the dt and divide by θ

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right) dt = r(V + S)dt,$$

or in shorthand notation

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV + rSV_S = 0. \tag{22}$$

2.2.3 Black-Scholes valuation formula

The following theorem states the most common valuation formula for standard European type options.

Theorem 2.5. *The time-t price $C(S_t, K, T)$ of a European call option with strike K,maturity $\tau = T - t$ written on a non-dividend paying stock with spot price S_t and volatility σ can be found by using the following formula*

$$C(S_t, K, T) = S_t\Phi(d1_1) - e^{-r\tau}K\Phi(d_2) \tag{23}$$

where

$$d_{1,2} = \frac{\ln\frac{S_t}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \tag{24}$$

and $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt$ is the cumulative distribution function of the standard normal distribution.

As mentioned above we show two derivations a rather short but elegant one by Feynmann-Kac and a more detailed one based on the heat equation which will be useful when deriving the value of a Digital Barrier Option later on.

Proof. **Black-Scholes by the Feynmann-Kac Theorem**
Note that C satisfies equation (17), with the generator of the geometric brownian motion $Af(x) = rS_t f'(x) + \frac{1}{2}\sigma^2 S_t^2 f''(x)$ and boundary condition $C(S_T, K, T) = (S_T - K)^+$. Therefore we can apply the theorem and the value of the European call is

$$\begin{aligned} C(S_t, K, T?\,) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r\ du}\ C(S_T, K, T)|\mathcal{F}_t \right] \\ &= e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+ \right]. \end{aligned}$$

The expectation can be evaluated by straightforward integration in the same way as for example in [16]. □

Proof. **Black-Scholes by the Heat Equation**
To convert the BSM PDE into the heat equation we use the following transformations:

$$\begin{aligned} x &= \ln \frac{S}{K}, \\ \tau &= \frac{\sigma^2}{2}(T - t), \text{and} \\ u(x, \tau) &= \frac{1}{K}V(S, t) = \frac{1}{K}V(Ke^x, T - 2\tau/\sigma^2). \end{aligned}$$

The next step is to convert the partial derivatives:

$$\begin{aligned} V_t &= Ku_\tau \tau_t = \frac{-K\sigma^2}{2}u_\tau, \\ V_S &= Ku_x x_S = \frac{K}{S}u_x = e^{-x}u_x, \\ V_{SS} &= -\frac{K}{S^2}u_x + \frac{K}{S^2}u_{xx} \\ &= \frac{e^{-2x}}{K}\left(u_{xx} - u_x\right). \end{aligned}$$

Inserting into (22) and simplifying results in

$$
\begin{aligned}
0 &= \frac{-K\sigma^2}{2}u_\tau + rKe^x e^{-x}u_x + \frac{1}{2}\sigma^2 K^2 e^{2x}\frac{e^{-2x}}{K}\left(u_x x - u_x\right) - ru \\
&= -u_\tau + (\frac{2r}{\sigma^2} - 1)u_x + u_{xx} - \frac{2r}{\sigma^2}u. \qquad (25)
\end{aligned}
$$

The final condition for V is naturally the final payoff $V(S_T, T) = (S_T - K)^+$, which transforms into an inital condition for u we therefore get $u(x_T, 0) = \frac{1}{K}V\left(S_T - K\right)^+ = \frac{1}{K}\left(Ke^{x_T} - K\right)^+ = \left(e^{x_t} - 1\right)^+$.

Since (25) still does not resemble the heat equation we have to make an additional transformation. Define $\alpha = \frac{1}{2}(k-1)$ and $\beta = \frac{1}{2}(k+1)$, where $k = \frac{2r}{\sigma^2}$ then set

$$
w(x, \tau) = e^{\alpha x + \beta^2 \tau}u(x, \tau). \qquad (26)
$$

Again the partial derivatives of u in terms of w have to be calculated

$$
\begin{aligned}
u_t &= e^{\alpha x + \beta^2 \tau}\left(w_\tau - w\beta^2\right), \\
u_S &= e^{\alpha x + \beta^2 \tau}\left(w_x - \alpha w\right), \\
u_{xx} &= e^{\alpha x + \beta^2 \tau}\left(\alpha^2 w - 2\alpha w_x + w_{xx}\right).
\end{aligned}
$$

Substituting back into (26)

$$
\beta^2 w - w_\tau + (k-1)\left[-\alpha w + w_x\right] + \alpha w - 2\alpha w_x + w_{xx} - kw = 0,
$$

and simplifying we get the heat equation

$$
w_\tau = w_{xx}. \qquad (27)
$$

The initial condition is $w(x_t, 0) = e^{\alpha x_T}u(x_T, 0) = \left(e^{\beta x_T} - e^{\alpha x_T}\right)^+$.

To get from V to w simply set

$$
V(S, t) = \frac{1}{K}e^{-\alpha x - \beta^2 \tau}w(x, \tau). \qquad (28)
$$

We have transformed the BS-PDE into the heat equation, so the solution to this partial-differential equation is (19),hence

$$\begin{aligned} w(x,\tau) &= \frac{1}{\sqrt{4\pi\tau}} \int\limits_{\mathbb{R}} e^{-(x-y)^2/4\tau} w_0(y)\ dy \\ &= \frac{1}{\sqrt{4\pi\tau}} \int\limits_{\mathbb{R}} e^{-(y-x)^2/4\tau} \left(e^{\beta y} - e^{\alpha y}\right)^+\ dy. \end{aligned}$$

A change of variables helps evaluating this integral, set $z = \frac{y-x}{\sqrt{2\tau}}$ so that $y = \sqrt{2\tau}z + x$ and $dy = \sqrt{2\tau}dz$.

$$w(x,\tau) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \exp\left\{-\frac{1}{2}z^2\right\} \exp\left\{\beta(\sqrt{2\tau}z + x) - \alpha(\sqrt{2\tau}z + x)\right\}^+\ dz$$

Observe that the integral is non-zero only if the second exponent is greater than zero, when $z > \frac{-x}{\sqrt{2\pi}}$, therefore we can split the integral into two parts

$$\begin{aligned} w(x,\tau) = &\frac{1}{\sqrt{2\tau}} \int\limits_{-x/\sqrt{2\tau}}^{\infty} \exp\left\{-\frac{1}{2}z^2\right\} \exp\left\{\beta(\sqrt{2\tau}z + x)\right\}\ dz \\ &- \frac{1}{\sqrt{2\pi}} \int\limits_{-x/\sqrt{2\tau}}^{\infty} \exp\left\{-\frac{1}{2}z^2\right\} \exp\left\{\alpha(\sqrt{2\tau}z + x)\right\}\ dz \\ &= I_1 - I_2. \end{aligned}$$

To evaluate the first integral I_1 complete the square so that

$$I_1 = e^{\beta x + \beta^2 \tau} \frac{1}{\sqrt{2\pi}} \int\limits_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}(z-\beta\sqrt{2\tau})^2}\ dz.$$

Finally for the last transformation set $\nu = z - \beta\sqrt{2\tau}$

$$
\begin{aligned}
I_1 &= e^{\beta x+\beta^2\tau}\frac{1}{\sqrt{2\pi}}\int\limits_{-x/\sqrt{2\tau}-\beta\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}y^2}\,dy \\
&= e^{\beta x+\beta^2\tau}\left(1-\Phi\left(-\frac{x}{\sqrt{2\tau}}-\beta\sqrt{2\tau}\right)\right) \\
&= e^{\beta x+\beta^2\tau}\Phi\left(\frac{x}{\sqrt{2\tau}}+\beta\sqrt{2\tau}\right).
\end{aligned}
$$

The second integral I_2 follows easily just replace β with α

$$
I_2 = e^{\alpha x+\alpha^2\tau}\Phi\left(\frac{x}{\sqrt{2\tau}}+\alpha\sqrt{2\tau}\right).
$$

Let's take a look at the term $\frac{x}{\sqrt{2\tau}}+\beta\sqrt{2\tau}$, and transform them back into the original variables, we get

$$
\frac{x}{\sqrt{2\tau}}+\beta\sqrt{2\tau} = \frac{\ln\frac{S}{K}+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_1,
$$

and

$$
\frac{x}{\sqrt{2\tau}}+\alpha\sqrt{2\tau} = d_1-\sigma\sqrt{T-t} = d_2.
$$

So the solution for the heat equation w reads

$$
\begin{aligned}
w(x,\tau) &= I_1 - I_2 \\
&= e^{\beta x+\beta^2\tau}\Phi(d_1) - e^{\alpha x+\alpha^2\tau}\Phi(d_2).
\end{aligned}
$$

To obtain the price for a call option use (28)

$$
\begin{aligned}
V(S,t) &= Ke^{-\alpha x-\beta^2\tau}w(x,\tau) \\
&= Ke^{-\alpha x-\beta^2\tau}(I_1-I_2) \\
&= Ke^{-\alpha x-\beta^2\tau}e^{\beta x+\beta^2\tau}\Phi(d_1) - Ke^{-\alpha x-\beta^2\tau}e^{\alpha x+\alpha^2\tau}\Phi(d_2) \\
&= Ke^{\beta-\alpha}\Phi(d_1) - Ke^{(\alpha^2-\beta^2)\tau}\Phi(d_2) \\
&= S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_1),
\end{aligned}
$$

since $\beta-\alpha=1$ and $\alpha^2-\beta^2=-\frac{2r}{\sigma^2}$. □

2.3 Short introduction into large deviations theory

For Monte Carlo simulation of the barrier digital we need to take into account that using a discrete setting our process can hit the barrier during two time steps.

Baldi et al. [3] derived formulas for the exit probability using sharp large deviation theory. In this chapter we will cover the basic theory behind large deviation techniques and state Cramér's theorem. Note that we do not give an exhaustive overview of the methods Baldi et al. used to prove their theorem. The application to the Monte Carlo simulation of barrier options will be stated in Chapter 4.

Large deviation is a part of probability theory that deals with the mathematics of rare events.

Consider a random variable X. Now we want to estimate the probability $p := \mathbb{P}[X \geq l]$. For example X could be the value of a portfolio or for risk management purposes we could be interested in a loss of at least l.

To estimate this probability we could resort to Monte Carlo simulation and generate n iid r.v. $X_1, X_2, \ldots, X_n$ and look at the following estimator

$$\widehat{S}_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{(l,\infty)}(X_i)$$

The strong law of large numbers tells us that

$$\widehat{S}_n \underset{n \to \infty}{\longrightarrow} p, \ \mathbb{P} \ a.s.$$

Due to the central limit theorem we can determine the rate of convergence via

$$\mathbb{P}\left[\left|\widehat{S}_n - p\right| \geq \frac{a}{\sqrt{n}}\right] \underset{n \to \infty}{\longrightarrow} 2\Phi\left(-\frac{a}{\sqrt{p(1-p)}}\right)$$

for every $a > 0$, and $\phi(\cdot)$ is the cumulative distribution function of a standard normal distribution.

The theory of large deviations, i.e. Cramér's theorem gives us a slightly better approximation.

For every $a > 0$ there exists suitable constants C and γ such that

$$\mathbb{P}\left[\left|\widehat{S}_n - p\right| \geq a\right] \simeq C \cdot e^{-\gamma n}.$$

We now give a rigorous definition of Cramér's theorem.

Theorem 2.6. *Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with values in $\mathbb{R}$. Define $\varphi(t) := \mathbb{E}\left[e^{tX}\right] < \infty$ for all $t \in \mathbb{R}$, $\mathbb{V}[X] > 0$ and $S_n = \sum_{i=1}^{n} X_i$. Then for every $x > \mathbb{E}[X]$ it holds that*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}[S_n \geq nx] = -I(x), \tag{29}$$

where $I(\cdot)$ is the Legendre transform[4] *of $\log \varphi$.*

Proof. w.l.o.g. we can assume that $x = 0$ (consequently $\mathbb{E}[X] < 0$). Otherwise look at the family $X_1 - x, X_2 - x, \ldots$ and show that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}[S_n \geq 0] = \log \rho,$$

where $\rho = \inf_{t\in R} \varphi(t)$.
The rest of the proof is to consider three cases, (i) the r.v. are strictly negative, i.e. $\mathbb{P}[X_1 < 0] = 1$ (ii) the r.v. are not positive almost surely, i.e. $\mathbb{P}[X_1 \leq 0] = 1$ and (iii) the r.v. take positive values with positive probability, i.e. $\mathbb{P}[X_1 > 0] > 0$.

(i) In this case $t \mapsto \varphi(t)$ is strictly decreasing and therefore $\lim_{t\to\infty} \varphi(t) = 0$. Additionally $\mathbb{P}[S_n \geq 0] = 0$ whereby the statement follows.
(ii) Now φ is still decreasing however $\rho = \lim_{t\to\infty} \varphi(t) = \mathbb{P}[X_1 = 0] > 0$. Then again

$$\mathbb{P}[S_n \geq 0] = \mathbb{P}[X_1 = 0, \ldots, X_n = 0] = \rho^n,$$

[4] The map $I : \mathbb{R} \to \mathbb{R}, x \mapsto \sup_{t\in R} (tx - \log \varphi(t))$ is called the Legendre transform

from which it follows that $\frac{1}{n}\log\mathbb{P}\left[S_n \geq 0\right] \underset{n\to\infty}{\to} \log\rho$ for $n \to \infty$.
(iii) Since $\mathbb{E}\left[X_1\right] < 0$ X_1 can take negative values with postive probability. Moreover φ is strictly convex and $\underset{t\to\pm\infty}{lim}\varphi(t) = \infty$, φ has a minimum in a point t_0 and $\varphi(t)' = \mathbb{E}\left[Xe^{t_0X}\right] = 0$.

"$\leq$" For all $t \in \mathbb{R}$:

$$\mathbb{P}\left[S_n \geq 0\right] = \mathbb{P}\left[e^{tS_n} \leq 1\right] \leq \varphi(t)^n,$$

thus $\underset{n\to\infty}{lim\,sup}\frac{1}{n}\log\mathbb{P}\left[S_n \geq 0\right] \leq \underset{t\in R}{inf}\log\varphi(t) = \log\rho$.

"$\geq$" This is the hard part. We will only state the general idea and leave the proof to the interested reader.

Consider a series of *iid* r.v. $\widehat{X_1}, \widehat{X_2}, \ldots$ whose distribution is given by the Radon-Nikodym density

$$\mathbb{P}\left[\widehat{X_1} \in dx\right] = \rho^{-1}e^{t_0x}\mathbb{P}\left[X_i \in dx\right].$$

The right side is indeed a probability distribution and we can write down the moment generating function. Next we need to transform the sum S_n and give a lower estimate of the inequality. The last step is to apply the central limit theorem to complete the proof. □

For the purpose of this paper we need a more abstract concept of large deviations. Let (E, d) be a metrical space with $\mathcal{B}(E)$ the Borel-σ algebra on E. Compared to the previous results we want to relax the iid assumption as well as derive asymptotic results for sets other than $[a, \infty)$.

Definition 2.11. *A function $f : E \to [-\infty, \infty]$ is called lower semi-continuous if one of the following conditions is satisfied*

(i) For all sequences $(x_n)_{n\in\mathbb{N}}$ with $x_n \to x$ for $n \to \infty$ and $x \in E$ follows that:

$$\underset{n\to\infty}{lim\,inf} f(x_n) \geq f(x).$$

(i) For all $x \in E$

$$\underset{\epsilon \downarrow 0}{lim}\ \underset{B(x,\epsilon)}{inf}\ f(y) = f(x).$$

where $B(x, \epsilon)$ is the open ball with radius ϵ.

(i) f has closed lower level sets, i.e. for all $c \in \mathbb{R}$ the following sets are closed

$$f^{-1}([-\infty, c]) := \{x \in E : f(x) \geq c\}.$$

Before we can state a large deviations principle the concept of a rate function is introduced.

Definition 2.12. *A function $I : E \to [0, \infty]$ is called a rate function if*

(a) $I \not\equiv \infty$.

(a) I has compact lower level sets.

The next definition goes back to Varadhan [21] and is the main result in the theory of large deviations

Definition 2.13. *A sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on E is said to follow the principle of large deviations with speed $(\gamma_n)_{n\in\mathbb{N}}$ and rate function I if*

(LD1) I is a rate function as defined in (2.12).

(LD2) $\underset{n\to\infty}{lim\,sup} \frac{1}{\gamma_n} \log \mu(C) \leq -I(C)$ for all closed sets $C \subseteq E$.

(LD3) $\underset{n\to\infty}{lim\,inf} \frac{1}{\gamma_n} \log \mu(O) \geq -I(O)$ for all open sets $O \subseteq E$.

Lemma 2.2. *If $(\mu_n)_{n\in\mathbb{N}}$ meets (LD1)-(LD3) with speed $(\gamma_n)_{n\in\mathbb{N}}$ then the rate function is uniquely determined.*

Proof. Take two rate functions I and J which meet (2.13) with speed $(\gamma_n)_{n\in\mathbb{N}}$ and show $I \equiv J$ using (LD1)-(LD3). $\square$

These results form the basis of large deviation theory, Baldi et al.([3]) use this theory in the following way.

Consider for a positive parameter ϵ the solution X^{ϵ} of the following SDE

$$dX_t^{\epsilon} = b^{\epsilon}(X_t^{\epsilon}, t)dt, + \sqrt{\epsilon}\sigma(X_t^{\epsilon})dB_t \ s < t. \tag{30}$$
$$X_s^{\epsilon} = x \in \mathbb{R}^n \tag{31}$$

We require the usual Lipschitz conditions for the drift and that the diffusion $\sigma : \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$ is a Lipschitz continuous matrix field, $\forall \xi \in \mathbb{R}^n$

$$\langle a(x)\xi, \xi \rangle \geq a_0 \langle \xi, \xi \rangle,$$

for some $a_0 > 0$ and $a(x) = \sigma(x)\sigma^T(x)$.

The law of the process is denoted by P_x^{ϵ}, s. Now suppose that there exists a Lipschitz continuous vector field b on $\mathbb{R}^n$ such that

$$\lim_{\epsilon \to 0} b^{\epsilon} = b,$$

uniformly on compact sets. In this setting, there holds a Large Deviation principle for the familiy $\left\{P_{x,s}^{\epsilon}\right\}$, with speed ϵ^{-1} and rate function $I = I_{x,s} : \mathcal{C}([s, \mathcal{T}], \mathbb{R}^n)$ defined by

$$I(\varphi) = \int_s^{\mathcal{T}} L(t, \varphi_t, \varphi_t') \ dt$$

if $\varphi(x)$ is absolutely continuous and $\varphi_s = x, I(\varphi) = +\infty$ otherwise. The integrand $L(\cdot)$ is

$$L(t, x, u) = \frac{1}{2}\langle a^{-1}(x)(u - b(x, r)), u - b(x, r)\rangle.$$

Based on this setting Baldi et al. [3] derived the probability that the underlying S hits one of the barriers during an unobserved interval $[T_0, T_0 + \epsilon]$. The exact expression for the exit probability is stated in Chapter 4.

3 Barrier Options

In the previous section we derived the price of an standard European call option, the aim of this section is to study more complicated options, namely path-dependent options. The payoff of these instruments depends on whether the underlying price stays under/over or inside a certain barrier, hence the name barrier options.

Barrier options are mainly traded in foreign exchange markets and are considerably cheaper than standard FX options. Let's look at an example to make things clear.

An investor believes that the USD will strengthen against the YEN over the next six months (current spot \$99). She purchases an ordinary 6 month USD call option at a strike of \$99. This would cost 350 basis points.
An alternative is to purchase a USD at the money call (\$99) with a knockout at \$109,a so called *up and out call*. This would reduce the premium to only 100 basis points.

Now there are three possible outcomes:

- If the USD does strengthen but stays above \$109 over the life of the option, the call will expire worthless.
- If the USD strengthens, but never reaches \$109 over the life of the option, the call will behave like an ordinary call and the investor will exercise the call and make the same profit as the ordinary call.
- If the USD does not close above the strike, the option will expire worthless.

An *up and out call* is very attractive for an investor with such believes.

Barrier options were first studied by Hui [12] who derived prices for two types of barrier options in the Black-Scholes setting. These two types are i) front-end knock-out options, i.e. options where the barrier is active from start time to a certain time t and ii) rear-end knock-out options with an active barrier from time t to option maturity T. He covers single barrier as well as double barrier options.

As an example for a standard barrier option we provide the value function of a rear-end double-barrier knock-out call option. The life of the option is divided into two segments, in the first period from $t = 0$ to time t it is an standard call option, the second period from time t to maturity T there are two barriers B_{low} and B_{up} with $B_{\text{low}} < B_{\text{up}}$. If the underlying crosses either barrier the option becomes worthless.

Theorem 3.1. *The value function of a rear-end double barrier knock-out call option $BC_r(S, K, t, B_{\text{low}}, B_{\text{up}})$ with barriers B_{low} and B_{up} and strike K can be obtained by*

case when $K > B_{\text{low}}$

$$
\begin{aligned}
BC_r = \sum_{n=1}^{\infty} K \left(\frac{S}{K}\right)^{-(1/2)(k_1-1)} &\left\{ \frac{2}{L\left[\frac{1}{4}(k_1+1)^2 + \left(\frac{n\pi}{L}\right)^2\right]} \right. \\
&\left[\frac{1}{2}(k_1+1)\sin\left(\frac{n\pi}{L}\ln\frac{B_{\text{low}}}{K}\right) + \frac{n\pi}{L}\cos\left(\frac{n\pi}{L}\ln\frac{B_{\text{low}}}{K}\right)\right. \\
&\left. - \frac{n\pi}{L}(-1)^n\left(\frac{B_{\text{up}}}{K}\right)^{(1/2)(k_1+1)}\right] - \frac{2}{L\left[\frac{1}{4}(k_1-1)^2 + \left(\frac{n\pi}{L}\right)^2\right]}\left[\frac{1}{2}(k_1 - \right. \\
&1)\sin\left(\frac{n\pi}{L}\ln\frac{B_{\text{low}}}{K}\right) + \frac{n\pi}{L}\cos\left(\frac{n\pi}{L}\ln\frac{B_{\text{low}}}{K}\right) \\
&\left.\left. - \frac{n\pi}{L}(-1)^n\left(\frac{B_{\text{up}}}{K}\right)^{(1/2)(k_1-1)}\right]\right\}\left[b_{1n}\sin\left(\frac{n\pi}{L}\ln\frac{S}{B_{\text{low}}}\right)\right. \\
&\left. + b_{2n}\cos\left(\frac{n\pi}{L}\ln\frac{S}{B_{\text{low}}}\right)\right]\exp\left\{-(1/2)(n\pi/L)^2\sigma^2(T-t) + (1/2)\beta\sigma^2 T\right\}
\end{aligned}
\tag{32}
$$

case when $K < B_{\text{low}}$

$$BC_r = \sum_{n=1}^{\infty} K \left(\frac{S}{K}\right)^{-(1/2)(k_1-1)} \left\{ \frac{2n\pi}{L^2\left[\frac{1}{4}(k_1+1)^2 + \left(\frac{n\pi}{L}\right)^2\right]} \left[\left(\left(\frac{B_{\text{low}}}{K}\right)^{(1/2)(k_1+1)} - (-1)^n \left(\frac{B_{\text{up}}}{K}\right)^{(1/2)(k_1+1)}\right)\right] - \frac{2n\pi}{L^2\left[\frac{1}{4}(k_1-1)^2 + \left(\frac{n\pi}{L}\right)^2\right]} \left[\left(\left(\frac{B_{\text{low}}}{K}\right)^{(1/2)(k_1-1)} - (-1)^n \left(\frac{B_{\text{up}}}{K}\right)^{(1/2)(k_1-1)}\right)\right]\right\} \left[b_{1n}\sin\left(\frac{n\pi}{L}\ln\frac{S}{B_{\text{low}}}\right) + b_{2n}\cos\left(\frac{n\pi}{L}\ln\frac{S}{B_{\text{low}}}\right)\right] \exp\left\{-(1/2)(n\pi/L)^2\sigma^2(T-t) + (1/2)\beta\sigma^2 T\right\} \tag{33}$$

where

$$L = \ln\left(\frac{B_{\text{up}}}{B_{\text{low}}}\right),$$
$$k = \frac{2r}{\sigma^2}$$

and

$$\begin{aligned} b_{1n} &= \int_{a_1}^{a_2} \cos\left(\tfrac{n\pi x}{L}\sqrt{\sigma^2 t}\right) e^{-(1/2)x^2}\, dx, \\ b_{2n} &= \int_{a_1}^{a_2} \sin\left(\tfrac{n\pi x}{L}\sqrt{\sigma^2 t}\right) e^{-(1/2)x^2}\, dx, \\ a_1 &= \tfrac{\ln(B_{\text{low}}/S)}{\sqrt{\sigma^2 t}}, a_2 = \tfrac{\ln(B_{\text{up}}/S)}{\sqrt{\sigma^2 t}} \end{aligned} \tag{34}$$

Proof. The proof is to transform the BS PDE into the heat equation with appropriate boundary conditions. For a detailed proof refer to Hui [12]. □

As in the case of standard options one can look at so-called digital (binary) options. For example a cash-or-nothing digital call option pays a certain amount of currency if the price of the underlying is greater than the strike at maturity, $DC_{CoN} = X\mathbb{1}_{\{S_T > K\}}$.

We now want to derive the value function for a digital barrier option with two barriers, a double barrier digital. Such a contract pays an amount of currency only if the underlying stays between the barriers during the time period when the barriers are active. We will consider first the case where the option has only one single period and afterwards we extend the results to arbitrarily many barrier periods.

3.1 One Period Double Barrier Digital

A one period double barrier digital pays out one unit of currency at maturity if the underlying has stayed within the barriers between a pre-specified time. Denote by $P > 0$ the length of the barrier period and T_0 the time when the barriers are active, then the payoff at $T_0 + P$ is

$$C_1 := \mathbb{1}_{\{B_{\text{low}} < S_t < B_{\text{up}},\ t \in [T_0, T_0 + P]\}}.$$

Then the price of this digital can be found by

$$BD(S_t, t; \{T_0\}, P, B_{\text{low}}, B_{\text{up}}, r) := e^{-r(T_0+P-t)}\mathbb{E}\left[C_1 | \mathcal{F}_t\right], \tag{35}$$

where $\mathbb{E}$ is the expectation w.r.t. the pricing measure $\mathbf{P}$.

The barriers are active towards the end of the contract so it can be viewed as a *rear-end* barrier option, c.f. Hui ([12]).

For the multiple barrier case the derivation with PDE methods is more appropriate so we present the idea for the one period case here.

Let us define the value function by

$$f(S, t) := BD(S, t; \{T_0\}, P, B_{\text{low}}, B_{\text{up}}, r),$$

which satisfies the Black-Scholes PDE (22) with terminal condition $f(S, T_0 + P) = 1$, for $S \in (B_{\text{low}}, B_{\text{up}})$ and boundary conditions $f(B_{\text{low}}, t) = f(B_{\text{up}}, t) = 0$ for $t \in [T_0, T_0 + P]$.

To transform the Black-Scholes PDE into the heat equation,

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial \tau}. \tag{36}$$

we apply the same transformations as before, e.g. $f(S, t) = e^{\alpha x + \beta \tau} U(x, \tau)$ with

$$x := \log(S/B_{\text{low}}), \qquad \tau := \tfrac{1}{2}\sigma^2(T_0 + P - t), \tag{37}$$
$$\alpha := -\frac{1}{2}\left(\frac{2}{\sigma^2}r - 1\right), \qquad \beta := -\frac{2r}{\sigma^2} - \alpha^2.$$

The original time points $(0, T_0, T_0+P)$ are now mapped to $(\frac{1}{2}\sigma^2(T_0+P), p, 0)$, where $p := \frac{1}{2}\sigma^2 P$ is the transformed barrier period length.

The boundary as well as the terminal condition need to be transformed as well. Therefore in the new coordinates we get

$$U(0,\tau) = U(L,\tau) = 0, \qquad \tau \in [0,p], \tag{38}$$

where $L := \log(B_{\text{up}}/B_{\text{low}})$, for the boundary condition and

$$U(x,0) = e^{-\alpha x}, \qquad x \in (0,L). \tag{39}$$

as initial condition.

The next proposition shows how to price a one-period digital barrier option.

Proposition 3.1. *For $0 < t < T_0$, the price of a barrier digital with barrier period $[T_0, T_0 + P]$ and payoff C_1 at $T_0 + P$ is*

$$\begin{aligned} BD(S,t;\{T_0\},P,B_{\text{low}},B_{\text{up}},r) = \sqrt{2\pi}\left(\frac{S}{B_{\text{low}}}\right)^{\alpha} \\ \cdot \sum_{k=1}^{\infty} k \frac{1-(-1)^k e^{-\alpha L}}{\alpha^2 L^2 + k^2\pi^2} e^{-(\frac{k\pi}{L})^2 p + \beta\tau} \\ \cdot \int_{-\frac{x}{\sqrt{2(\tau-p)}}}^{\frac{L-x}{\sqrt{2(\tau-p)}}} \sin\left(\frac{k\pi}{L}(x + y\sqrt{2(\tau-p)})\right) e^{-y^2/2} dy. \end{aligned} \tag{40}$$

Proof. We have to solve the problem (36)–(39). First consider the rectangle $(0,L) \times (0,p)$. There the solution, which is unique [8, p. 358], can be found by separation of variables [8, Section 4.1]:

$$U(x,\tau) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L}x\right) e^{-(\frac{k\pi}{L})^2\tau}, \qquad (x,\tau) \in (0,L) \times (0,p), \tag{41}$$

where

$$b_k := \frac{2}{L}\int_0^L e^{-\alpha x_1}\sin\left(\frac{k\pi}{L}x_1\right)dx_1 = 2k\pi\frac{1-(-1)^k e^{-\alpha L}}{\alpha^2L^2+k^2\pi^2}$$

are the Fourier coefficients of the boundary function $U(x,0) = e^{-\alpha x}$. To verify that this is indeed the solution, it suffices to appeal to the standard criterion for exchanging derivative and series [18, p. 152].

At $\tau = p$, the solution is given by (41) for $0 < x < L$ and vanishes otherwise. Inserting $\tau = p$ into (41) yields

$$U(x,p) = \begin{cases} \sum_{k=1}^{\infty} 2k\pi\frac{1-(-1)^k e^{-\alpha L}}{\alpha^2L^2+k^2\pi^2}\sin(\frac{k\pi}{L}x)e^{-(\frac{k\pi}{L})^2 p}, & 0 < x < L \\ 0, & x \le 0 \text{ or } x \ge L. \end{cases} \tag{42}$$

Now we solve for U in the region $\mathbb{R}\times(p, \frac{1}{2}\sigma^2(T_0+P))$. There are no boundary conditions here, since the barriers are not active in the interval $(0, T_0)$ (in the original time scale). A solution is found by convolving the initial condition (42) with the heat kernel [8, p. 47]:

$$\begin{aligned} U(x,\tau) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(x+y\sqrt{2(\tau-p)},p)e^{-y^2/2}dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2(\tau-p)}}}^{\frac{L-x}{\sqrt{2(\tau-p)}}} U(x+y\sqrt{2(\tau-p)},p)e^{-y^2/2}dy. \end{aligned} \tag{43}$$

Inserting (42) and rearranging yields (40). It remains to argue that the solution (43) is the right one, i.e., that it indeed equals the transformation of the value function (35). By Tikhonov's classical uniqueness theorem [13, p. 216f], the solution in the strip $\mathbb{R}\times(p, \frac{1}{2}\sigma^2(T_0+P))$ is unique if we restrict attention to functions admitting bounds of the form $c_1\exp(c_2|x|^2)$ with positive constants c_1 and c_2. Now note that (42), and hence also (43), is bounded by a constant, and that the solution U we seek is of at most exponential growth, since our value function $f(S,t) = e^{\alpha x+\beta\tau}U(x,\tau)$ is bounded. □

3.2 Multiple Period Double Barrier Digital

Consider a contract that pays one unit of currency at maturity if the underlying has remained between the two barriers B_{low} and B_{up} during pre-specified non-overlapping time intervals. This means we look at n tenor dates

$$0 < T_0 < \cdots < T_{n-1}$$

and a fixed period length $P > 0$, the time intervals are in this notation $[T_i, T_i + P]$, $i = 0, \ldots, n-1$.
By the risk-neutral pricing formula, the price of this "multi-period double barrier digital" is given by

$$BD(S_t, t; \{T_0, \ldots, T_{n-1}\}, P, B_{\text{low}}, B_{\text{up}}, r) := e^{-r(T_{n-1}+P-t)} \mathbb{E}\left[\prod_{i=1}^{n} C_i \middle| \mathcal{F}_t\right], \tag{44}$$

where

$$C_i := \mathbb{1}_{\{B_{\text{low}} < S_t < B_{\text{up}},\ t \in [T_{i-1}, T_{i-1}+P]\}}.$$

To calculate the price, we use once again the coordinate change (37) (with T_{n-1} in place of T_0). The n barrier periods $[T_i, T_i + P]$ are mapped to $[\tau_i, \tau_i + p]$, where

$$\tau_i := \tfrac{1}{2}\sigma^2(T_{n-1} - T_{i-1}), \qquad i = n, \ldots, 1,$$

are the images of the barrier period endpoints under the coordinate change (see Figure 1).

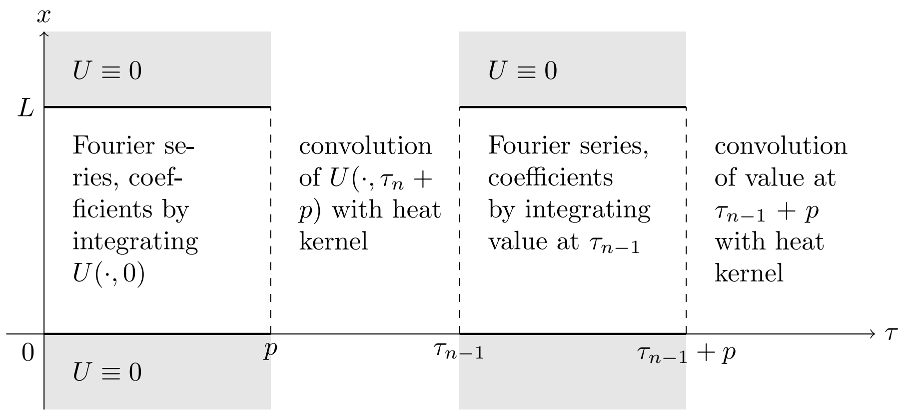

Figure 1: For an arbitrary number of barrier periods consider the coordinate change (37) and solve the boundary value problem by Fourier series within the barrier period and by convolving with the heat kernel when the barriers are not active.

The following proposition contains the pricing formula for the multiple period case. The first formula (45) is for time points inside a barrier period, whereas the second expression (46) holds for valuation times where the barriers are not active.

Proposition 3.2. *The value function* (44) *equals* $e^{\alpha x+\beta\tau}U(x,\tau)$*, where for* $0 \le j < n$*,* $\tau_{n-j} \le \tau \le \tau_{n-j}+p$*,* $0 < x < L$*, we have*

$$U(x,\tau) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{j} \underbrace{\int_{0}^{L} \cdots \int_{0}^{L}}_{j+1} \sum_{k_1=0}^{\infty} \cdots \sum_{k_{j+1}=0}^{\infty}$$

$$g_j(k_1,\ldots,k_{j+1};x_1,\ldots,x_{j+1};y_1,\ldots,y_j;x,\tau)dx_1\ldots dx_{j+1}dy_1\ldots dy_j, \quad (45)$$

whereas for $0 \le j < n$*,* $\tau_{n-j}+p < \tau < \tau_{n-(j+1)}$ *(with* $\tau_0 := \infty$*),* $x \in \mathbb{R}$*, we have*

$$U(x,\tau) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{j+1} \underbrace{\int_{0}^{L} \cdots \int_{0}^{L}}_{j+1} \sum_{k_1=0}^{\infty} \cdots \sum_{k_{j+1}=0}^{\infty}$$

$$h_j(k_1,\ldots,k_{j+1};x_1,\ldots,x_{j+1};y_1,\ldots,y_{j+1};x,\tau)dx_1\ldots dx_{j+1}dy_1\ldots dy_{j+1}. \quad (46)$$

where the auxiliary functions are defined as follows:

$$\begin{aligned} &h_j(k_1,\ldots,k_{j+1};x_1,\ldots,x_{j+1};y_1,\ldots,y_{j+1};x,\tau) \\ &\quad := \tfrac{1}{\sqrt{2\pi}} e^{-y_{j+1}^2/2}\, \mathbf{1}_{\left[-\frac{x}{\sqrt{2(\tau-(\tau_{n-j}+p))}},\frac{L-x}{\sqrt{2(\tau-(\tau_{n-j}+p))}}\right]}(y_{j+1}) \\ &\cdot g_j(k_1,\ldots,k_{j+1};x_1,\ldots,x_{j+1};y_1,\ldots,y_j;x+y_{j+1}\sqrt{2(\tau-(\tau_{n-j}+p))},\tau_{n-j}+p) \end{aligned}$$

and

$$\begin{aligned} &g_j(k_1,\ldots,k_{j+1};x_1,\ldots,x_{j+1};y_1,\ldots,y_j;x,\tau) \\ &\qquad := \tfrac{2}{L} \sin \tfrac{k_{j+1}\pi x_{j+1}}{L} \sin \tfrac{k_{j+1}\pi x}{L} e^{-(k_{j+1}\pi/L)^2(\tau-\tau_{n-j})} \\ &\qquad\quad \cdot h_{j-1}(k_1,\ldots,k_j;x_1,\ldots,x_j;y_1,\ldots,y_j;x_{j+1},\tau_{n-j}), \end{aligned}$$

with the recursion starting at

$$g_0(k_1;x_1;;x,\tau) := \tfrac{2}{L} e^{-\alpha x_1} \sin \tfrac{k_1\pi x_1}{L} \sin \tfrac{k_1\pi x}{L} e^{-(k_1\pi/L)^2\tau}. \quad (47)$$

Proof. The idea is to iterate the argument of Proposition 3.1 (see Figure 1). We use separation of variables in the barrier periods, and convolution with the heat kernel for the periods in between. The required initial condition at the left boundary comes from the previous step of the iteration (for $j = 0$

also from the payoff, of course). The discussion of existence and uniqueness is analogous to the proof of Proposition 3.1, and we omit the details.

For $j = 0$, formula (45) is identical to (41). To show (46) for $j = 0$, let $p < \tau < \tau_{n-1}$ (recall that $\tau_n = 0$) and $x \in \mathbb{R}$, and use (43) and (41) to obtain

$$
\begin{aligned}
U(x,\tau) &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\frac{x}{\sqrt{2(\tau-p)}}}^{\frac{L-x}{\sqrt{2(\tau-p)}}} U(x + y_1\sqrt{2(\tau-p)}, p) e^{-y_1^2/2} dy_1 \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \mathbf{1}_{\left[-\frac{x}{\sqrt{2(\tau-p)}}, \frac{L-x}{\sqrt{2(\tau-p)}}\right]}(y_1) \\
&\qquad \int\limits_0^L \sum_{k_1=0}^{\infty} g_0(k_1; x_1; ; x + y_1\sqrt{2(\tau-p)}, p) e^{-y_1^2/2} dx_1 dy_1 \\
&= \int\limits_{-\infty}^{\infty} \int\limits_0^L \sum_{k_1=0}^{\infty} h_0(k_1; x_1; y_1; x, \tau) dx_1 dy_1.
\end{aligned}
$$

This is (46) for $j = 0$.

Next consider a rectangle

$$
(\tau, x) \in (\tau_{n-j}, \tau_{n-j} + p) \times (0, L), \qquad 1 \le j < n. \tag{48}
$$

At the left boundary, the solution is $x_{j+1} \mapsto U(x_{j+1}, \tau_{n-j})$. By the induction hypothesis, it equals (46) with j replaced by $j - 1$:

$$
\begin{aligned}
U(x_{j+1}, \tau_{n-j}) = \underbrace{\int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty}}_{j} \underbrace{\int\limits_0^L \cdots \int\limits_0^L}_{j} \sum_{k_1=0}^{\infty} \cdots \sum_{k_j=0}^{\infty} \\
h_{j-1}(k_1, \dots, k_j; x_1, \dots, x_j; y_1, \dots, y_j; x_{j+1}, \tau_{n-j}) dx_1 \dots dx_j dy_1 \dots dy_j.
\end{aligned} \tag{49}
$$

The solution in the rectangle (48) is thus obtained by separation of variables as

$$
U(x,\tau) = \sum_{k_{j+1}=0}^{\infty} b_{k_{j+1}} \sin\left(\frac{k_{j+1}\pi}{L} x\right) e^{-(\frac{k_{j+1}\pi}{L})^2(\tau - \tau_{n-j})}, \tag{50}
$$

where

$$b_{k_{j+1}} := \frac{2}{L} \int_0^L U(x_{j+1}, \tau_{n-j}) \sin\left(\frac{k_{j+1}\pi}{L} x_{j+1}\right) dx_{j+1} \tag{51}$$

denote now the Fourier coefficients of $x_{j+1} \mapsto U(x_{j+1}, \tau_{n-j})$. Inserting (49) into (51) and then (51) into (50) yields (45), by the definition of g_j.

Finally, consider a strip

$$(\tau, x) \in (\tau_{n-j} + p, \tau_{n-(j+1)}) \times \mathbb{R}, \qquad 1 \le j < n. \tag{52}$$

At the left boundary, we use (45) as induction hypothesis. The solution thus vanishes for $x \notin (0, L)$, and for $\tau = \tau_{n-j} + p$ and $x \in (0, L)$ it is

$$U(x, \tau_{n-j} + p) = \underbrace{\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty}}_{j} \underbrace{\int_0^L \dots \int_0^L}_{j+1} \sum_{k_1=0}^{\infty} \dots \sum_{k_{j+1}=0}^{\infty}$$
$$g_j(k_1, \dots, k_{j+1}; x_1, \dots, x_{j+1}; y_1, \dots, y_j; x, \tau_{n-j} + p) dx_1 \dots dx_{j+1} dy_1 \dots dy_j. \tag{53}$$

As above, the solution in the strip (52) is found by convolution with the heat kernel:

$$U(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{1}_{\left[-\frac{x}{\sqrt{2(\tau-(\tau_{n-j}-p))}}, \frac{L-x}{\sqrt{2(\tau-(\tau_{n-j}-p))}}\right]}(y_{j+1})$$
$$U(x + y_{j+1}\sqrt{2(\tau - (\tau_{n-j} - p))}, \tau_{n-j} - p) e^{-y_{j+1}^2/2} dy_{j+1}.$$

Now insert (53), with x replaced by $x + y_{j+1}\sqrt{2(\tau - (\tau_{n-j} + p))}$, and use the definition of h_j to conclude (46). $\square$

The one-period case can be also be priced with Proposition 3.2, which is simply (46) for $j = 0$.

A different option with the same barrier conditions can easily be priced by replacing the quantity $e^{-\alpha x_1}$ in (47) by the appropriate payoff $U(x_1, 0)$.

3.3 Structure Floors

As noted earlier a digital double barrier option makes sense as a stand-alone product but a more practical contract is a structured note with several coupons. Often these products feature an aggregate floor at maturity, i.e. the final coupon date, where the holder receives an additional payoff dependent on the coupons but in any case an agreed upon amount.

For this section we assume that the tenor structure satisfies $T_{i-1} + P = T_i$ for $1 \leq i < n$, and define $T_n := T_{n-1} + P$.

We consider a structured note with n coupons, where the i-th coupon is similar to the payoff for double barrier digital and consists of a payment

$$C_i = \mathbb{1}_{\{B_{\mathrm{low}} < S_t < B_{\mathrm{up}},\ t \in [T_{i-1}, T_i]\}}, \qquad 1 \leq i \leq n, \tag{54}$$

at time T_i. These coupons can be priced by Proposition 3.1 (replace T_0 by T_{i-1}). In addition, the holder receives the terminal premium

$$\left(F - \sum_{i=1}^{n} C_i \right)^{+} \tag{55}$$

at T_n, where $F > 0$. This means that the aggregate payoff $A := \sum\limits_{i=1}^{n} C_i$ of the note is floored at F.

While the individual coupons are straightforward to valuate, it is less obvious how to get a handle on the law of A. We now show that this law is linked to barrier options with several barrier periods. Indeed, the following result is based on the fact that the moments

$$\mathbb{E}\left[A^{\nu}\right] = \sum_{i=0}^{n} i^{\nu} \mathbb{P}\left[A = i\right], \qquad 1 \leq \nu < n, \tag{56}$$

of A are linear combinations of multi-period double barrier option prices, with coefficients

$$c(\nu, J) := \sum_{\substack{0 \leq i_1, \ldots, i_n \leq \nu \\ \mathrm{supp}(\mathbf{i}) = J}} \binom{\nu}{i_1, \ldots, i_n}, \qquad J \subseteq \{1, \ldots, n\}. \tag{57}$$

(The notation $\mathrm{supp}(\mathbf{i}) = J$ means that J is the set of indices such that the corresponding components of the vector $\mathbf{i} = (i_1, \ldots, i_n)$ are non-zero.) W.l.o.g. we assume that the valuation time is $t = 0$.

Theorem 3.2. *The price of the structure floor* (55) *at time* $t = 0$ *can be expressed as*

$$e^{-rT_n}\mathbb{E}\left[(F-A)^+\right] = e^{-rT_n}\sum_{i=0}^{n\wedge\lfloor F\rfloor}(F-i)\mathbb{P}\left[A=i\right], \tag{58}$$

where

$$\mathbb{P}\left[A=n\right] = BD(S_0, 0; \{T_0\}, T_n - T_0, B_{\text{low}}, B_{\text{up}}, 0). \tag{59}$$

The other point masses $\mathbb{P}[A = i]$ *in* (58) *can be recovered from the moments of* A *by solving* (56) *(including* $\nu = 0$*). The moments in turn can be computed from barrier digital prices by (*$1 \le \nu < n$*)*

$$\mathbb{E}\left[A^\nu\right] = \sum_{J\subseteq\{1,\dots,n\}} c(\nu, J)\cdot BD(S_0, 0; \{T_j : j\in J\}, P, B_{\text{low}}, B_{\text{up}}, 0), \tag{60}$$

where the coefficients $c(\nu, J)$ *are defined in* (57)*.*

Proof. The expression (58) is obvious. The event in (59) means that the underlying stayed within the barriers and all of the n coupons (54) are paid. By our assumption that $T_i = T_{i-1} + P$, its risk-neutral probability is the (undiscounted) price of a double barrier digital with one barrier period $[T_0, T_n]$, which yields (59). To show (60), we calculate

$$\begin{aligned}
\mathbb{E}\left[A^\nu\right] = \mathbb{E}\left[\left(\sum_{i=1}^n C_i\right)^\nu\right] &= \sum_{i_1,\dots,i_n}\binom{\nu}{i_1,\dots,i_n}\mathbb{E}\left[C_1^{i_1}\cdots C_n^{i_n}\right] \\
&= \sum_{i_1,\dots,i_n}\binom{\nu}{i_1,\dots,i_n}\mathbb{E}\Bigg[\prod_{\substack{j=1\\ i_j>0}}^n C_j\Bigg] \\
&= \sum_{J\subseteq\{1,\dots,n\}}\Bigg(\sum_{\substack{i_1,\dots,i_n\\ \text{supp}(\mathbf{i})=J}}\binom{\nu}{i_1,\dots,i_n}\Bigg)\mathbb{E}\left[\prod_{j\in J}C_j\right].
\end{aligned}$$

Now observe that $\prod_{j\in J} C_j$ is the payoff of a double barrier digital with barrier periods $[T_j, T_j + P]$ for $j \in J$. □

3.4 Corridor Options

Later in this section we show how the price of the structure floor (55) can be approximated by another contract, namely a corridor option. For this purpose we show how to price this instrument following the work of Fusai [9].

A corridor option is a exotic derivative which pays at maturity an amount that depends on the time spent by the underlying between two barriers or inside a corridor.

The price of the underlying can be described by the usual SDE (20).
Now define a random variable by

$$\tau(t, B_{\text{low}}, B_{\text{up}}) := \int_0^t \mathbb{1}_{\{B_{\text{low}}<S(z)<B_{\text{up}}\}}\, dz. \tag{61}$$

Then a corridor (put) option has a payoff at maturity by $(\tau - K)^+$, with strike K. Fusai [9] studied a call option of this type.

If we look at the integrand in (61) we notice that

$$\begin{aligned}\mathbb{1}_{\{B_{\text{low}}<S(z)<B_{\text{up}}\}} &= \mathbb{1}_{\left\{B_{\text{low}}<S\exp\left(r-\frac{\sigma^2}{2}\right)z+\sigma W(z))<B_{\text{up}}\right\}} = \\ &= \mathbb{1}_{\left\{\frac{1}{\sigma}\log\{B_{\text{low}}/S\}<\frac{1}{\sigma}\left(r-\frac{\sigma^2}{2}\right)z+W(z)<\frac{1}{\sigma}\log\{B_{\text{up}}/S\}\right\}}.\end{aligned}$$

So we can calculate the density function with the help of the occupation time of the Brownian motion inside the barriers $u = \log(L)/\sigma$, $l = \log(U)/\sigma$ and starting value $x = \log(S)/\sigma$.

Proposition 3.3. *The price of a corridor option at time t is given by*

$$\begin{aligned}P\left(t, K, B_{\text{low}}, B_{\text{up}}\right) := \int_0^K (K-\tau)\, f_\tau(\tau, x, t, u, l)\; d\tau + (K-\tau)^+ \\ \times \mathbb{P}_{x\in(l,u)}\left[\tau(t,u,l)=t\right].\end{aligned} \tag{62}$$

where $f_\tau(\tau, x, t, u, l)$ is the density function of the r.v. $\tau(\cdot)$.

Since the expression for the density function $f(\cdot)$ is rather involved and for our purpose we do not need an explicit expression we do not reproduce it at this point.[5] Let's look at the first term in (62), we can apply the Laplace transformation and using the fact that it transforms convolution into multiplication we get

[5] c.f. [10]

$$\mathcal{L}_K\left\{\int_0^K (K-\tau)\, f_\tau(\tau,x,t,u,l)\; d\tau\right\}(\mu) = \frac{\Omega(t,\mu,x,l,u,m)}{\mu},$$

where

$$\begin{aligned}\Omega(t,\mu,x,l,u,m) &:= \int_0^t e^{i\mu\tau} f_\tau(\tau,x,t,u,l)\; d\tau \\ &= e^{-mx-\frac{m^2}{2}t}\mathcal{L}_\gamma^{-1}\left\{\omega(\gamma,\mu,x,l,u,m)\right\}(t),\end{aligned}$$

as defined in Appendix A.1. That means that the price of a put can be found by a double Laplace inversion of $\omega(\gamma,\mu) := \omega(\gamma,\mu,x,l,u,m)/\mu^2$ plus the probability that the underlying stays inside the corridor the whole time.

3.4.1 Laplace inversion

To price a corridor option we need a method for inverting Laplace transforms in two dimensions. Since there are virtually no closed form inverses we apply a very accurate and stable method to numerically invert Laplace transforms by a Padé rational function proposed by Singhal et al. ([20]). Laplace transforms are a useful method for finding solutions to partial differential equation, too.

Definition 3.1. *The two dimensional Laplace transform is simply an extension of of the standard well-known transformation. It is given as*

$$F(s_1,s_2) = \mathcal{L}\left[f(t_1,t_2),s_1,s_2\right] = \int_0^\infty\int_0^\infty f(t_1,t_2)e^{-s_1t_1-s_2t_2}\; dt_1\; dt_2,$$

whereas the Laplace inversion formula has the following form

$$f(t_1,t_2) = \left(\frac{1}{2\pi i}\right)^2 \int_{c-i\infty}^{c+i\infty}\int_{c-i\infty}^{c+i\infty} F(s_1,s_2)e^{s_1t_1+s_2t_2}\; ds_1\; ds_2.$$

Now as said before, closed form solutions are very hard to find so we need a method to evaluate above expression. The idea behind the approach suggested by [20] is to approximate the exponential function by a Padé rational function

$$e^{z_k} \simeq \frac{\sum\limits_{i=0}^{M_k} (M_k + N_k - i)! \binom{N_k}{i} z_k^i}{\sum\limits_{i=0}^{N_k} (-1)^i (M_k + N_k - i)! \binom{M_k}{i} z_k^i}$$

where N_k and M_k should be selected such that $N_k < M_k$. Using complex calculus Singhal et al. [20] arrive at the following form which can be easily implemented

$$\widehat{f}(t_1, t_2) = \frac{1}{t_1 t_2} \sum_{i=1}^{M_1} \sum_{j=1}^{M_1} K_{1i} K_{2j} F\left(\frac{z_{1i}}{t_1}, \frac{z_{2j}}{t_j}\right) \tag{63}$$

where $z_{nk}, n \in \{1, 2\}$ are the poles of the approximation and $K_{nk}, n \in \{1, 2\}$ the corresponding residues. In the next section we show that this method works well for functions which are sufficiently smooth.

3.5 Approximation by a Corridor Option

Theorems 3.2 and 3.2 express the price of the structure floor (55) by iterated sums and integrals. Due to the factors of order $e^{-k_j^2}$, the infinite series $\sum_{k_j}$ may be truncated after just a few terms. Still, numerical quadrature may be too involved for a large number of coupons, so we present an approximation. Let us fix a maturity $T = T_n$ and assume that the n coupon periods

$$\mathcal{T}_i^n :=]\tfrac{i-1}{n}T, \tfrac{i}{n}T], \qquad 1 \leq i \leq n,$$

have length T/n. For large n, the proportion of intervals during which the underlying stays inside the barrier interval

$$\mathcal{B} := [B_{\text{low}}, B_{\text{up}}]$$

is similar to the proportion of time that the underlying spends inside $\mathcal{B}$, i.e., the occupation time. A somewhat related problem has been studied in [10] (continuous vs. discrete monitoring for occupation time derivatives). Our reasoning is made precise in the following result, which holds not only for the Black-Scholes model, but for virtually any continuous model. Note that the level sets of geometric Brownian motion have a.s. measure zero (cf. [14, Theorem 2.9.6]).

Theorem 3.3. *Let $(S_t)_{t\geq 0}$ be a continuous stochastic process such that for each real c the level set $\{t \geq 0 : S_t = c\}$ has a.s. Lebesgue measure zero. Then we have a.s.*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{S_t \in \mathcal{B} \ \forall t \in \mathcal{T}_i^n\}} = \frac{1}{T} \int_0^T \mathbf{1}_{\mathcal{B}}(S_t) dt.$$

Proof. For $1 \leq i \leq n$, define processes $(X_{ni}(t))_{0\leq t \leq T}$ by

$$X_{ni}(t) := \begin{cases} 1 & \text{if } t \in \mathcal{T}_i^n \text{ and } S_u \in \mathcal{B} \ \forall u \in \mathcal{T}_i^n \\ 0 & \text{otherwise.} \end{cases}$$

Put $X_n := \sum_{i=1}^n X_{ni}$. We claim that, a.s., the function $X_n(\cdot)$ converges pointwise on the set $[0, T] \setminus \{t : S_t = B_{\text{low}} \text{ or } S_t = B_{\text{up}}\}$, with limit $\mathbf{1}_{\mathcal{B}}(S_\cdot)$. Indeed, if $t \in [0, T]$ is such that $S_t \notin \mathcal{B}$, then $X_n(t) = 0$ for all n. If, on the other hand, $S_t \in \text{int}(\mathcal{B})$, then t has a neighborhood V such that $S_u \in \mathcal{B}$ for all $u \in V$, by continuity. Hence $X_n(t) = 1$ for large n. Since we have pointwise convergence on a set of (a.s.) full measure, we can apply the dominated convergence theorem to conclude

$$\lim_{n\to\infty} \int_0^T X_n(t) dt = \int_0^T \mathbf{1}_{\mathcal{B}}(S_t) dt, \qquad \text{a.s.}$$

But this is the desired result, since

$$\begin{aligned}
\int_0^T X_n(t) dt &= \sum_{i=1}^{n} \int_0^T X_{ni}(t) dt \\
&= \sum_{i=1}^{n} \int_{\mathcal{T}_i^n} X_{ni}(t) dt \\
&= \sum_{i=1}^{n} |\mathcal{T}_i^n| \ \mathbf{1}_{\{S_t \in \mathcal{B} \ \forall t \in \mathcal{T}_i^n\}} = \frac{T}{n} \sum_{i=1}^{n} \mathbf{1}_{\{S_t \in \mathcal{B} \ \forall t \in \mathcal{T}_i^n\}}.
\end{aligned}$$

□

Theorem 3.3 suggests the approximation

$$e^{-rT}\mathbb{E}\left[(F - A)^+\right] \approx e^{-rT} \frac{n}{T} \mathbb{E}\left[\left(\frac{FT}{n} - \int_0^T \mathbf{1}_{\mathcal{B}}(S_t) dt\right)^+\right] \tag{64}$$

for the price of the structure floor (55). It is obtained from replacing F by F/n in the relation

$$\mathbb{E}\left[(nF - A)^+\right] \sim n\mathbb{E}\left[\left(F - \frac{1}{T}\int_0^T \mathbf{1}_{\mathcal{B}}(S_t)dt\right)^+\right], \quad n \to \infty,$$

which follows from Theorem 3.3 (recall that $A = \sum_{i=1}^n C_i$ denotes the sum of the coupons). On the right hand side of (64) we recognize the price of a put on the occupation time of S, also called a corridor option. Fusai [9] studied such options in the Black-Scholes model. In particular, his Theorem 1 gives an expression for the characteristic function of $\int_0^T \mathbf{1}_{\mathcal{B}}(S_t)dt$.

The approximation (64) holds for period lengths tending to zero. One could also let the number of coupons tend to infinity for a fixed period length P, so that maturity increases linearly with n. As seen from their definition in (54), the dependence of the random variables C_i and C_j decreases for large $|i-j|$, and so it is a natural question whether a central limit theorem holds, i.e., whether

$$\frac{A - \mathbb{E}[A]}{\sqrt{\mathbf{Var}[A]}}$$

converges in law to a standard normal random variable as $n \to \infty$. Note that $\mathbb{E}[A] = \sum_{i=1}^n \mathbb{E}[C_i]$ and $\mathbb{E}[A^2] = \mathbb{E}[A] + 2\sum_{i<j}\mathbb{E}[C_iC_j]$ can be easily computed from Proposition 3.1 respectively Theorem 3.2. The structure floor (55) could then be approximately valuated by a Bachelier-type put price formula. We were not able, though, to verify any of the mixing conditions [6] that could lead to a central limit result. Numerical experiments also cast some doubt on the existence of a Gaussian limit law. This is therefore left for future research.

4 Numerical Results

Here we present numerical results to the theoretical concepts derived in the previous chapters. We will price one and multi period double barrier digitals by the semi-analytical formulas as well as with Monte-Carlo simulation to compare the results. Furthermore we will also price a structured floor and show the convergence to a corridor option. For the implementation of the value function we have chosen the computer algebra system Mathematica. It offers symbolic capabilities, which are especially helpful for defining the auxiliary functions h_j and g_j from Theorem 3.2, as well as fast multidimensional numerical integration.
The corresponding code can be found in Appendix A.2

4.1 One Period Double Barrier Digital

For the first example of a one periode double barrier digital we chose the following parameters, $S = 100$, $t = 0$, $r = 0.01$, $\sigma = 0.15$, $P = 1$, $B_{\text{low}} = 80$, $B_{\text{up}} = 120$.

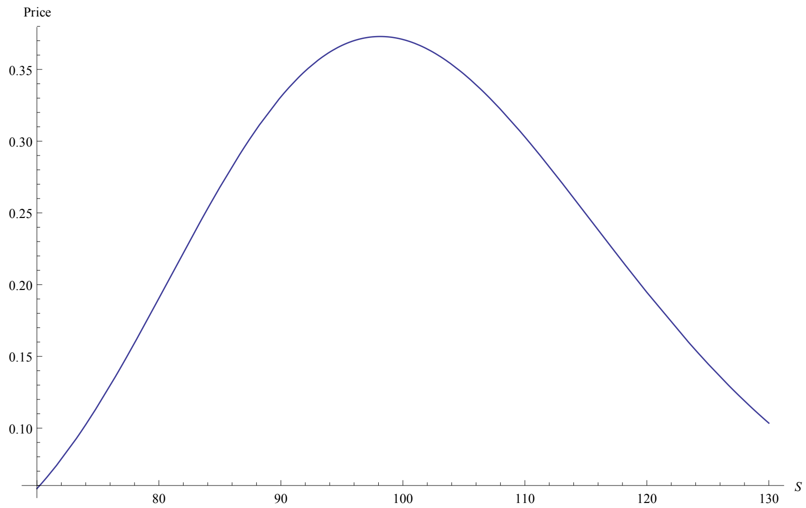

Figure 2: Price of a one period barrier digital at different underlying prices

It is clearly visible that the price of the underlying is the highest at the spot, which is the mean of the upper and lower boundary. At this point the probability of reaching the barrier is the smallest and therefore the price the highest.

The next plot shows the price of the option against time. As time passes the price goes up to 1, which is what we would expect since at maturity the option pays out 1.

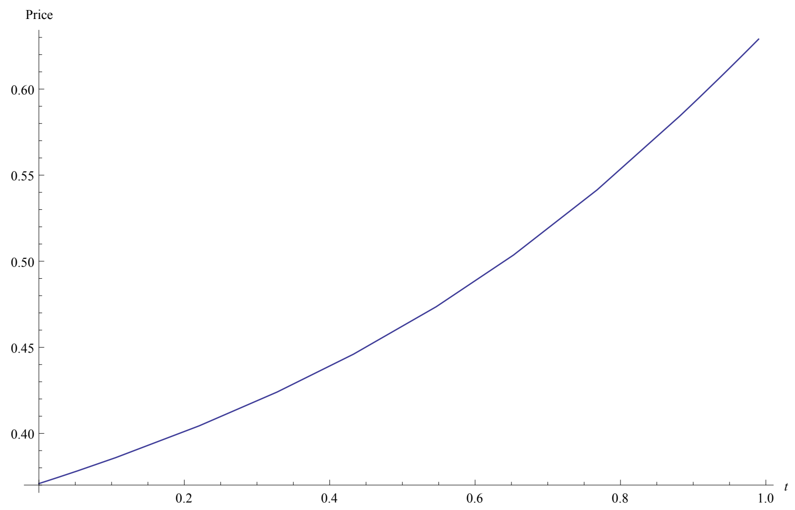

Figure 3: Price of a one period barrier digital as time goes by

We checked the pricing formula (3.1) against a Monte-Carlo simulation of the digital, the next table shows our result.

number of simulations	price	MC price	standard error
100	0.37086	0.38228	0.18701
500	0.37086	0.37512	0.06893
1000	0.37086	0.37136	0.18701
5000	0.37086	0.37072	0.02156
10000	0.37086	0.37073	0.01536

Table 1: Approximation of a one period barrier digital with Monte-Carlo simulation and increasing path size. The parameters are the same as in the previous example.

4.2 Multiple Period Double Barrier Digital

The implementation of the pricing algorithm consists of the following steps:

- Define the auxiliary functions h_j, g_j as well as the first two functions g_0 and h_0.
- Apply the coordinate change (37) to the input variables.

- Determine in which period the valuation time lies, i.e., calculate j.
- Define the integration variables and the integration limits.
- Plug in the values and integrate the functions.

In the last step we used `NIntegrate[]` to avoid symbolic integration and speed up the calculations. Furthermore, memoization should be used to save computation time when calculating the recursion. (This means storing function return values instead of repeating function calls for the same input; see, e.g., [7].) For the infinte sums we found that truncating after five summands suffices to get a reasonably accurate value.

In Figure 4 the value of a double barrier digital with two barrier periods, varying time-to-maturity and underlying price, is shown.[6] The parameters are $r = 0.01$, $\sigma = 0.15$, $B_{\text{low}} = 80$, $B_{\text{up}} = 120$, $\{T_0, T_1\} = \{1, 6\}$, and $P = 2$. It is clearly visible that if the valuation takes place during a barrier period, e.g. $t \in [T_1, T_1 + P]$, then the value of the option is zero as soon as the underlying moves beyond the barriers.

Otherwise, between the periods, for example at $t = 4$, we have a positive value even if the price process is outside the barriers.

[6] With Mathematica 8.0 the calculation of one value takes about 15s on a 2.83 GHz machine with four cores and 4 GB memory.

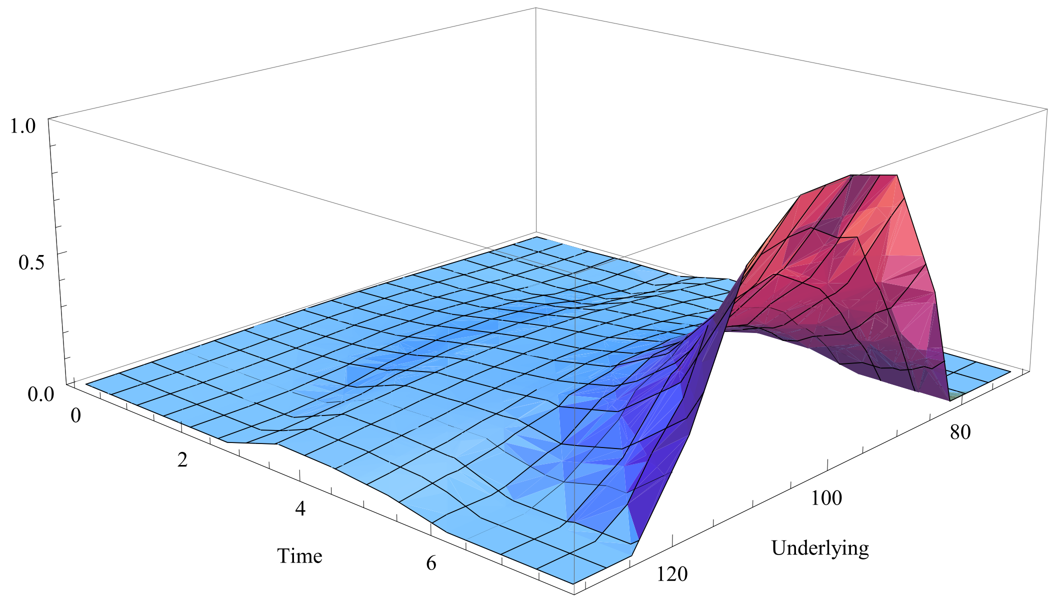

Figure 4: Value function of a double barrier digital with two barrier periods, $[1, 3]$ and $[6, 8]$. Observe that the value outside of the barriers is zero during a barrier period and takes on positive values if the underlying stays within the barriers.

The next plots shows the value if the underlying is already outside the barriers, e.g. $S = 135$. As in the plot above the price is zero during the barrier periods but can be positive outside.

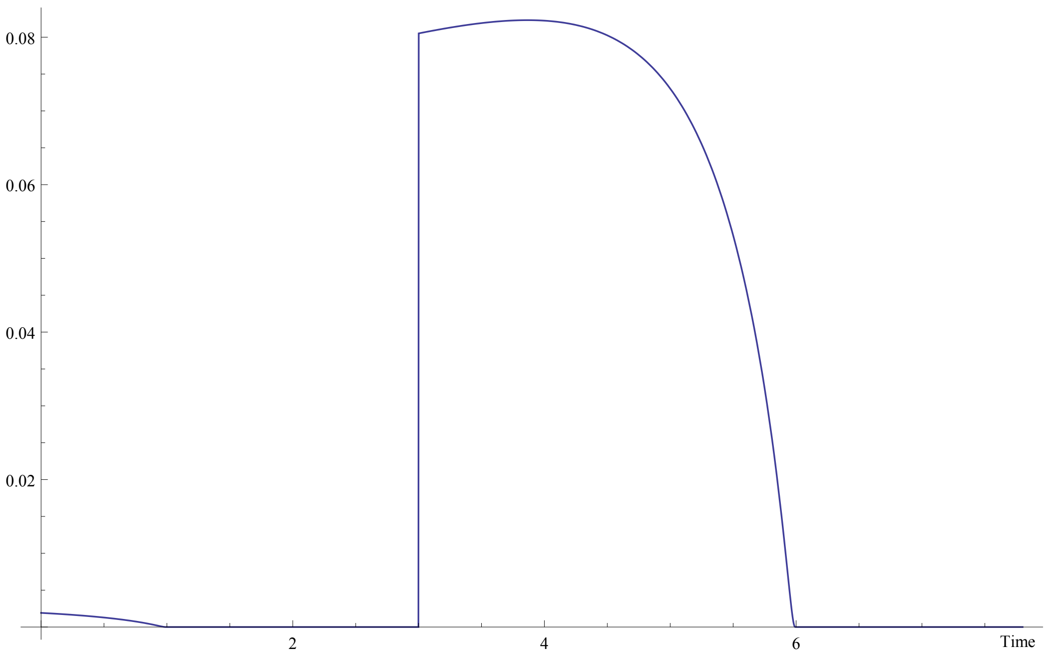

Figure 5: Value function of a double barrier digital with two barrier periods, $[1, 3]$ and $[6, 8]$, where the underlying is already out-of-the-money.

4.3 Monte Carlo simulation of barrier digital prices

Since the evaluation for multiple barrier periods is rather involved requires long computing times another option is to price them with Monte Carlo simulation. The idea is visualized by the following plot.

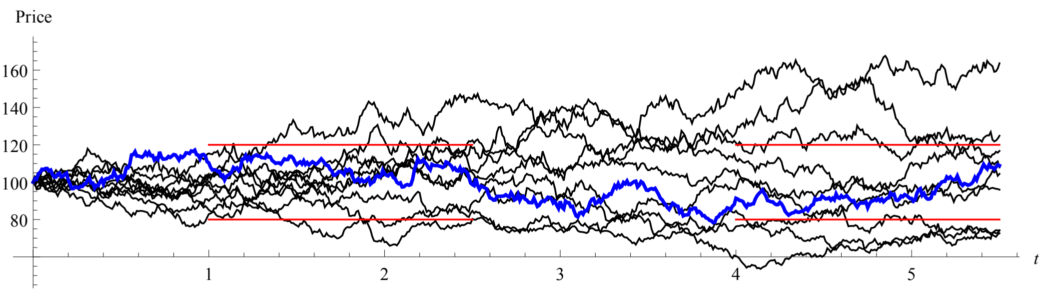

Figure 6: Monte Carlo simulation of a Barrier option with two barrier periods. The blue path is a valid path, because during barrier times it stays within the corridor.

If a path stays within the barriers during the barrier times its a valid path and we simply sum up the valid paths. Nevertheless Monte Carlo simulation of barrier digitals, as well as plain vanilla barrier options poses the problem, that because of the discrete monitoring of the underlying price process the barrier might have been crossed without being observed.

Compared to continuously monitored, Monte Carlo simulation gives an overestimated hitting time and thus misprice the option. If we consider a barrier option with only one barrier, e.g. a knock-out call, this problem has been studied before and can be solved with the help of the Brownian bridge. The idea is to use the law of the maximum of the Brownian bridge to evaluate the probability that the process hits the barrier between each step of the simulation. As mentioned this is not possible for double barrier options.

The paper of Baldi et al. [3] deals with this type of problem, they derive, with large deviation methods, simple formulas to approximate the exit probability for double barrier options where the underlying process is driven by a diffusion process.

We recall here their main theorem, cf [3] Theorem 2.1.

Theorem 4.1. *Consider a partition $t_0 = 0 < t_1 < \cdots < t_n = T$ of the time interval $[0, T]$ with $t_i - t_{i-1} = \epsilon$ for $i = 0, \ldots n-1$.*
For a time $T_0 \in [0, T]$ denote by $p^{\epsilon}_{U,L}(T_0, \zeta, y)$ the probability that the process S, 20, hits one of the barriers in the interval $[T_0, T_0+\epsilon]$, given the observations $\log S_{T_0} = \zeta$ and $\log S_{T_0+\epsilon} = y$. Then

$$p^{\epsilon}_{U,L}(T_0, \zeta, y) = \begin{cases} \exp\left\{-\frac{2}{\sigma^2\epsilon}(U-\zeta)(U-y)\right\}(1+o(\epsilon^k))) & \text{if } \zeta + y > U + L \\ \exp\left\{-\frac{2}{\sigma^2\epsilon}(\zeta-L)(y-L)\right\}(1+o(\epsilon^k))) & \text{if } \zeta + y < U + L \end{cases} \tag{65}$$

for every $k \in \mathbb{N}$.

Using this theorem we can account for the discrete sampling and are able to obtain a better estimate for the barrier digital.

The following table shows the value of a two period barrier option calculated with standard Monte Carlo simulation and with Monte Carlo simulation using the above discretization bias correction. Here we discard a path if the probability is bigger than .55. The value of the barrier option calculated with theorem 3.2 is 0.03493.

number of simulations	simple MC	standard error	MC correction	standard error
100	0.04026	0.03785	0.18169	0.12051
500	0.03995	0.03755	0.09220	0.09232
1000	0.03812	0.03712	0.06634	0.05479
5000	0.03964	0.03586	0.02792	0.01271
10000	0.03739	0.03504	0.01998	0.00804

Table 2: A two period barrier digital is approximated using simple Monte Carlo simulation and using the discretization bias correction. The results show that with the correction the error due to overestimation is reduced

4.4 Structure Floor

Here we present the numerical results for the approximation of a structure floor with a corridor option. We checked (64) numerically for up to ten coupons, with reasonable results, see Table 3. The maturity is $T = 4$, and the structure floor is at $F = 10$. The other model parameters are the same as in Figure 4. The left hand side of (64) was evaluated by a Monte Carlo simulation with 10.000 paths, using the discretization bias correction mentioned in the previous chapter (with a threshold probability of .55 for discarding a path).

coupons	structure floor	corridor option	relative error	standard error
$n = 1$	7.63696	9.91563	0.22980	0.14436
$n = 2$	7.52979	9.24883	0.18586	0.16975
$n = 3$	7.42262	8.66698	0.14357	0.14522
$n = 4$	7.31545	8.06291	0.09270	0.15126
$n = 5$	7.20827	7.44886	0.03229	0.18958
$n = 6$	7.10110	7.31880	0.02974	0.20350
$n = 7$	6.99393	7.18558	0.02667	0.21811
$n = 8$	6.88677	7.03704	0.02135	0.24764
$n = 9$	6.77962	6.92288	0.02069	0.26579
$n = 10$	6.67232	6.80399	0.01935	0.28013

Table 3: Numerical approximation of the structure floor by the corridor option (64) with maturity $T = 4$, structure floor level $F = 10$, and n coupons. The other parameters are $r = 0.01$, $\sigma = 0.15$, $B_{\text{low}} = 80$, and $B_{\text{up}} = 120$. This results show a reasonable approximation to the corridor option for larger n.

A Appendix

A.1 Corridor option expressions

In the following we give the expressions for the corridor option as described in section 3.4. To price the corridor option a double Laplace inversion of the expression $\omega(\gamma, \mu, x, l, u, m)$ defined as follows

$$\omega(\gamma, \mu, x, l, u, m)$$
$$= \begin{cases} \mathbb{1}_{x \geq u} \exp\left\{-\sqrt{2}(x-u)\sqrt{\gamma}\right\} \mathcal{L}_t \left\{y(t,1)\right\}(\gamma) \\ \mathbb{1}_{l<x<u} \frac{1}{\sinh(a\pi)} [\mathcal{L}_t \left\{y(t,0)\right\}(\gamma) \sinh\left(a\pi \frac{u-x}{u-l}\right) \\ \qquad + \mathcal{L}_t \left\{y(t,1)\right\}(\gamma) \sinh\left(a\pi \frac{x-l}{u-l}\right)] \\ \mathbb{1}_{x \leq l} \exp\left\{-\sqrt{2}(l-x)\sqrt{\gamma}\right\} \mathcal{L}_t \left\{y(t,0)\right\}(\gamma) \end{cases}$$

where

$$\mathcal{L}_t \left\{y(t,1)\right\}(\gamma) = \frac{e^{mu}}{\sqrt{\gamma}(\sqrt{\gamma} - m/\sqrt{2})} - \frac{c}{2\sqrt{\gamma}}(s(\gamma) + d(\gamma)),$$
$$\mathcal{L}_t \left\{y(t,0)\right\}(\gamma) = \frac{e^{ml}}{\sqrt{\gamma}(\sqrt{\gamma} + m/\sqrt{2})} + \frac{c}{2\sqrt{\gamma}}(s(\gamma) + d(\gamma))$$

with

$$\begin{aligned} \frac{c}{\sqrt{\gamma}} d(\gamma) = & \frac{\sqrt{\gamma+\mu}\sinh(a\pi)}{\sqrt{\gamma+\mu}\sinh(a\pi) + \sqrt{\gamma}(\cosh(a\pi)+1)} \\ & \times \Bigg(\frac{e^{mu}}{\sqrt{\gamma}(\sqrt{\gamma} - m/\sqrt{2})} + \frac{e^{ml}}{\sqrt{\gamma}(\sqrt{\gamma} + m/\sqrt{2})} \\ & \quad + \frac{1}{\sqrt{\gamma+\mu}\sinh(a\pi)(\gamma+\mu-m^2/2)} \\ & \quad \times \left[\frac{-m}{\sqrt{2}}(e^{ml} - e^{mu})(\cosh(a\pi)+1) - \sqrt{\gamma+\mu}(e^{mu} - e^{ml})\sinh(a\pi) \right] \Bigg), \end{aligned}$$

$$
\begin{aligned}
\frac{c}{\sqrt{\gamma}} s(\gamma) = {} & \frac{\sqrt{\gamma+\mu}\sinh(a\pi)}{\sqrt{\gamma+\mu}\sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)-1)} \\
& \times \left(\frac{e^{mu}}{\sqrt{\gamma}(\sqrt{\gamma}-m/\sqrt{2})} + \frac{e^{ml}}{\sqrt{\gamma}(\sqrt{\gamma}+m/\sqrt{2})} \right. \\
& \quad - \frac{1}{\sqrt{\gamma+\mu}\sinh(a\pi)(\gamma+\mu-m^2/2)} \\
& \quad \left. \times \left[\frac{-m}{\sqrt{2}}(e^{ml}+e^{mu})(\cosh(a\pi)-1) - \sqrt{\gamma+\mu}(e^{mu}-e^{ml})\sinh(a\pi) \right] \right),
\end{aligned}
$$

$$
a\pi = \sqrt{\frac{\gamma+\mu}{c^2}}; \quad \alpha = -m; \quad \beta = \frac{-m^2}{2}; \quad c^2 = \frac{1}{2(u-l)^2}
$$

A.2 Mathematica Code

Here we present the mathematica code used to compute the results in the previous chapters.

One period double barrier digital

The following function calculates the value for a one period double barrier digital. The input parameters are the underlying asset price S, the valuation time t, the time when the barriers are activated $T0$, how long the barrier period lasts P, the upper and lower barriers $Blow$ and Bup, interest rate r, volatiliy of the underlying σ and how many terms should be considered in the summation lim.

$$
\begin{aligned}
&\mathtt{BD[S\_,\ t\_,\ T0\_,\ P\_,\ Blow\_,\ Bup\_,\ r\_,\ \sigma\_,\ lim\_]\ :=}\\
&\mathtt{Module}\Big[\Big\{\alpha = -\frac{1}{2}\left(\frac{2\,\mathtt{r}}{\sigma^2}-1\right),\ \tau = \frac{1}{2}\sigma^2\,(\mathtt{T0}+\mathtt{P}-\mathtt{t}),\ \mathtt{x} = \mathtt{N}\Big[\mathtt{Log}\Big[\frac{\mathtt{S}}{\mathtt{Blow}}\Big]\Big],\\
&\beta = -\frac{2\,\mathtt{r}}{\sigma^2} - \left(-\frac{1}{2}\left(\frac{2\,\mathtt{r}}{\sigma^2}-1\right)\right)^2,\ \mathtt{L} = \mathtt{N}\Big[\mathtt{Log}\Big[\frac{\mathtt{Bup}}{\mathtt{Blow}}\Big]\Big],\ \mathtt{p} = \frac{\sigma^2\,\mathtt{P}}{2}\Big\},\\
&\sqrt{2\,\pi}\,\left(\frac{\mathtt{S}}{\mathtt{Blow}}\right)^{\alpha}\ \mathtt{NSum}\Big[\frac{1}{\alpha^2\,\mathtt{L}^2+\mathtt{k}^2\,\pi^2}\,\mathtt{k}\,\left(1-(-1)^{\mathtt{k}}\,\mathtt{Exp}[-\alpha\,\mathtt{L}]\right)\,\mathtt{Exp}\Big[-\left(\frac{\mathtt{k}\,\pi}{\mathtt{L}}\right)^2\mathtt{p}+\beta\,\tau\Big]\\
&\mathtt{NIntegrate}\Big[\mathtt{Sin}\Big[\frac{\mathtt{k}\,\pi\,\left(\mathtt{x}+\mathtt{y}\sqrt{2\,(\tau-\mathtt{p})}\right)}{\mathtt{L}}\Big]\,\mathtt{Exp}\Big[-\frac{\mathtt{y}^2}{2}\Big],\ \Big\{\mathtt{y},\ -\frac{\mathtt{x}}{\sqrt{2\,(\tau-\mathtt{p})}},\ \frac{\mathtt{L}-\mathtt{x}}{\sqrt{2\,(\tau-\mathtt{p})}}\Big\}\Big],\\
&\qquad \{\mathtt{k},\ 1,\ \mathtt{lim}\}\Big]\Big]
\end{aligned}
$$

Monte Carlo one period double barrier digital

The function to price a one period barrier digital with monte carlo simulation needs basically the same input parameters as well as the number of simulations n.

```
BDMC[S_, t_, T0_, P_, Blow_, Bup_, r_, σ_, n_] := Block[{data2, MC},
  data2 = RandomFunction[
     GeometricBrownianMotionProcess[r, σ, S], {t, T0 + P, .01}, n];
  MC = Select[#, #[[1]] >= T0 &] & /@ data2["Paths"];
  Return[
   Exp[-r * (T0 + P)] * 1 / n * Total[Table[If[Max[Flatten[Table[MC[[j]][[i]][[2]],
          {i, 1, Length[MC[[1]]]}]]] < Bup &&
        Min[Flatten[Table[MC[[j]][[i]][[2]],
          {i, 1, Length[MC[[1]]]}]]] > Blow, 1, 0] , {j, 1, n}]]]]
```

Multiple period double barrier digital

The following functions valuate a multiple barrier digital, first the auxiliary functions as well as a helper function for determining the appropriate index are defined and then the main function valuates the digital. Here again *lim* is the number of summation steps, as already mentioned above for the infinte sums we found that truncating after five summands suffices to get a reasonably accurate value.

```
(*Define helper function for coordinate change*)
τj[x_] := Module[{const = ConstantArray[x[[Length[x]]], Length[x]]}, const - x];
(*Auxiliary functions*)
g0[k1_, x1_, yi___, x_, τ_, τj_, p_, 0, α_, L_] :=
  (2 Exp[-α x1] Sin[k1 π x1/L] Sin[k1 π x/L] Exp[-(k1 π/L)^2 τ])/L;
h0[k1_, x1_, y1_, x_, τ_, τj_, p_, 0, n_, α_, L_] :=
  1/Sqrt[2 π] Exp[-y1^2/2] *
   g0[k1, x1, 0, x + y1 Sqrt[2 (τ - (τj[[n - 0]] + p))], τj[[n - 0]] + p, τ, p, 0, α, L];
gj[kj__, xj__, yj__, x_, τ_, τj_, p_, j_, α_, L_, n_] :=
  If[j == 0, g0[kj, xj, yj, x, τ, τj, p, j, α, L],
   gj[kj, xj, yj, x, τ, τj, p, j, α, L, n] =
    Module[{kjp1 = kj[[j + 1]], xjp1 = xj[[j + 1]], τjnmj = τj[[n - j]]},
     1/L 2 Sin[(kjp1 π xjp1)/L] Sin[(kjp1 π x)/L] Exp[-((kjp1 π)/L)^2 (τ - τjnmj)]
      hj[Most[kj], Most[xj], yj, xjp1, τjnmj, τj, p, j - 1, α, L, n]]];
hj[kj__, xj__, yj__, x_, τ_, τj_, p_, j_, α_, L_, n_] :=
  If[j == 0, h0[kj, xj, yj, x, τ, τj, p, j, n, α, L],
   hj[kj, xj, yj, x, τ, τj, p, j, α, L, n] =
    Module[{yjp1 = yj[[j + 1]], τjnmj = τj[[n - j]]},
     1/Sqrt[2 π] Exp[-yjp1^2/2] * gj[kj, xj, Most[yj], x + yjp1 Sqrt[2 (τ - (τjnmj + p))],
       τjnmj + p, τj, p, j, α, L, n]]];
```

```
(*Main Pricing Function
  lim= #Terms for sum*)
BDMult[S_, t_, Ti_, P_, Blow_, Bup_, r_, σ_, lim_] :=
 Module[{n, x, τ, α, β, p, L, tj, gxvars, gxrange, gyvars,
   gyrange, gkvars, gkrange, hxvars, hxrange, hyvars, hyrange, hkvars,
   hkrange, expression0, expression1, j, value, i, k},
  n = Length[Ti]; x = N[Log[S/Blow]]; τ = 1./2. σ^2 (Ti[[n]] + P - t);
  α = -1./2. (2. r/σ^2 - 1); β = -2. r/σ^2 - α^2; p = σ^2 P/2.; L = N[Log[Bup/Blow]];
  tj = 1. / 2. * σ^2 * τj[Ti]; If[Length[Pick[#, tj[[n - (# - 1)]] +
              p < τ < If[# == n, Infinity,
              tj[[n - (#)]]] & /@ #] &[Range[Length[tj]]] - 1] != 0,
   j = Pick[#, tj[[n - (# - 1)]] + p < τ < If[# == n, Infinity, tj[[n - (#)]]] & /@ #]
      &[Range[Length[tj]]] - 1,
   j = Pick[#, tj[[n - (#)]] <= τ ≤ tj[[n - (#)]] + p & /@ #]
     &[Range[0, Length[tj] - 1]]];
  j = j[[1]]; (*Define integration/summation variables for two cases*)
  gxvars = Table[Symbol["x" <> ToString[i]], {i, 1, j + 1}];
  gxrange = Table[{gxvars[[i]], 0, L}, {i, 1, j + 1}];
  gyvars = Table[Symbol["y" <> ToString[i]], {i, 1, j}];
  gyrange =
   Quiet[Diagonal[Table[{gyvars[[k]], -x / Sqrt[2 * (τ - (tj[[n - i]] + p))],
       (L - x) / Sqrt[2 * (τ - (tj[[n - i]] + p))]}, {k, 1, j}, {i, 0, j - 1}]]];
  gkvars = Table[Symbol["k" <> ToString[i]], {i, 1, j + 1}];
  gkrange = Table[{gkvars[[i]], 0, lim}, {i, 1, j + 1}];
  (*hj:*)
  hxvars = Table[Symbol["x" <> ToString[i]], {i, 1, j + 1}];
  hxrange = Table[{hxvars[[i]], 0, L}, {i, 1, j + 1}];
  hyvars = Table[Symbol["y" <> ToString[i]], {i, 1, j + 1}];
  hyrange =
   Quiet[Diagonal[Table[{hyvars[[k]], -x / Sqrt[2 * (τ - (tj[[n - i]] + p))],
       (L - x) / Sqrt[2 * (τ - (tj[[n - i]] + p))]}, {k, 1, j + 1}, {i, 0, j}]]];
  hkvars = Table[Symbol["k" <> ToString[i]], {i, 1, j + 1}];
  hkrange = Table[{hkvars[[i]], 0, lim}, {i, 1, j + 1}];
```

```
(*Main Pricing Expression*)
 If[tj[[n - j]] ≤ τ && tj[[n - j]] + p >= τ, If[j == 0,
   expression0 = Sum[gj[gkvars, gxvars, gyvars, x, τ, tj, p, j, α, L, n],
     Evaluate[Sequence @@ gkrange]];
   expression1 = Integrate[expression0, Sequence @@ gxrange];
   value = Re[Exp[α * x + β * τ] * expression1],
   expression0 = Sum[gj[gkvars, gxvars, gyvars, x, τ, tj, p, j, α, L, n],
     Evaluate[Sequence @@ gkrange]];
   expression1 = Integrate[expression0, Sequence @@ gxrange];
   value =
    Re[Exp[α * x + β * τ] * NIntegrate[expression1, Evaluate[Sequence @@ gyrange],
       Method → {Automatic, "SymbolicProcessing" → 0}]]],
  expression0 = Sum[hj[hkvars, hxvars, hyvars, x, τ, tj, p, j, α, L, n],
    Evaluate[Sequence @@ hkrange]];
  expression1 = Integrate[expression0, Evaluate[Sequence @@ hxrange]];
  value =
   Re[Exp[α * x + β * τ] * NIntegrate[expression1, Evaluate[Sequence @@ hyrange],
      Method → {Automatic, "SymbolicProcessing" → 0}]]]]
```

Monte Carlo multiple period double barrier digital

The function *prob* calculates the exit probability as described in Chapter 4.3, *HittingTime* evaluates the simulated Brownian Motion (simulated by function *GBMPathCompiled*) and if a certain probability (parameter *ra*) is crossed dismisses the path. The same logic applies to *PartialBarrierTest* which evaluates if the path stays inside the barriers during the times when the barriers are active.

```
prob[S0_, S1_, Blow_, Bup_, σ_, dt_] := Module[{ξ, y, l, u}, l = Log[Blow];
  u = Log[Bup]; ξ = Log[S0]; y = Log[S1];
   Return[Exp[-(Piecewise[{{2 / σ^2 * (u - ξ) * (u - y), (ξ + y) > (u + l)},
      {2 / σ^2 * (ξ - l) * (y - l), (ξ + y) < (u + l)}}]) / dt]] ];
HittingTime[path_, iSt_, iFn_, Blow_, Bup_, σ_, dt_, ra_] :=
  Module[{subpath = path[(iSt - 1) ;; (iFn + 1)], probs, i},
   probs = Table[prob[subpath[[i]], subpath[[i + 1]], Blow, Bup, σ, dt],
   {i, 1, Length[subpath] - 1}]; And[Max[probs] <= ra]];
GBMPathCompiled =
  Compile[{{S0, _Real}, {drift, _Real}, {diff, _Real}, {nSteps, _Integer}},
   FoldList[(#1 drift Exp[diff #2]) &, S0,
    RandomVariate[NormalDistribution[0, 1], nSteps]]];
PartialBarrierTest[path_, iSt_, iFn_, U_, L_] :=
  Module[{subPath = path[[iSt ;; iFn]], subMax, subMin},
   {subMax, subMin} = {Max[subPath], Min[subPath]};
   (And[Max[subPath] <= U, Min[subPath] >= L])];
BDMultMC[S_, t_, Ti_, P_, Blow_, Bup_, r_, σ_, n_, dt_, ra_] :=
  Module[{T, drift, diff, paths, τ, remainingPaths, value, j},
   T = Last[Ti] + P; {drift, diff} = {Exp[(r - σ^2 / 2) dt], diff = σ Sqrt[dt]};
   paths = Table[GBMPathCompiled[S, drift, diff, (T / dt)], {n}];
   τ = Table[{Ti[[i]] / dt, (Ti[[i]] + P) / dt}, {i, 1, Length[Ti]}];
   remainingPaths = paths;
   For[j = 1, j <= Length[τ], j++,
    remainingPaths =
      Select[remainingPaths,
       PartialBarrierTest[#, Floor[τ[[j]][[1]]], Floor[τ[[j]][[2]]]
         , Bup, Blow] &];    remainingPaths = Select[remainingPaths,
       HittingTime[#, Floor[τ[[j]][[1]]], Floor[τ[[j]][[2]]],
         Bup, Blow, σ, dt, ra] &];];
   value = Exp[-r * T] * N[1 / n * Length[remainingPaths]]];
```

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