



TECHNISCHE  
UNIVERSITÄT  
WIEN  
Vienna University of Technology

DIPLOMARBEIT

# Convergence Analysis of the Longstaff-Schwartz Algorithm

ausgeführt am

Institut für Wirtschaftsmathematik  
der Technischen Universität Wien

unter Anleitung von

Privatdoz. Dipl.-Ing. Dr. Stefan GERHOLD

durch

Emanuel GUDAT, BSc.

Rembrandtstraße 9/10

1020 Wien

Wien, Februar 2015



## STATUTORY DECLARATION

I declare that I have authored this thesis independently, that I have not used other than the declared sources/resources, and that I have explicitly marked all material which has been quoted either literally or by content from the used sources.

Vienna, on February 6th, 2015



Signature



**ABSTRACT** We analyse convergence properties of the Longstaff-Schwartz algorithm, a routine used in pricing American options. After a description and a numerical example of the algorithm, we will present an introduction to statistical learning theory and give a rigorous proof of Pollard's inequality. Having established the Vapnik-Chervonenkis dimension, we pass on to prove an inequality about the error that occurs within one step of the LS-Algorithm. We use all this to establish convergence theorems, even in settings where the approximation spaces are not convex, closed or linear, as long as they are uniformly bounded and have a finite Vapnik-Chervonenkis dimension. The rest of the thesis deals with applications of the convergence theorems. We are going to use polynomial approximations architectures and artificial neural networks.

**KEYWORDS** American option pricing, least-squares Monte Carlo, Longstaff-Schwartz algorithm, statistical learning, Vapnik-Chervonenkis dimension, artificial neural networks



# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>The Longstaff-Schwartz Algorithm</b>	<b>2</b>
2.1	Description of the Algorithm . . . . .	2
2.2	An Numerical Example . . . . .	3
<b>3</b>	<b>Statistical Learning</b>	<b>7</b>
3.1	Function Estimation Model . . . . .	7
3.2	Pollard's Inequality . . . . .	10
3.3	VC Dimension . . . . .	16
<b>4</b>	<b>Error Propagation</b>	<b>22</b>
<b>5</b>	<b>Error Estimates</b>	<b>28</b>
5.1	The two main results . . . . .	28
<b>6</b>	<b>Polynomial Approximations</b>	<b>39</b>
<b>7</b>	<b>Approximation with Artificial Neural Networks</b>	<b>43</b>
7.1	Introduction to ANN . . . . .	43
7.2	Neural Network Error Estimates . . . . .	44
7.3	The VC-dimension of sigmoid networks . . . . .	52
7.4	Using Neural Networks in the LS-Algorithm . . . . .	54

# 1 Introduction

The valuation of American style options remains one of the most relevant problems in pricing theory. Since no closed form solution exists, numerical procedures must be applied. The methods that are largely in use in the financial industry, such as finite difference and binomial techniques, struggle if more than one risk factor is involved in the modelling process [20]. In their article *Valuing American Options by Simulation: A Simple Least-Squares Approach*, which was published in 2001, the authors Francis A. Longstaff and Eduardo S. Schwartz presented a way to value American derivatives using simulation techniques. The key concept is to use least squares to model the conditional expected payoff of the derivative, given that it is not exercised immediately. They are referred to as continuation values and used to find a return maximising exercising strategy. As all numerical methods based on Monte-Carlo simulation, also the Longstaff-Schwartz (LS) algorithm can only handle a finite set of exercising points. It therefore approximates the American derivative by a Bermudan one.

Since then the algorithm gained popularity among practitioners, as it has many advantages. One of them being that it easily allows the implementation of parallel computing and therefore the increasing of computational speed [5]. In 2002 Clément, Lamberton and Protter proved the convergence of the algorithm, if the approximation spaces are of finite dimension and linear. Later, in 2005, Egloff established more general error estimates, but he assumes, among other things, convexity. This thesis analysis how a further generalisation was achieved by Zanger in 2009. He is able, using concepts from statistical learning theory, to drop the assumption of convexity. The approximation spaces can be any subsets of  $L^2$ , as long as they have a finite VC-dimension and their members are uniformly bounded by a fixed, finite constant [25].

In chapter 2 we will introduce the algorithm and the underlying probabilistic setup. In order to gain a better understanding of the LS-algorithm, a numerical example will be given. Chapter 3 introduces all the necessary results from statistical learning theory, including Pollard's inequality and the concept of the Vapnik-Chervonenkis dimension. Afterwards we analyse the error propagation that occurs within one iteration step in the algorithm. This gives us all the needed tools to present the main results, what will be done in chapter 5. The sixth chapter deals with an application of those theorems, in cases where the continuation values possess some regularity conditions. Finally, in chapter 6, we are going to introduce artificial neural networks and use them within our main results.



## 2 The Longstaff-Schwartz Algorithm

### 2.1 Description of the Algorithm

We start with the definition of the discrete time optimal stopping problem. As already mentioned in the introduction, we have to replace the continuous time problem by a discrete one. Fix a  $K \in \mathbb{N}$ , which is the time horizon, i.e. the expiration date of the derivative. Let  $(F, \mathcal{F}, (\mathcal{F}_k)_{k=0, \dots, K}, \mu)$  be a filtered probability space, where the filtration is generated by a stochastic process  $(X_k)_{k=0, \dots, K}$ . We think of the  $X_k$  as risk factors or underlyings of our derivative. They map to the state space  $\mathcal{S} \subseteq \mathbb{R}^d$  and are assumed to be a  $(\mathcal{F}_k)$ - Markov chain. Denote the image measure that is induced on  $(\mathcal{S}, \mathcal{B}_{\mathcal{S}})$  by  $X_k$  as  $\rho_k$ , where  $\mathcal{B}_{\mathcal{S}}$  is the Borel  $\sigma$ -algebra on  $\mathcal{S}$ . Assume that the start value  $X_0$  is deterministic. For every  $k = 0, \dots, K$  the payoff process  $Z_k$  of the derivative is modelled by a Borel function, such that  $Z_k$  is square integrable, i.e. we have  $Z_k = f_k(X_k) \in L^2(\mathcal{S}, \mathcal{B}_{\mathcal{S}}, \rho_k)$ . The optimal stopping problem is to find the value process

$$V_k = \operatorname{ess\,sup}_{\mathcal{T}_{k,K}} \mathbb{E}[Z_{\tau} | \mathcal{F}_k]$$

where  $\mathcal{T}_{k,K}$  is the set of all stopping times taking values in  $\{k, \dots, K\}$ . A stopping  $\tau_k \in \mathcal{T}_{k,K}$  is optimal, if

$$V_k = \mathbb{E}[Z_{\tau_k} | \mathcal{F}_k] \tag{2.1}$$

Once the value process is known, we can write  $\tau_k$  as

$$\tau_k = \min\{j \geq k : V_j \leq Z_j\}.$$

From optimal stopping theory it is known [5], that

$$\begin{aligned} V_K &= Z_K \\ V_k &= \max(Z_k, \mathbb{E}[V_{k+1} | \mathcal{F}_k]) \end{aligned} \tag{2.2}$$

Now combine for  $k=0$  (2.2) with (2.1) to get

$$\begin{aligned} V_0 &= \max(Z_0, \mathbb{E}[V_1 | \mathcal{F}_0]) \\ &= \max(Z_0, \mathbb{E}[\mathbb{E}[Z_{\tau_1} | \mathcal{F}_1] | \mathcal{F}_0]) \\ &= \max(Z_0, \mathbb{E}[Z_{\tau_1} | \mathcal{F}_0]) \end{aligned} \tag{2.3}$$

where we used the tower property of conditional expectation. Since the process  $(X_k)_{k=0, \dots, K}$  is a Markov chain, we have

$$\mathbb{E}[Z_{\tau_{k+1}} | \mathcal{F}_k] = \mathbb{E}[Z_{\tau_{k+1}} | X_k]$$

This gives us, together with (2.3) and the fact that  $X_0$  is deterministic,

$$V_0 = \mathbb{E}[Z_{\tau_0}] = \max(Z_0, \mathbb{E}[Z_{\tau_1} | X_0]) = \max(Z_0, \mathbb{E}[Z_{\tau_1}]) \tag{2.4}$$

We can reformulate the optimal stopping problem (2.2) in terms of optimal stopping times as follows:

$$\tau_K = K \quad (2.5)$$

$$\tau_k = k\mathbb{I}_{[Z_k \geq \mathbb{E}[Z_{\tau_{k+1}} | X_k]]} + \tau_{k+1}\mathbb{I}_{[Z_k < \mathbb{E}[Z_{\tau_{k+1}} | X_k]]} \quad (2.6)$$

with  $1 \leq k \leq K - 1$ . The values  $\mathbb{E}[Z_{\tau_{k+1}} | X_k]$  are known as continuation values, and they will be the quantity that we are trying to approximate. Denote the approximation of  $\mathbb{E}[Z_{\tau_{k+1}} | X_k]$  with  $\mathcal{A}\left(E[Z_{\tau_{k+1}^A} | X_k]\right)$ , realized by a given sequence of square integrable functions  $\{r_k\}_{k=0, \dots, K-1}$ , such that for  $x \in \mathcal{S}$

$$r_k(x) = \mathcal{A}\left(E[Z_{\tau_{k+1}^A} | X_k = x]\right).$$

For each  $k = 0, \dots, K - 1$  we require the approximator to lie in given approximation spaces  $\mathcal{H}_k \subseteq L^2(\mathcal{S}, \mathcal{B}_{\mathcal{S}}, \rho_k)$ . The approximated stopping rules are recursively defined as

$$\tau_K^A = K \quad (2.7)$$

$$\tau_k^A = k\mathbb{I}_{[Z_k \geq \mathcal{A}\left(\mathbb{E}[Z_{\tau_{k+1}^A} | X_k]\right)]} + \tau_{k+1}^A\mathbb{I}_{[Z_k < \mathcal{A}\left(E[Z_{\tau_{k+1}^A} | X_k]\right)]} \quad (2.8)$$

where  $1 \leq k \leq K - 1$ . We also get an approximated counterpart of equation (2.4), that is

$$\mathcal{A}\left(\mathbb{E}[Z_{\tau_0^A}]\right) = \max(Z_0, \mathcal{A}\left(\mathbb{E}[Z_{\tau_1^A} | X_0]\right)) = \max(Z_0, \mathcal{A}\left(\mathbb{E}[Z_{\tau_1^A}]\right)) \quad (2.9)$$

Thus the Longstaff-Schwartz algorithm is performed as in algorithm 1. We don't need the approximation spaces  $\mathcal{H}_k \subseteq L^2(\mathcal{S}, \mathcal{B}_{\mathcal{S}}, \rho_k)$  to be linear, that is why we may refer to it as the generalized LS-algorithm [5, 25].

## 2.2 An Numerical Example

Assume that we have two stocks  $X^1$  and  $X^2$  with given price paths, which are assumed to be uncorrelated geometric Brownian motions. The first one has a volatility of 20% and starts at  $X_0^1 = 1$ , the second one a volatility of 30% and starts at  $X_0^2 = 0.5$ . We assume that the risk less interest rate is 0. Table 1 shows 10 simulated price paths with  $K = 2$  of this underlying. The derivative we want to model is a call on the maximum of  $X^1$  and  $2X^2$  with strike 1.1, thus the payoff  $Z_k$  is given by  $Z_k = (\max(X_k^1, 2X_k^2) - 1.1)^+$ . We start by setting the stopping time  $\tau = K = 2$  on all paths. Next we fit a polynomial with a maximal degree of 2. The independent variables are  $(X_1^1, X_1^2)$  and the dependent variable is  $Z_2$ . This gives us

$$\begin{aligned} \mathcal{A}\left(\mathbb{E}[Z_2 | X_1 = (X_1^1, X_1^2)]\right) &= 0.45(X_1^1)^2 - 0.93X_1^1X_1^2 \\ &\quad - 0.07X_1^1 + 3.17(X_1^2)^2 - 1.54X_1^2 + 0.09 \end{aligned} \quad (2.10)$$

---

**Algorithm 1** The Longstaff-Schwartz algorithm

---

Simulate  $N$  independent sample paths of the Markov chain  $(X_k)_{k=0,\dots,K}$  and denote them by  ${}^i x_0, \dots, {}^i x_K$   $i = 1, \dots, N$

For all  $k = 1, \dots, K - 1$  define the approximation space  $\mathcal{H}_k$

**for**  $k = K - 1$  to 0 **do**

**if**  $k = K - 1$  **then**

    Set  $\tau_K^A = K$  and for  $i = 1, \dots, N$  calculate  ${}^i z_{\tau_K^A} = f_K({}^i x_K)$

**else**

    On each of the  $N$  sample paths calculate  ${}^i z_{\tau_{k+1}^A}$  using  $\mathcal{A}\left(\mathbb{E}[Z_{\tau_{k+2}^A} | X_{k+1}]\right) = r_{k+1}({}^i x_{k+1})$  and (2.8)

**if**  ${}^i z_{\tau_{k+1}^A} = k + 1$  **then**

      set  ${}^i z_{\tau_{k+1}^A} = f_{k+1}({}^i x_{k+1})$

**else**

      In this case we have  ${}^i z_{\tau_{k+1}^A} = {}^i z_{\tau_{k+2}^A}$  and accordingly

      set  ${}^i z_{\tau_{k+1}^A} = {}^i z_{\tau_{k+2}^A}$

**end if**

**end if**

**if**  $k \geq 1$  **then**

    Set  $\mathcal{A}\left(\mathbb{E}[Z_{\tau_{k+1}^A} | X_k]\right) := r_k$  where  $r_k \in \mathcal{H}_k$  is any function that satisfies

$$\sum_{i=1}^N \left({}^i z_{\tau_{k+1}^A} - r_k({}^i x_k)\right)^2 \leq \epsilon_0 + \inf_{r \in \mathcal{H}_k} \sum_{i=1}^N \left({}^i z_{\tau_{k+1}^A} - r({}^i x_k)\right)^2$$

and  $\epsilon_0$  is a constant which may be chosen to be 0 if the infimum is in fact attained for all  $k$ .

**else**

    Let  $\mathcal{A}\left(\mathbb{E}[Z_{\tau_1^A} | X_0]\right) = \frac{1}{N} \sum_{i=1}^N {}^i z_{\tau_1^A}$  and

    set  $\mathcal{A}\left(\mathbb{E}[Z_{\tau_0^A}]\right) = \max\left(Z_0, \mathcal{A}\left(\mathbb{E}[Z_{\tau_1^A} | X_0]\right)\right)$

**end if**

**end for**

---

See figure 1 for the graph of the polynomial. The column  $\mathcal{A}(\mathbb{E}[Z_2|X_1])$  contains the values of this polynomial, evaluated at the simulated points. If we compare this column with  $Z_2$ , we can calculate  $\tau$  according to (2.8). Thus on each path we get paid  $Z_2$  if  $\tau = 2$  or we receive  $Z_1$  if  $\tau = 1$ . This gives us the cash flows  $CF_k$  at time steps  $k = 1, 2$  from table 1. We calculate  $\mathcal{A}(\mathbb{E}[Z_{\tau_1}^{\mathcal{A}}])$  as the empirical mean over each path's total cash flows. Since the option is out of the money at  $k = 0$ , this value dominates  $Z_0$  and therefore equals the approximated option price  $\mathcal{A}(\mathbb{E}[Z_{\tau_0}^{\mathcal{A}}]) = 0.18$ .

Path	$(X_1^1, X_1^2)$	$Z_1$	$(X_2^1, X_2^2)$	$Z_2$	$\mathcal{A}(\mathbb{E}[Z_2 X_1])$	$\tau$	$CF_1$	$CF_2$
1	0.93 0.65	0.20	0.72 0.77	0.44	0.44	2	0	0.44
2	0.98 0.46	0.00	1.15 0.41	0.05	0.06	2	0	0.05
3	0.97 0.33	0.00	0.96 0.28	0.00	0.00	2	0	0
4	0.70 0.59	0.08	0.69 0.62	0.13	0.11	2	0	0.13
5	1.09 0.33	0.00	1.37 0.32	0.27	0.26	2	0	0.27
6	1.12 0.45	0.02	0.94 0.55	0.00	0.05	2	0	0
7	1.25 0.62	0.15	1.47 0.62	0.37	0.37	2	0	0.37
8	0.82 0.51	0.00	0.87 0.66	0.21	0.19	2	0	0.21
9	0.66 0.72	0.34	0.45 0.63	0.17	0.18	1	0.34	0
10	0.89 0.42	0.00	0.84 0.49	0.00	-0.03	1	0	0

Table 1: The prices of the underlying  $X_k$ , the payoff  $Z_k$  and the generated cash flows  $CF_k$  at time steps  $k = 1, 2$ .  $\tau$  is the optimal stopping time for each path and  $\mathcal{A}(\mathbb{E}[Z_2|X_1])$  the approximated continuation value.

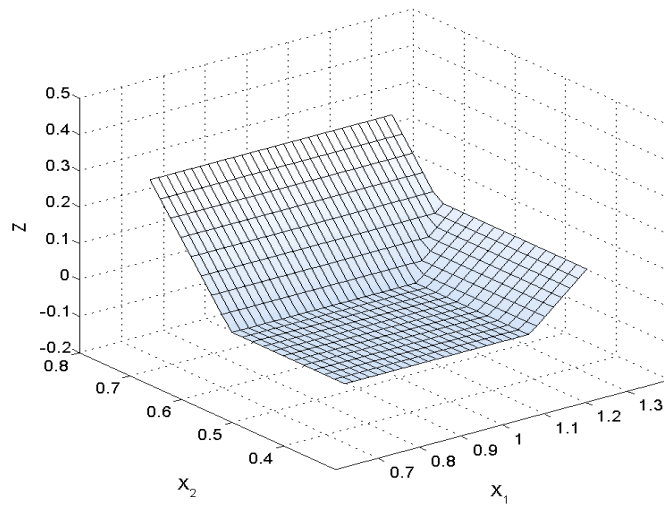
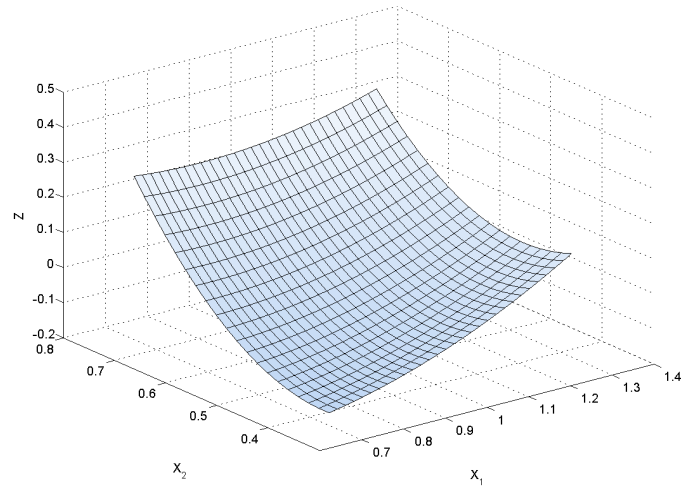


Figure 1: Comparison of the approximated continuation value (on top) and the actual payoff

### 3 Statistical Learning

This chapter will provide an introduction to various concepts in statistical learning theory. The convergence analysis of the LS-algorithm can then be seen as an application of theorems we are going to deduce on the following pages.

#### 3.1 Function Estimation Model

Let  $\mathcal{S}$  be a compact domain in Euclidean space, equipped with its Borel  $\sigma$ -algebra denoted by  $\mathcal{B}_{\mathcal{S}}$ . Likewise let  $(R, \mathcal{B}_R)$  denote a measurable space, where we choose  $R = \mathbb{R}$ . Define the measure space  $(\Omega = \mathcal{S} \times R, \mathcal{B}, \rho)$  with a Borel measure  $\rho$ . Interpret a point  $s \in \mathcal{S}$  as the input which generates an output  $r \in R$ . The measure  $\rho$  allows us to draw samples  $\omega = (s, r)$ . The aim of the statistical learning theory presented here is to model the process that generates the corresponding output to an input  $s$ , by using a function  $f : \mathcal{S} \mapsto R$ . When we analyse the LS-algorithm, we will see the risk factors as inputs and the payoff of the optimally exercised derivative as the output.

A key concept is the Mean Square Error, the error generated by using  $f$  as a model for the generating process, and its empirical companion [7].

**Definition 3.1.** The Mean Square Error (MSE) is defined as the  $\rho$ -average squared error  $(f(s) - r)^2$  over all pairs  $(s, r)$

$$\text{MSE}(f) = \int_{\Omega} (f(s) - r)^2 d\rho(s, r).$$

For a sample  $\omega = ((s_1, r_1), \dots, (s_N, r_N)) \in \Omega^N$  drawn independently according to  $\rho$ , the Empirical Mean Square Error (EMSE) is defined to be

$$\text{EMSE}_{\omega}(f) = \frac{1}{N} \sum_{i=1}^N (f(s_i) - r_i)^2.$$

Clearly our aim will be to find a minimizer and an upper bound of the error. To do so we define the marginal probability measure  $\rho_{\mathcal{S}}$  on the Borel  $\sigma$ -algebra  $\mathcal{B}_{\mathcal{S}}$  by

$$\rho_{\mathcal{S}}(S) = \rho(S \times R) \text{ for all } S \in \mathcal{B}_{\mathcal{S}} \quad (3.11)$$

To give a proper definition of the function that minimizes the MSE we will need the definition of a transition measure [19].

**Definition 3.2.** A transition measure on  $\mathcal{B}_R$  is any function  $\mu : \mathcal{S} \times \mathcal{B}_R \mapsto [0, \infty]$  that satisfies

1.  $\forall s \in \mathcal{S}$  the function  $B \in \mathcal{B}_R \mapsto \mu(s, B)$  is a measure on  $\mathcal{B}_R$

2.  $\forall B \in \mathcal{B}_R$  the function  $s \mapsto \mu(s, B)$  is measurable.

The product of a transition measure  $\mu$  and a  $\sigma$ -finite measure  $\nu : \mathcal{S} \mapsto [0, \infty]$  simplifies to

$$\forall W \in \mathcal{B} : (\mu \times \nu)(W) = \int_{\mathcal{S}} \mu(s, W_s) d\nu(s)$$

with  $W_s = \{r \in \mathbb{R} : (s, r) \in W\}$  being the projection of  $W$  onto the real line.

**Definition 3.3.** A transition measure  $\mu$  that satisfies  $\rho = \mu \times \rho_{\mathcal{S}}$  is called a conditional probability measure. In this case we write  $\rho_{R|s}(\cdot) := \mu(s, \cdot)$

A version of Fubini's theorem now states that for all integrable functions  $\phi : \Omega \mapsto \mathbb{R}$  we have

$$\int_{\Omega} \phi(\omega) d\rho(\omega) = \int_{\mathcal{S}} \int_{\mathbb{R}} \phi(s, r) d\rho_{R|s}(r) d\rho_{\mathcal{S}}(s)$$

We now have the necessary tools for

**Definition 3.4.** Let  $\rho_{R|s}$  be a transition measure on  $(R = \mathbb{R}, \mathcal{B}_R)$ . Then the regression function  $f_{\rho} : \mathcal{S} \mapsto \mathbb{R}$  is defined by

$$f_{\rho}(s) = \int_{\mathbb{R}} r d\rho_{R|s}(r)$$

and its MSE is denoted by  $\sigma_{\rho}^2 := \text{MSE}(f_{\rho})$ .

As we shall see now, the regression function is the minimizer of the error over all  $L^2$  functions. [3]. For this to become clear we decompose the MSE as follows.

**Proposition 3.5.** For every function  $f : \mathcal{S} \mapsto \mathbb{R}$

$$\text{MSE}(f) = \|f - f_{\rho}\|_{L^2}^2 + \sigma_{\rho}^2 \quad (3.12)$$

where  $\|\cdot\|_{L^2}$  is the norm on  $L^2(\mathcal{S}, \mathcal{B}_{\mathcal{S}}, \rho_{\mathcal{S}})$ .

*Proof.* To show equation (3.12) write

$$\begin{aligned} \text{MSE}(f) &= \int_{\Omega} (f(s) - r)^2 d\rho(s, r) \\ &= \int_{\mathcal{S}} \int_{\mathbb{R}} (f(s) - f_{\rho}(s) + f_{\rho}(s) - r)^2 d\rho_{R|s}(r) d\rho_{\mathcal{S}}(s) \\ &= \int_{\mathcal{S}} 1(f(s) - f_{\rho}(s))^2 d\rho_{\mathcal{S}}(s) \\ &\quad + \int_{\mathcal{S}} \int_{\mathbb{R}} 2(f(s) - f_{\rho}(s))(f_{\rho}(s) - r) + (f_{\rho}(s) - r)^2 d\rho_{R|s}(r) d\rho_{\mathcal{S}}(s) \\ &= \|f - f_{\rho}\|_{L^2}^2 + 2 \int_{\mathcal{S}} (f(s) - f_{\rho}(s)) \int_{\mathbb{R}} f_{\rho}(s) - r d\rho_{R|s}(r) d\rho_{\mathcal{S}}(s) + \sigma_{\rho}^2 \\ &= \|f - f_{\rho}\|_{L^2}^2 + \sigma_{\rho}^2 + 0 \end{aligned}$$

To see the last step notice that  $f - f_\rho$  is independent of  $r$ , so fix a  $s \in \mathcal{S}$  and one gets

$$\begin{aligned} \int_{\mathbb{R}} f_\rho(s) - r d\rho_{R|s}(r) &= \int_{\mathbb{R}} f_\rho(s) d\rho_{R|s}(r) - \int_{\mathbb{R}} r d\rho_{R|s}(r) \\ &= f_\rho(s) - \int_{\mathbb{R}} r d\rho_{R|s}(r) \\ &= 0 \end{aligned}$$

□

Proposition 3.5 shows that  $\sigma_\rho^2$  is a lower bound on the MSE, attained by choosing  $f_\rho$  as a model. It can be seen as a measure how well conditioned  $\rho$  is. If we instead choose  $f$  as a model we additionally have the error term  $\|f - f_\rho\|_{L^2}^2$ .

**Definition 3.6.** For a sample  $\omega \in \Omega^N$  and  $\epsilon_0 \geq 0$ , we will call any function in the approximation space  $\mathcal{H}$ , that fulfils

$$\text{EMSE}_\omega(f_{\omega, \epsilon_0}) \leq \epsilon_0 + \inf_{f \in \mathcal{H}} \text{EMSE}_\omega(f) \quad (3.13)$$

an approximator and denote it by  $f_{\omega, \epsilon_0}$ .

The next proposition will give us a bound on the  $L^2$ -distance of the approximator  $f_{\omega, \epsilon_0}$  and the regression function  $f_\rho$ , provided that we can limit the absolute deviation over all  $f \in \mathcal{H}$  of the theoretic and empirical mean square error.

**Proposition 3.7.** Let  $\epsilon > 0$  and  $\delta \in [0, 1]$  such that

$$\mathbb{P} \left[ \sup_{f \in \mathcal{H}} |\text{MSE}(f) - \text{EMSE}_\omega(f)| \leq \epsilon \right] \geq 1 - \delta \quad (3.14)$$

Then we have, for any  $\epsilon_0 \geq 0$

$$\mathbb{P} \left[ \|f_{\omega, \epsilon_0} - f_\rho\|_{L^2}^2 \leq 2\epsilon + \epsilon_0 + \inf_{f \in \mathcal{H}} \|f - f_\rho\|_{L^2}^2 \right] \geq 1 - \delta \quad (3.15)$$

*Proof.* Since  $f_{\omega, \epsilon_0} \in \mathcal{H}$  assumption (3.14) gives us

$$\begin{aligned} 1 - \delta &\leq \mathbb{P} [|\text{MSE}(f_{\omega, \epsilon_0}) - \text{EMSE}_\omega(f_{\omega, \epsilon_0})| \leq \epsilon] \\ &\leq \mathbb{P} [\text{MSE}(f_{\omega, \epsilon_0}) - \text{EMSE}_\omega(f_{\omega, \epsilon_0}) \leq \epsilon] \\ &= \mathbb{P} [\text{MSE}(f_{\omega, \epsilon_0}) \leq \text{EMSE}_\omega(f_{\omega, \epsilon_0}) + \epsilon] \end{aligned}$$

and accordingly for any  $f \in \mathcal{H}$  we get

$$1 - \delta \leq \mathbb{P} [\text{EMSE}_\omega(f) \leq \text{MSE}(f) + \epsilon]$$



From the definition of  $f_{\omega, \epsilon_0}$ , equation (3.13), we have for every  $f \in \mathcal{H}$

$$\text{EMSE}_{\omega}(f_{\omega, \epsilon_0}) \leq \text{EMSE}_{\omega}(f) + \epsilon_0$$

Putting it all together we get with probability at least  $1 - \delta$

$$\begin{aligned} \text{MSE}(f_{\omega, \epsilon_0}) &\leq \text{EMSE}_{\omega}(f_{\omega, \epsilon_0}) + \epsilon \\ &\leq \text{EMSE}_{\omega}(f) + \epsilon_0 + \epsilon \\ &\leq \text{MSE}(f) + 2\epsilon + \epsilon_0 \end{aligned}$$

To get (3.15) perform the decomposition of the MSE as in proposition 3.5 and cancel out  $\sigma_{\rho}^2$  on both sides.  $\square$

### 3.2 Pollard's Inequality

For what is to come next the concept of covering numbers and some basic properties are needed.

**Definition 3.8.** Let  $A \subset \mathbb{R}^N$  be bounded. For any  $\epsilon > 0$  define the  $\ell_1$ -covering number  $\mathcal{N}(\epsilon, A)$  to be the cardinality of the smallest finite subset  $B \subset A$ , such that  $\forall \vec{a} \in A : \exists \vec{b} \in B : 1/N \|\vec{a} - \vec{b}\|_1 := 1/N \sum_{i=1}^N |a_i - b_i| < \epsilon$ . The set  $B$  is called  $\epsilon$ -cover or net [13].

If  $\mathcal{G}$  is a set of uniformly bounded functions from  $\Sigma \subset \mathbb{R}^m$  to  $\mathbb{R}$ , then for any vector  $\vec{v} \in \Sigma^N$  the set

$$\mathcal{G}(\vec{v}) = \{(g(v_1), \dots, g(v_N)) | g \in \mathcal{G}\} \quad (3.16)$$

is bounded. Therefore we can consider its covering number  $\mathcal{N}(\epsilon, \mathcal{G}(\vec{v}))$ .

We will now prove some basic properties of covering numbers as stated in [10].

**Proposition 3.9.** 1. If  $A' \subseteq A$ , then  $\mathcal{N}(\epsilon, A') \leq \mathcal{N}(\epsilon, A)$ .

2. Let the collection  $\mathcal{G}$  of functions be uniformly bounded by  $G < \infty$  and denote by  $\mathcal{G}^2 = \{g^2 | g \in \mathcal{G}\}$ . Then we have

$$\mathcal{N}(\epsilon, \mathcal{G}^2(\vec{v})) \leq \mathcal{N}\left(\frac{\epsilon}{2G}, \mathcal{G}(\vec{v})\right).$$

3. For the covering number of  $\mathcal{G}_1 - \mathcal{G}_2 = \{g_1 - g_2 | g_1 \in \mathcal{G}_1, g_2 \in \mathcal{G}_2\}$  we get

$$\mathcal{N}(\epsilon, (\mathcal{G}_1 - \mathcal{G}_2)(\vec{v})) \leq \mathcal{N}\left(\frac{\epsilon}{2}, \mathcal{G}_1(\vec{v})\right) \mathcal{N}\left(\frac{\epsilon}{2}, \mathcal{G}_2(\vec{v})\right).$$

*Proof.* 1. Since the  $\epsilon$ -cover for  $A'$  is a subset of the one for  $A$  the statement follows.

2. Let  $G'$  stand for the  $\epsilon/2G$ -cover of  $\mathcal{G}(\vec{v})$ . Since  $\mathcal{G}$  is uniformly bounded by  $G$  we have for all  $\gamma \in G' : \|\gamma\|_\infty \leq G$  and for all  $g \in \mathcal{G} : \|g(\vec{v})\|_\infty \leq G$ . Now fix a  $g^2(\vec{v}) \in \mathcal{G}^2$ . We know that

$$\exists \gamma \in G' : \frac{1}{N} \|\gamma - g(\vec{v})\|_1 \leq \frac{\epsilon}{2G}.$$

And we can therefore conclude that

$$\begin{aligned} & \frac{1}{N} \|\gamma^2 - g^2(\vec{v})\|_1 \\ &= \frac{1}{N} \|(\gamma^2 - g(\vec{v})) + (\gamma^2 + g(\vec{v}))\|_1 \\ &\leq (G + G) \frac{1}{N} \|\gamma - g(\vec{v})\|_1 \\ &\leq 2G \frac{\epsilon}{2G} = \epsilon. \end{aligned}$$

We showed that the set  $\{\gamma^2 | \gamma \in G'\}$  is an  $\epsilon$ -approximation of  $\mathcal{G}^2(\vec{v})$  and thereby establishing the statement.

3. Fix an arbitrary element  $(g_1 - g_2)(\vec{v})$  from  $(\mathcal{G}_1 - \mathcal{G}_2)(\vec{v})$ . Denote for  $i = 1, 2$  the  $\epsilon/2$ -net of  $\mathcal{G}_i$  by  $G_i$ . Then we have

$$\begin{aligned} \exists \gamma_1 \in G_1 : \frac{1}{N} \|\gamma_1 - g_1(\vec{v})\|_1 &\leq \frac{\epsilon}{2} \\ \exists \gamma_2 \in G_2 : \frac{1}{N} \|\gamma_2 - g_2(\vec{v})\|_1 &\leq \frac{\epsilon}{2}. \end{aligned}$$

And therefore

$$\begin{aligned} & \frac{1}{N} \|(\gamma_1 - \gamma_2) - (g_1 - g_2)(\vec{v})\|_1 \\ &\leq \frac{1}{N} \|\gamma_1 - g_1(\vec{v})\|_1 + \frac{1}{N} \|\gamma_2 - g_2(\vec{v})\|_1 \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Thus the set  $\{\gamma_1 - \gamma_2 | \gamma_1 \in G_1, \gamma_2 \in G_2\}$  is an  $\epsilon$ -approximation for  $(\mathcal{G}_1 - \mathcal{G}_2)(\vec{v})$  and therefore its cardinality dominates  $\mathcal{N}(\epsilon, (\mathcal{G}_1 - \mathcal{G}_2)(\vec{v}))$ .  $\square$

In order to prove the main result of this section, theorem 3.12, we need two inequalities which are going to be presented without proof in the next two lemmas. The first one is a version of a Grüss type integral inequality, that can be found in [9].

**Lemma 3.10.** Let  $f, g, k$  be measurable functions on a measure space  $(\Omega, \Sigma, \mu)$ , such that  $fg, fk, gk \in L^2(\Omega, \mu)$  and  $\|k^2\|_{L^2} = 1$ . If we can find real numbers  $\gamma, \Gamma, \phi, \Phi$  such that

$$\gamma k \leq f \leq \Gamma k \text{ and } \phi k \leq g \leq \Phi k$$

holds a.s., we have the following inequality

$$\begin{aligned} & \left| \int_{\Omega} f(x)g(x)d\mu(x) - \int_{\Omega} f(x)k(x)d\mu(x) \int_{\Omega} g(x)k(x)d\mu(x) \right| \\ & \leq \frac{1}{4}(\Gamma - \gamma)(\Phi - \phi) \end{aligned}$$

We are only going to use a special version of this lemma in a probabilistic setting, which is sometimes called the Popoviciu inequality. If the measure space is a probability space, we can choose  $k(x) = 1$ . Let  $X$  be a random variable with distribution  $\mu$  and  $f = g$ , where  $|g(x)| \leq G \in \mathbb{R}$ . Then we get by lemma 3.10

$$\mathbb{V}[g(X)] \leq \frac{(2G)^2}{4} = G^2 \quad (3.17)$$

The second inequality is due to Hoeffding [16]. It gives an upper bound for the probability, that the sum of bounded and independent random variables deviates from its expected value more than a constant.

**Lemma 3.11.** Let  $X_1, X_2, \dots, X_N$  be independent random variables and denote their sum by  $S_N$ . If  $a_1, a_2, \dots, a_N$  and  $b_1, b_2, \dots, b_N$  are real numbers such that for all  $i = 1, 2, \dots, N$  we have with probability one that  $X_i \in [a_i, b_i]$ , then

$$\mathbb{P} \left[ \left| \frac{S_N}{N} - \frac{\mathbb{E}[S_N]}{N} \right| \geq t \right] \leq 2 \exp \left( - \frac{2N^2 t^2}{\sum_{i=1}^N (b_i - a_i)^2} \right)$$

Next we state Pollard's inequality which may be found in [8]. It is due to this theorem that we will later on be able to bound the approximation error by means of covering numbers. Note that if  $\vec{X} = (X_1, \dots, X_N)$  are random variables,  $\mathcal{N}(\epsilon, \mathcal{G}(\vec{X}))$  is one as well.

**Theorem 3.12.** Let  $\mathcal{G}$  denote a set of functions as above, that is uniformly bounded by  $G$ .  $X, X_1, \dots, X_N$  are independent and identically distributed random variables and we set  $\vec{X} = (X_1, \dots, X_N)$ . Then we have  $\forall \epsilon > 0$

$$\begin{aligned} & \mathbb{P} \left[ \sup_{g \in \mathcal{G}} \left| \frac{1}{N} \sum_{i=1}^N g(X_i) - \mathbb{E}[g(X)] \right| > \epsilon \right] \\ & \leq 8\mathbb{E} \left[ \mathcal{N} \left( \frac{\epsilon}{8}, \mathcal{G}(\vec{X}) \right) \right] \exp \left( - \frac{N\epsilon^2}{512G^2} \right). \end{aligned} \quad (3.18)$$

*Proof.* We will use the notation  $\overline{g(\vec{X})} = 1/N \sum_{i=1}^N g(X_i)$ . Define the random variables  $X'_1, \dots, X'_N$  to be a ghost sample, i.e. they are independent of the  $X_i$ 's and  $X$  and identically distributed. Throughout the proof assume that

$N\epsilon^2 \geq 8G^2$ , otherwise the bound is trivial. For any  $g \in \mathcal{G}$  we have by Chebyshev's inequality and (3.17)

$$\mathbb{P} \left[ \left| \overline{g(\vec{X}')} - \mathbb{E}[g(X)] \right| \leq \frac{\epsilon}{2} \right] \geq 1 - \frac{\mathbb{V}[g(X)]}{N} \frac{4}{\epsilon^2} \geq 1 - \frac{G^2}{N} \frac{4}{\epsilon^2} \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

Let  $g^* \in \mathcal{G}$  be any function, such that

$$\left| \overline{g^*(\vec{X})} - \mathbb{E}[g^*(X)] \right| > \epsilon$$

if it exists. Otherwise  $g^*$  is any arbitrary function in  $\mathcal{G}$ . Note that the choice of  $g^*$  depends only on  $\vec{X}$  and not on  $\vec{X}'$  or  $X$ . Therefore we have

$$\mathbb{P} \left[ \left| \overline{g^*(\vec{X}')} - \mathbb{E}[g^*(X)] \right| \leq \frac{\epsilon}{2} \mid \vec{X} \right] \geq \frac{1}{2}$$

and hence

$$\begin{aligned} & \mathbb{P} \left[ \sup_{g \in \mathcal{G}} \left| \overline{g(\vec{X})} - \overline{g(\vec{X}')} \right| > \frac{\epsilon}{2} \right] \\ & \geq \mathbb{P} \left[ \left| \overline{g^*(\vec{X})} - \overline{g^*(\vec{X}')} \right| > \frac{\epsilon}{2} \right] \\ & \geq \mathbb{P} \left[ \left| \overline{g^*(\vec{X})} - \mathbb{E}[g^*(X)] \right| > \epsilon, \left| \overline{g^*(\vec{X}')} - \mathbb{E}[g^*(X)] \right| \leq \frac{\epsilon}{2} \right] \\ & = \mathbb{E} \left[ \mathbb{E} \left[ \mathbb{I}_{\left[ \left| \overline{g^*(\vec{X})} - \mathbb{E}[g^*(X)] \right| > \epsilon \right]} * \mathbb{I}_{\left[ \left| \overline{g^*(\vec{X}')} - \mathbb{E}[g^*(X)] \right| \leq \frac{\epsilon}{2} \right]} \mid \vec{X} \right] \right] \\ & = \mathbb{E} \left[ \mathbb{I}_{\left[ \left| \overline{g^*(\vec{X})} - \mathbb{E}[g^*(X)] \right| > \epsilon \right]} \mathbb{P} \left[ \left| \overline{g^*(\vec{X}')} - \mathbb{E}[g^*(X)] \right| \leq \frac{\epsilon}{2} \mid \vec{X} \right] \right] \\ & \geq \frac{1}{2} \mathbb{P} \left[ \left| \overline{g^*(\vec{X})} - \mathbb{E}[g^*(X)] \right| > \epsilon \right] \\ & = \frac{1}{2} \mathbb{P} \left[ \sup_{g \in \mathcal{G}} \left| \overline{g^*(\vec{X})} - \mathbb{E}[g^*(X)] \right| > \epsilon \right]. \end{aligned} \tag{3.19}$$

Introduce  $\sigma_1, \dots, \sigma_N$ , which are independent random variables, all identically distributed with  $\mathbb{P}[\sigma_i = 1] = \mathbb{P}[\sigma_i = -1] = 1/2$ . With their help we can define the so called Rademacher processes  $\frac{1}{N} \sum_{i=1}^N \sigma_i g(X_i)$  [18]. This additional randomization will allow us later to use Hoeffding's inequality in a much nicer way. Since  $X_1, \dots, X_N, X'_1, \dots, X'_N$  are all i.i.d., the joint distribution of  $\vec{X}$  and  $\vec{X}'$  does not change if we randomly exchange corresponding

components of  $\vec{X}$  and  $\vec{X}'$ . Thus we get [14]

$$\begin{aligned}
& \mathbb{P} \left[ \sup_{g \in \mathcal{G}} \left| \overline{g(\vec{X})} - \overline{g(\vec{X}')} \right| > \frac{\epsilon}{2} \right] \\
&= \mathbb{P} \left[ \sup_{g \in \mathcal{G}} \left| \frac{1}{N} \sum_{i=1}^N (g(X_i) - g(X'_i)) \right| > \frac{\epsilon}{2} \right] \\
&= \mathbb{P} \left[ \sup_{g \in \mathcal{G}} \left| \frac{1}{N} \sum_{i=1}^N \sigma_i (g(X_i) - g(X'_i)) \right| > \frac{\epsilon}{2} \right] \\
&\leq \mathbb{P} \left[ \sup_{g \in \mathcal{G}} \left| \frac{1}{N} \sum_{i=1}^N \sigma_i g(X_i) \right| > \frac{\epsilon}{4} \right] + \mathbb{P} \left[ \sup_{g \in \mathcal{G}} \left| \frac{1}{N} \sum_{i=1}^N \sigma_i g(X'_i) \right| > \frac{\epsilon}{4} \right] \\
&= 2\mathbb{P} \left[ \sup_{g \in \mathcal{G}} \left| \frac{1}{N} \sum_{i=1}^N \sigma_i g(X_i) \right| > \frac{\epsilon}{4} \right]. \tag{3.20}
\end{aligned}$$

Until now we showed, combining (3.19) and (3.20), that

$$\mathbb{P} \left[ \sup_{g \in \mathcal{G}} \left| \overline{g(\vec{X})} - \mathbb{E}[g(X)] \right| > \epsilon \right] \leq 4\mathbb{P} \left[ \sup_{g \in \mathcal{G}} \left| \frac{1}{N} \sum_{i=1}^N \sigma_i g(X_i) \right| > \frac{\epsilon}{4} \right].$$

From now on till the end of the proof we will condition on  $\vec{X} = \vec{x} \in \Sigma^N$ , so the probability in (3.20) gets equal to

$$\mathbb{P} \left[ \exists g \in \mathcal{G} : \left| \frac{1}{N} \sum_{i=1}^N \sigma_i g(x_i) \right| > \frac{\epsilon}{4} \right]. \tag{3.21}$$

Denote by  $\mathcal{G}_{\frac{\epsilon}{8}}$  a minimal  $\frac{\epsilon}{8}$ -cover of  $\mathcal{G}(\vec{x})$ , which is a set of  $M = \mathcal{N}(\frac{\epsilon}{8}, \mathcal{G}(\vec{x}))$  functions  $\{g_1, \dots, g_M\}$ , such that for every  $g \in \mathcal{G}$ , we find a  $g' \in \mathcal{G}_{\frac{\epsilon}{8}}$  with

$$\frac{1}{N} \sum_{i=1}^N |g(x_i) - g'(x_i)| < \frac{\epsilon}{8}.$$

This gives us

$$\begin{aligned}
& \left| \frac{1}{N} \sum_{i=1}^N \sigma_i g(x_i) \right| \\
&\leq \left| \frac{1}{N} \sum_{i=1}^N \sigma_i g'(x_i) + \frac{1}{N} \sum_{i=1}^N \sigma_i (g(x_i) - g'(x_i)) \right| \\
&\leq \left| \frac{1}{N} \sum_{i=1}^N \sigma_i g'(x_i) \right| + \frac{1}{N} \sum_{i=1}^N |(g(x_i) - g'(x_i))| \\
&\leq \left| \frac{1}{N} \sum_{i=1}^N \sigma_i g'(x_i) \right| + \frac{\epsilon}{8}
\end{aligned}$$

which we will now use to bound (3.21) by

$$\begin{aligned} & \mathbb{P} \left[ \exists g' \in \mathcal{G}_{\frac{\epsilon}{8}} : \left| \frac{1}{N} \sum_{i=1}^N \sigma_i g'(x_i) \right| + \frac{\epsilon}{8} > \frac{\epsilon}{4} \right] \\ & \leq \sum_{g \in \mathcal{G}_{\frac{\epsilon}{8}}} \mathbb{P} \left[ \left| \frac{1}{N} \sum_{i=1}^N \sigma_i g(x_i) \right| > \frac{\epsilon}{8} \right] \\ & \leq \mathcal{N} \left( \frac{\epsilon}{8}, \mathcal{G}(\vec{x}) \right) \max_{g \in \mathcal{G}_{\frac{\epsilon}{8}}} \mathbb{P} \left[ \left| \frac{1}{N} \sum_{i=1}^N \sigma_i g(x_i) \right| > \frac{\epsilon}{8} \right]. \end{aligned}$$

In the last step we are going to apply Hoeffding's inequality. Note that  $\sigma_1 g'(x_1), \dots, \sigma_N g'(x_N)$  are independent and identically distributed random variables with expectation zero. Furthermore they are bounded by  $|\sigma_i g'(x_i)| \leq G$ . Therefore we get by lemma 3.11 for every  $g \in \mathcal{G}_{\frac{\epsilon}{8}}$

$$\mathbb{P} \left[ \left| \frac{1}{N} \sum_{i=1}^N \sigma_i g(x_i) \right| > \frac{\epsilon}{8} \right] \leq 2 \exp \left( -\frac{2N^2(\epsilon/8)^2}{N(2G)^2} \right) = 2 \exp \left( -\frac{N\epsilon}{128G^2} \right).$$

Taking the expectation to get rid of the conditioning on  $\vec{X}$  finishes the proof.  $\square$

The expected covering number for a set of functions that is applied to a random variable is of course a quantity that will be hard to find. We will bound it with the soon to be introduced concept of the VC-dimension. To be more precise, a closely related concept, the so called packing numbers, will be bounded. We introduce it together with a generalisation of the covering number.

**Definition 3.13.** Let  $(\mathcal{M}, d)$  denote a pseudometric space, that is, we may possibly have  $d(x_1, x_2) = 0$  with  $x_1 \neq x_2$ . For a subset  $A \subset \mathcal{M}$  and any  $\epsilon > 0$  define the covering number  $\mathcal{N}(\epsilon, A, d)$  to be the cardinality of the smallest finite subset  $B \subseteq A$ , such that  $\forall \vec{a} \in A : \exists \vec{b} \in B : d(\vec{a}, \vec{b}) < \epsilon$ .

A set  $\{a_1, \dots, a_m\} \subseteq A$  is said to be  $\epsilon$ -separated if  $d(a_i, a_j) > \epsilon \quad \forall i \neq j$ . Define the packing number  $\mathcal{D}(\epsilon, A, d)$  as the largest integer  $m$  such that  $A$  contains an  $\epsilon$  separated set of cardinality  $m$ .

The following Lemma is from [24] and stated without proof. It states that the packing number is an upper bound for the covering number.

**Lemma 3.14.** For all  $A \subseteq \mathcal{M}$  and  $\epsilon > 0$ , it is true that

$$\mathcal{N}(\epsilon, A, d) \leq \mathcal{D}(\epsilon, A, d). \tag{3.22}$$

### 3.3 VC Dimension

In this section we are going to introduce the concept of the Vapnik - Chervonenkis (VC) dimension. For us its main use is to obtain bounds for covering numbers. We will first assign a VC-dimension to sets and then use this concept to extend the definition to classes of functions. To do so we start with

**Definition 3.15.** Let  $\mathbb{C}$  be a collection of subsets of a set  $\mathcal{X}$ . We say a finite set  $x = \{x_1, \dots, x_n\} \subseteq \mathcal{X}$  is shattered by  $\mathbb{C}$ , if

$$\Delta_{\mathbb{C}}(x) = |\{C \cap \{x_1, \dots, x_n\} : C \in \mathbb{C}\}| = 2^n$$

This means that for every possible subset  $B \subseteq \{x_1, \dots, x_n\}$ , we can find a set  $C \in \mathbb{C}$ , such that  $B = C \cap \{x_1, \dots, x_n\}$ . We will say that  $C$  picks out  $B$ .

**Example** Let  $\mathcal{X} = \{0, 1\}^3$  denote the set of all 0-1 sequences of length 3. Then the set  $x = \{001, 101\} \subset \mathcal{X}^2$  is shattered by

$$\mathbb{C} = \{\{011\}, \{011, 001\}, \{011, 101\}, \{011, 001, 101\}\}$$

since we can build all  $2^2 = 4$  subsets  $\{\{\}, \{001\}, \{101\}, \{001, 101\}\}$ . Note that no  $x \in \mathcal{X}^3$  can be shattered by  $\mathbb{C}$ .

With the concept of shattering we are now able to define the VC - dimension for sets.

**Definition 3.16.** The VC-dimension  $\dim_{VC}(\mathbb{C})$  of a not necessarily finite collection of sets is defined as

$$\dim_{VC}(\mathbb{C}) = \sup\{n \in \mathbb{N} : \exists x \subseteq \mathcal{X}, |x| = n : \Delta_{\mathbb{C}}(x) = 2^n\},$$

i.e. it is the largest integer  $n$  such that we can find a discrete subset of  $\mathcal{X}^n$  which can be shattered by  $\mathbb{C}$  [14].

If we identify a set with its indicator function, we get a VC-dimension for collections of binary valued functions.

**Example** The set  $\mathbb{C}$  from the previous example has a VC-dimension of two.

**Example** For  $\mathcal{X} = \mathbb{R}$ ,  $\mathbb{C} = \{(-\infty, c] : c \in \mathbb{R}\}$  and any  $x \in \mathbb{R} = \mathcal{X}^1$  the set  $\mathbb{C}$  shatters the set  $\{x\}$  since it can pick out  $\{\}$  and  $\{x\}$ . For  $x = (x_1, x_2) \in \mathbb{R}^2$  with  $x_1 < x_2$  it fails to pick out the larger element, so we have

$$\Delta_{\mathbb{C}}((x_1, x_2)) = |\{\{\}, \{x_1\}, \{x_1, x_2\}\}| = 3 \neq 2^2$$

and therefore  $\dim_{VC}(\mathbb{C}) = 1$ .

We will now extend the concept of VC-dimensions to collections of real valued functions using two equivalent definitions. For the first we need to

extend the idea of shattering. We denote by  $\Theta$  the Heaviside step function, which is defined as

$$\Theta(y): \mathbb{R} \rightarrow \{0, 1\}$$

$$y \mapsto \begin{cases} 0 & y < 0 \\ 1 & y \geq 0 \end{cases}$$

**Definition 3.17.** Let  $\mathcal{G}$  denote a class of functions mapping  $\Sigma \subseteq \mathbb{R}^m \mapsto \mathbb{R}$ . We say a subset  $\{y_1, \dots, y_n\} \subseteq \Sigma$  is (pseudo-) shattered by  $\mathcal{G}$ , if there exists a  $\vec{r} \in \mathbb{R}^n$  such that for every  $\vec{b} \in \{0, 1\}^n$  we can find a function  $g_{\vec{b}} \in \mathcal{G}$  fulfilling

$$\forall i \in \{1, \dots, n\} : g_{\vec{b}}(y_i) > r_i \text{ if } b_i = 1 \text{ and } g_{\vec{b}}(y_i) \leq r_i \text{ if } b_i = 0. \quad (3.23)$$

This can also be written, using the Heaviside function, as

$$\forall i \in \{1, \dots, n\} : \Theta(g_{\vec{b}}(y_i) - r_i) = b_i$$

In this case, we will say that the vector  $\vec{r}$  is a witness to the shattering [24]. The (pseudo-)VC-dimension  $\dim_{pVC}(\mathcal{G})$  of a class of functions is then the size of a largest subset that can be shattered, or more precisely

$$\dim_{pVC}(\mathcal{G}) = \sup\{n \in \mathbb{N} : \exists \{y_1, \dots, y_n\} \subseteq \Sigma \text{ shattered by } \mathcal{G}\}$$

Figure 2 helps to understand the concept of pseudo shattering. Note that the subscript of the functions in this figure coincides with the binary vector  $\vec{b}$  from (3.23).

We introduce the subgraph of a function for our second definition.

**Definition 3.18.** For a function  $g : \Sigma \subseteq \mathbb{R}^m \mapsto \mathbb{R}$  we define the subgraph to be

$$G_g = \{(y, r) \in \Sigma \times \mathbb{R} : r \leq g(y)\}$$

i.e. the area under the graph. The subgraph class of a set of functions  $\mathcal{G}$  is accordingly the set of all subgraphs  $G_{\mathcal{G}} = \{G_g : g \in \mathcal{G}\}$ . Assign the VC-dimension of the subgraph class to  $\mathcal{G}$

$$\dim_{VC}(\mathcal{G}) = \dim_{VC}(G_{\mathcal{G}})$$

Of course we would like the two definitions to agree. The next Lemma will show, that this is in fact true.

**Lemma 3.19.** Let  $\mathcal{G}$  denote a class of functions mapping  $\Sigma \subseteq \mathbb{R}^m \mapsto \mathbb{R}$ . Then the pseudo VC-dimension of  $\mathcal{G}$  equals the VC-dimension defined with the help of the subgraph class.



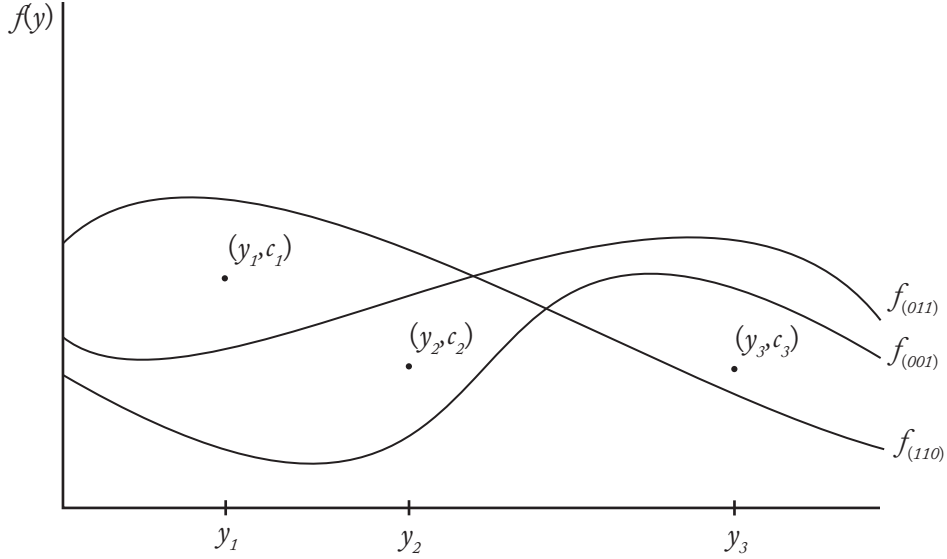


Figure 2: Demonstration of pseudo-shattering

*Proof.* Assume that  $\{y_1, \dots, y_n\}$  is pseudo shattered by  $\mathcal{G}$ . It follows, by definition, the existence of a witness vector  $\vec{r} \in \mathbb{R}^n$  and a function  $g_{\vec{b}}$ , such that (3.23) holds for a fixed  $\vec{b} \in \{0, 1\}^n$ . Observe that a point  $(y_i, r_i)$  is in the subgraph of  $g_{\vec{b}}$  if and only if  $b_i = 1$ . Thus the set  $\{(y_1, r_1), \dots, (y_n, r_n)\}$  is shattered by  $G_{\mathcal{G}}$  and we get  $\dim_{pVC}(\mathcal{G}) \leq \dim_{VC}(G_{\mathcal{G}}) = \dim_{VC}(\mathcal{G})$ . Now assume that the set  $\{z_1, \dots, z_n\} \subseteq \Sigma \times \mathbb{R}$  is shattered by  $G_{\mathcal{G}}$ . Partition every point  $z_i$  as  $z_i = (y_i, c_i)$  with  $y_i \in \Sigma$  and  $c_i \in \mathbb{R}$ . Choose  $\vec{c} = (c_1, \dots, c_n)$  as the witness vector, and it becomes clear, that the set  $\{y_1, \dots, y_n\} \subseteq \Sigma$  is pseudo shattered by  $\mathcal{G}$ . Therefore  $\dim_{VC}(\mathcal{G}) = \dim_{VC}(G_{\mathcal{G}}) \leq \dim_{pVC}(\mathcal{G})$  and the equivalence of the two definitions follows.  $\square$

From now on we will not distinguish between the two definitions and drop the notation  $\dim_{pVC}$ . Obtaining the VC-dimension is not an easy task, one very helpful tool is the following

**Lemma 3.20.** Let  $\mathcal{G}$  be a  $D$ -dimensional linear space of functions mapping  $\Sigma \subseteq \mathbb{R}^m \mapsto \mathbb{R}$ . Then we have

$$\dim_{VC}(\mathcal{G}) \leq D + 1$$

*Proof.* First note that if  $\mathcal{G}$  is a  $D$ -dimensional vector space, then  $\mathcal{F} := \{\alpha r + g(y) : g \in \mathcal{G}, \alpha \in \mathbb{R}\}$  has dimension  $D + 1$ . Introduce  $\mathcal{X} = \Sigma \times \mathbb{R}, x = (y, r)$ .

This gives us for the subgraph class of  $\mathcal{G}$ :

$$\begin{aligned} G_{\mathcal{G}} &= \{\{(y, r) \in \Sigma \times \mathbb{R} : r \leq g(y)\} : g \in \mathcal{G}\} \\ &\subseteq \{\{(y, r) \in \Sigma \times \mathbb{R} : \alpha r + g(y) \geq 0\} : g \in \mathcal{G}, \alpha \in \mathbb{R}\} \\ &= \{\{x : f(x) \geq 0\} : f \in \mathcal{F}\} \end{aligned}$$

The next steps will show that the VC-dimension of  $\{\{x : f(x) \geq 0\} : f \in \mathcal{F}\}$  is less or equal to  $D + 1$ , which proves the statement. To do so, we have to show that no subset of  $\mathcal{X}$  with size  $D + 2$  can be shattered by a set of the form  $\{x : f(x) \geq 0\}$ . Fix any set  $X = \{x_1, \dots, x_{D+2}\} \subset \mathcal{X}$  with distinct elements. Since  $\mathcal{F}$  is linear, the mapping  $L_X : \mathcal{F} \mapsto \mathbb{R}^{D+2}$ ,

$$L_X(f) = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_{D+2}) \end{pmatrix}$$

is linear as well. Furthermore  $L_X(\mathcal{F}) = \{L_X(f) : f \in \mathcal{F}\}$  is a linear subspace of  $\mathbb{R}^{D+2}$  with dimension less or equal to  $D + 1$ , which is the dimension of  $\mathcal{F}$ . Therefore the orthogonal complement of  $L_X(\mathcal{F})$  is not trivial. Thus we can pick a  $\vec{n} = (n_1, \dots, n_{D+2}) \in \mathbb{R}^{D+2}$ ,  $\vec{n} \neq 0$ , such that

$$\begin{aligned} \forall f \in \mathcal{F} : \langle \vec{n}, f(X) \rangle &= 0 \\ n_1 f(x_1) + \dots + n_{D+1} f(x_{D+1}) + n_{D+2} f(x_{D+2}) &= 0. \end{aligned} \quad (3.24)$$

W.l.o.g assume that at least one of the entries of  $\vec{n}$  is negative (if not, switch to  $-\vec{n}$ ). Define the index sets  $I = \{1, \dots, D + 1, D + 2\}$  and  $J = \{i \in I : n_i \geq 0\}$  and rewrite (3.24) as

$$\forall f \in \mathcal{F} : \sum_{i \in J} n_i f(x_i) = \sum_{i \in I \setminus J} (-n_i) f(x_i) \quad (3.25)$$

Now assume that we can find a  $f^* \in \mathcal{F}$ , such that the set  $\{x : f^*(x) \geq 0\}$  picks out exactly the  $x_i$ 's with  $n_i \geq 0$ , i.e.  $\{x : f^*(x) \geq 0\} \cap X = \{x_i : i \in J\}$ . This would lead to  $\sum_{i \in J} n_i f^*(x_i) \geq 0$  and  $\sum_{i \in I \setminus J} (-n_i) f^*(x_i) < 0$  since for  $i \in I \setminus J : -n_i > 0$  and  $f^*(x_i) < 0$ . This is obviously a contradiction to equation (3.25).  $\square$

Next we are going to state Haussler's inequality. Since it is subject of combinatorics the proof will be omitted here. The version given here is corollary 1 of [15].

**Lemma 3.21.** Let  $\mathbb{P}$  be any probability measure on  $\Sigma$  and  $\mathcal{G}$  a set of  $\mathbb{P}$ -measurable functions mapping  $\Sigma \mapsto \{0, 1\}$ . For  $f, g \in \mathcal{G}$  define the pseudo-metric  $\sigma_{\mathbb{P}}(f, g) = \mathbb{P}[f \neq g]$ . Then for any  $\epsilon > 0$

$$\mathcal{D}(\epsilon, \mathcal{G}, \sigma_{\mathbb{P}}) \leq e(\dim_{VC}(\mathcal{G}) + 1) \left(\frac{2e}{\epsilon}\right)^{\dim_{VC}(\mathcal{G})} \quad (3.26)$$

If we choose  $\mathcal{G}$  to be indicator functions of a collections of measurable sets  $\mathcal{C}$  we get equivalently [11]

$$\mathcal{D}(\epsilon, \mathcal{C}, d_{\mathbb{P}}) \leq e(\dim_{VC}(\mathcal{C}) + 1) \left( \frac{2e}{\epsilon} \right)^{\dim_{VC}(\mathcal{C})}$$

where

$$d_{\mathbb{P}}(A, B) := \mathbb{P}[A \Delta B] = \mathbb{P}[\mathbb{I}_A \neq \mathbb{I}_B] = \sigma_{\mathbb{P}}(\mathbb{I}_A, \mathbb{I}_B)$$

for  $A, B \in \mathcal{C}$ . Next we are going to generalize Lemma 3.21 to classes of uniformly bounded functions to get the version used in [25].

**Theorem 3.22.** Let  $\mathcal{G}$  be a class of functions uniformly bounded by  $G > 0$ , that is  $\forall g \in \mathcal{G} : \|g\|_{\infty} \leq G$ . For any  $\vec{v} \in \Sigma^N$ , any  $\epsilon > 0$  we get

$$\mathcal{N}(\epsilon, \mathcal{G}(\vec{v})) \leq e(\dim_{VC}(\mathcal{G}) + 1) \left( \frac{4eG}{\epsilon} \right)^{\dim_{VC}(\mathcal{G})}.$$

*Proof.* Define  $\mathbb{P} = 1/N \sum_{i=1}^N \delta_{v_i}$ , where  $\delta_{v_i}$  is the Dirac-measure, so  $\mathbb{P}$  is the discrete uniform distribution on the elements of  $\vec{v}$ . Also generalize the pseudo metric to the class of bounded functions by

$$\begin{aligned} \sigma_{\mathbb{P}}(f, g) &= \int_{\Sigma} |f - g| d\mathbb{P} \\ &= \frac{1}{N} \sum_{i=1}^N |f(v_i) - g(v_i)|. \end{aligned}$$

Thus we get

$$\begin{aligned} \mathcal{N}(\epsilon, \mathcal{G}(\vec{v})) &= \min\{n \in \mathbb{N} : n = |\Gamma|, \Gamma \subseteq \mathcal{G}(\vec{v}), \forall \vec{g} \in \mathcal{G}(\vec{v}) : \\ &\quad \exists \gamma \in \Gamma : 1/N \|\vec{g} - \gamma\|_1 < \epsilon\} \\ &= \min\{n \in \mathbb{N} : n = |\Gamma|, \Gamma \subseteq \mathcal{G}, \forall g \in \mathcal{G} : \exists \gamma \in \Gamma : \sigma_{\mathbb{P}}(f, g) < \epsilon\} \\ &= \mathcal{N}(\epsilon, \mathcal{G}, \sigma_{\mathbb{P}}). \end{aligned}$$

Let  $\lambda$  be the Lebesgue measure. Then we have for  $f, g \in \mathcal{G}$

$$\begin{aligned} (\mathbb{P} \times \lambda)[G_f \Delta G_g] &= \frac{1}{N} \sum_{i=1}^N \lambda[[f(v_i), g(v_i)] \cup [g(v_i), f(v_i)]] \\ &= \frac{1}{N} \sum_{i=1}^N |f(v_i) - g(v_i)| \\ &= \sigma_{\mathbb{P}}(f, g) \end{aligned}$$

where  $[a, b]$  is the interval from  $a$  to  $b$  and  $[a, b] = \{\}$  for  $a > b$ . Thus we get

$$\mathcal{N}(\epsilon, \mathcal{G}, \sigma_{\mathbb{P}}) = \mathcal{N}(\epsilon, G_{\mathcal{G}}, d_{\mathbb{P} \times \lambda}).$$

Define

$$\mathbb{Q} = \frac{\mathbb{P} \times \lambda}{2G}$$

which is a probability measure on  $\vec{v} \times [-G, G]$  and therefore we can apply Theorem 3.22 and Lemma 3.14

$$\begin{aligned} \mathcal{N}(\epsilon, G_{\mathcal{G}}, d_{\mathbb{P} \times \lambda}) &= \mathcal{N}\left(\frac{\epsilon}{2G}, G_{\mathcal{G}}, d_{\mathbb{Q}}\right) \\ &\leq \mathcal{D}\left(\frac{\epsilon}{2G}, G_{\mathcal{G}}, d_{\mathbb{Q}}\right) \\ &\leq e(\dim_{VC}(\mathcal{G}) + 1) \left(\frac{4eG}{\epsilon}\right)^{\dim_{VC}(\mathcal{G})} \end{aligned}$$

□

## 4 Error Propagation

We will now proceed to find an upper bound for the error propagation occurring during a (backward) step in the Longstaff-Schwartz algorithm. It will later be used to derive the main results. Throughout this chapter we write  $\|\cdot\|_{L_k^2}$  for the norm on  $L_k^2 := L^2(\mathcal{S}, \mathcal{B}_k, \rho_k)$ . We start this chapter with the following

**Proposition 4.1.** Let  $T$  denote the transition operator, defined on a function  $J \in L_{k+1}^2$  as

$$(TJ)(x) = \mathbb{E}[J(X_{k+1})|X_k = x]. \quad (4.27)$$

Then  $T$  is a  $L^2$  contraction, more precisely we have

$$\|TJ\|_{L_k^2} \leq \|J\|_{L_{k+1}^2} \quad (4.28)$$

In order to prove equation 4.28 we use, as in [23], Jensen's inequality and the tower property of the conditional expectation.

*Proof.*

$$\begin{aligned} \|TJ\|_{L_k^2}^2 &= \mathbb{E}\left[\left((TJ)(X_k)\right)^2\right] \\ &= \mathbb{E}\left[\left(\mathbb{E}[J(X_{k+1})|X_k]\right)^2\right] \\ &\leq \mathbb{E}\left[\mathbb{E}[J^2(X_{k+1})|X_k]\right] \\ &= \mathbb{E}[J^2(X_{k+1})] \\ &= \|J\|_{L_{k+1}^2}^2 \end{aligned}$$

□

**Proposition 4.2.**  $\mathbb{E}[Z_{\tau_k}|X_k]$ ,  $\mathbb{E}[Z_{\tau_{k+1}}|X_k]$ ,  $\mathbb{E}[Z_{\tau_{k+1}^A}|X_k]$  and  $\mathbb{E}[Z_{\tau_k^A}|X_k]$  are in  $L^2(\mathcal{S}, \rho_k)$ .

*Proof.* We will show the proof only for  $\mathbb{E}[Z_{\tau_k}|X_k]$  and  $\mathbb{E}[Z_{\tau_{k+1}}|X_k]$  since it works analogously for the approximated counterparts. Our aim is to find an  $L^2$ -integrable majorant. For every  $k \in \{0, \dots, K\}$  the random variable  $Z_k$  is in  $L^2$ , so consequently  $\sum_{j=k}^K \|Z_j\|_{L_j^2} < \infty$ . By the use of backwards induction and with proposition 4.1 we will demonstrate that for each  $k = 0, \dots, K - 1$

$$\|\mathbb{E}[Z_{\tau_{k+1}}|X_k]\|_{L_k^2} \leq \sum_{j=k+1}^K \|Z_j\|_{L_j^2} < \infty. \quad (4.29)$$

The base case for  $k = K - 1$  follows immediately by proposition 4.1. We will assume that (4.29) holds for all  $k \geq k_0$  and show the induction step  $k_0 \rightarrow k_0 - 1$ :

$$\begin{aligned}
& \left\| \mathbb{E}[Z_{\tau_{k_0}} | X_{k_0-1}] \right\|_{L^2_{k_0-1}} \\
&= \left\| \mathbb{E}[\mathbb{E}[Z_{\tau_{k_0}} | X_{k_0}] | X_{k_0-1}] \right\|_{L^2_{k_0-1}} \\
&\leq \left\| \mathbb{E}[Z_{\tau_{k_0}} | X_{k_0}] \right\|_{L^2_{k_0}} \\
&= \left\| \max(Z_{k_0}, \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}]) \right\|_{L^2_{k_0}} \\
&\leq \|Z_{k_0}\|_{L^2_{k_0}} + \left\| \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}] \right\|_{L^2_{k_0}} \\
&\leq \sum_{j=k_0}^K \|Z_j\|_{L^2_j}
\end{aligned}$$

□

The following proposition bounds the error that occurs within one step of the LS-algorithm and will be used to derive the main theorems.

**Proposition 4.3.**

$$\begin{aligned}
& \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k+1}^A} | X_k] \right) - \mathbb{E}[Z_{\tau_{k+1}} | X_k] \right\|_{L^2_k} \\
&\leq 2 \sum_{j=k}^{K-1} \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right) - \mathbb{E}[Z_{\tau_{j+1}} | X_j] \right\|_{L^2_j}
\end{aligned} \tag{4.30}$$

*Proof.* To prove (4.30) we will first show

$$\begin{aligned}
& \left\| \mathbb{E}[Z_{\tau_{k+1}^A} | X_k] - \mathbb{E}[Z_{\tau_{k+1}} | X_k] \right\|_{L^2_k} \\
&\leq 2 \sum_{j=k+1}^{K-1} \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right) - \mathbb{E}[Z_{\tau_{j+1}} | X_j] \right\|_{L^2_j}
\end{aligned} \tag{4.31}$$

for all  $0 \leq k \leq K - 2$  by backwards induction. Since the proof of the base case with  $k = K - 2$  follows the argumentation of the inductive step, we will show it afterwards. So let's assume that 4.31 is already shown for  $K - 2, \dots, k_0$  with  $K - 2 \geq k_0$  and we want to approve it for  $k_0 - 1$ . With the tower property we get

$$\left\| \mathbb{E}[Z_{\tau_{k_0}^A} - Z_{\tau_{k_0}} | X_{k_0-1}] \right\|_{L^2_{k_0-1}} \tag{4.32}$$

$$= \left\| \mathbb{E} \left[ \mathbb{E}[Z_{\tau_{k_0}^A} | X_{k_0}] - \mathbb{E}[Z_{\tau_{k_0}} | X_{k_0}] | X_{k_0-1} \right] \right\|_{L^2_{k_0-1}} \tag{4.33}$$

and now proposition 4.2 allows us to apply (4.28)

$$\begin{aligned}
&\leq \left\| \mathbb{E}[Z_{\tau_{k_0}^A} | X_{k_0}] - \mathbb{E}[Z_{\tau_{k_0}} | X_{k_0}] \right\|_{L_{k_0}^2} \\
&= \left\| \sum_{i=1}^4 \mathbb{I}_{S_i} \left( \mathbb{E}[Z_{\tau_{k_0}^A} | X_{k_0}] - \mathbb{E}[Z_{\tau_{k_0}} | X_{k_0}] \right) \right\|_{L_{k_0}^2} \tag{4.34}
\end{aligned}$$

where  $S_1, \dots, S_4$  is a partition of  $\Omega$  in events as follows:

$$\begin{aligned}
S_1 &: [\tau_{k_0} = k_0] \cap [\tau_{k_0}^A = \tau_{k_0+1}^A] \\
&= [Z_{k_0} \geq \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}]] \cap [Z_{k_0} < \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right)] \\
S_2 &: [\tau_{k_0} = \tau_{k_0+1}] \cap [\tau_{k_0}^A = k_0] \\
&= [Z_{k_0} < \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}]] \cap [Z_{k_0} \geq \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right)] \\
S_3 &: [\tau_{k_0} = \tau_{k_0+1}] \cap [\tau_{k_0}^A = \tau_{k_0+1}^A] \\
&= [Z_{k_0} < \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}]] \cap [Z_{k_0} < \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right)] \\
S_4 &: [\tau_{k_0} = k_0] \cap [\tau_{k_0}^A = k_0] \\
&= [Z_{k_0} \geq \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}]] \cap [Z_{k_0} \geq \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right)]
\end{aligned}$$

Our next step is to provide upper bounds for the expressions

$$\mathbb{I}_{S_i} \left| \mathbb{E}[Z_{\tau_{k_0}^A} | X_{k_0}] - \mathbb{E}[Z_{\tau_{k_0}} | X_{k_0}] \right|.$$

On the event  $S_1$  we get, applying the triangle inequality

$$\begin{aligned}
&\mathbb{I}_{S_1} \left| \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] - \mathbb{E}[Z_{\tau_{k_0}} | X_{k_0}] \right| \\
&= \mathbb{I}_{S_1} \left| \mathbb{E}[Z_{\tau_{k_0}^A} | X_{k_0}] - Z_{k_0} \right| \\
&\leq \mathbb{I}_{S_1} \left| \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] - \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right) \right| \\
&\quad + \mathbb{I}_{S_1} \left| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right) - Z_{k_0} \right| \\
&= \mathbb{I}_{S_1} \left| \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] - \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right) \right| \\
&\quad + \mathbb{I}_{S_1} \left| Z_{k_0} - \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right) \right|
\end{aligned}$$

and by the rule  $a \leq b \leq c \Rightarrow |b - c| \leq |a - c|$

$$\begin{aligned}
&\leq \mathbb{I}_{S_1} \left| \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] - \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right) \right| \\
&\quad + \mathbb{I}_{S_1} \left| \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}] - \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right) \right|
\end{aligned}$$

For all real numbers we have  $|d - c| + |a - c| \leq |d - c| + |a - c - d + c| + |d - c| = 2|d - c| + |d - a|$ , which gives us

$$\begin{aligned}
& \mathbb{I}_{S_1} \left| \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] - \mathbb{E}[Z_{\tau_{k_0}} | X_{k_0}] \right| \\
& \leq 2\mathbb{I}_{S_1} \left| \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] - \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right) \right| \\
& \quad + \mathbb{I}_{S_1} \left| \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] - \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right) \right|
\end{aligned} \tag{4.35}$$

Regarding the event  $S_2$  we find - applying the properties of  $S_2$  for the first inequality and later the triangular inequality - the following:

$$\begin{aligned}
& \mathbb{I}_{S_2} \left| \mathbb{E}[Z_{\tau_{k_0}^A} | X_{k_0}] - \mathbb{E}[Z_{\tau_{k_0}} | X_{k_0}] \right| \\
& = \mathbb{I}_{S_2} \left| Z_{k_0} - \mathbb{E}[Z_{\tau_{k_0}} | X_{k_0}] \right| \\
& \leq \mathbb{I}_{S_2} \left| \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}] - \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right) \right| \\
& \leq \mathbb{I}_{S_2} \left| \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}] - \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right| \\
& \quad + \mathbb{I}_{S_2} \left| \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] - \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right) \right|
\end{aligned}$$

To bring it in the desired form the first term will be multiplied by 2, and we finally get:

$$\begin{aligned}
& \mathbb{I}_{S_2} \left| \mathbb{E}[Z_{\tau_{k_0}^A} | X_{k_0}] - \mathbb{E}[Z_{\tau_{k_0}} | X_{k_0}] \right| \\
& \leq 2\mathbb{I}_{S_2} \left| \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] - \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right) \right| \\
& \quad + \mathbb{I}_{S_2} \left| \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] - \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}] \right|
\end{aligned} \tag{4.36}$$

And on  $S_3$  we easily find

$$\begin{aligned}
& \mathbb{I}_{S_3} \left| \mathbb{E}[Z_{\tau_{k_0}^A} | X_{k_0}] - \mathbb{E}[Z_{\tau_{k_0}} | X_{k_0}] \right| \\
& = \mathbb{I}_{S_3} \left| \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] - \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}] \right| \\
& \leq 2\mathbb{I}_{S_3} \left| \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] - \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right) \right| \\
& \quad + \mathbb{I}_{S_3} \left| \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] - \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}] \right|
\end{aligned} \tag{4.37}$$

Since on  $S_4$  both stopping times are  $k_0$ , the difference between the expectations vanishes, so it is sufficient to sum over  $i = 1, 2, 3$  in (4.34). If we plug



in (4.35),(4.36) and (4.37) into (4.34) we get

$$\begin{aligned}
& \left\| \mathbb{E}[Z_{\tau_{k_0}^A} - Z_{\tau_{k_0}} | X_{k_0-1}] \right\|_{L_{k_0-1}^2} \\
& \leq \left\| \sum_{i=1}^3 \mathbb{I}_{S_i} \left( \mathbb{E}[Z_{\tau_{k_0}^A} | X_{k_0}] - \mathbb{E}[Z_{\tau_{k_0}} | X_{k_0}] \right) \right\|_{L_{k_0}^2} \\
& \leq \left\| 2 \left| \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] - \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right) \right| \right. \\
& \quad \left. + \left| \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] - \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}] \right| \right\|_{L_{k_0}^2} \\
& \leq 2 \left\| \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] - \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right) \right\|_{L_{k_0}^2} \\
& \quad + \left\| \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] - \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}] \right\|_{L_{k_0}^2} \tag{4.38}
\end{aligned}$$

were we used Minowski's inequality in the last manipulation to finish the inductive step. The base case with  $k = K - 2$  follows by setting  $k_0 = K - 1$  in (4.38), and if we keep in mind that  $\tau_K^A = \tau_K = K$  it simplifies to

$$\begin{aligned}
& \left\| \mathbb{E}[Z_{\tau_{K-1}^A} - Z_{\tau_{K-1}} | X_{K-2}] \right\|_{L_{K-2}^2} \\
& \leq \left\| 2 \left| \mathbb{E}[Z_{\tau_K^A} | X_{K-1}] - \mathcal{A} \left( \mathbb{E}[Z_{\tau_K^A} | X_{K-1}] \right) \right| \right. \\
& \quad \left. + \left| \mathbb{E}[Z_{\tau_K^A} | X_{K-1}] - \mathbb{E}[Z_{\tau_K} | X_{K-1}] \right| \right\|_{L_{K-1}^2} \\
& = 2 \left\| \mathbb{E}[Z_{\tau_K^A} | X_{K-1}] - \mathcal{A} \left( \mathbb{E}[Z_{\tau_K^A} | X_{K-1}] \right) \right\|_{L_{K-1}^2} \tag{4.39}
\end{aligned}$$

It easily seen that the statement of the proposition holds for  $k = K - 1$ , again we use  $\tau_K^A = \tau_K = K$  and get

$$\begin{aligned}
& \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_K^A} | X_{K-1}] \right) - \mathbb{E}[Z_{\tau_K} | X_{K-1}] \right\|_{L_{K-1}^2} \\
& = \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_K^A} | X_{K-1}] \right) - \mathbb{E}[Z_{\tau_K^A} | X_{K-1}] \right\|_{L_{K-1}^2} \\
& \leq 2 \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_K^A} | X_{K-1}] \right) - \mathbb{E}[Z_{\tau_K^A} | X_{K-1}] \right\|_{L_{K-1}^2}
\end{aligned}$$

It remains to show that for all  $k \in \{0, \dots, K - 2\}$  equation (4.30) follows directly from (4.31). To do so we fix a  $k$  and add

$$2 \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k+1}^A} | X_k] \right) - \mathbb{E}[Z_{\tau_{k+1}^A} | X_k] \right\|_{L_k^2}$$

on both sides of (4.31). This gives us

$$\begin{aligned}
& 2 \sum_{j=k+1}^{K-1} \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{j+1}^{\mathcal{A}}} | X_j] \right) - \mathbb{E}[Z_{\tau_{j+1}^{\mathcal{A}}} | X_j] \right\|_{L_j^2} \\
& \geq \left\| \mathbb{E}[Z_{\tau_{k+1}^{\mathcal{A}}} | X_k] - \mathbb{E}[Z_{\tau_{k+1}} | X_k] \right\|_{L_k^2} \\
& \quad + 2 \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k+1}^{\mathcal{A}}} | X_k] \right) - \mathbb{E}[Z_{\tau_{k+1}^{\mathcal{A}}} | X_k] \right\|_{L_k^2} \\
& \geq \left\| \mathbb{E}[Z_{\tau_{k+1}^{\mathcal{A}}} | X_k] - \mathbb{E}[Z_{\tau_{k+1}} | X_k] + \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k+1}^{\mathcal{A}}} | X_k] \right) - \mathbb{E}[Z_{\tau_{k+1}^{\mathcal{A}}} | X_k] \right\|_{L_k^2} \\
& = \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k+1}^{\mathcal{A}}} | X_k] \right) - \mathbb{E}[Z_{\tau_{k+1}} | X_k] \right\|_{L_k^2}
\end{aligned}$$

□

**Corollary 4.4.** The upper bound for the error of the continuation value at the start point is given by

$$\left| \mathcal{A} \left( \mathbb{E}[Z_{\tau_0^{\mathcal{A}}}] \right) - \mathbb{E}[Z_{\tau_0}] \right| \leq 2 \sum_{j=0}^{K-1} \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{j+1}^{\mathcal{A}}} | X_j] \right) - \mathbb{E}[Z_{\tau_{j+1}^{\mathcal{A}}} | X_j] \right\|_{L_j^2} \quad (4.40)$$

*Proof.* Recall that we have

$$\mathbb{E}[Z_{\tau_0}] = \max(Z_0, \mathbb{E}[Z_{\tau_1}]) \text{ and } \mathcal{A} \left( \mathbb{E}[Z_{\tau_0^{\mathcal{A}}}] \right) = \max(Z_0, \mathcal{A} \left( \mathbb{E}[Z_{\tau_1^{\mathcal{A}}}] \right))$$

and apply the inequality

$$|\max(a, b) - \max(a, c)| \leq |b - c|$$

Together with (4.30) for  $k = 0$  the statement follows. □

## 5 Error Estimates

In this section we are going to deduce some error estimates for the Longstaff-Schwartz algorithm by applying our results of the statistical learning chapter. We begin with a recapitulation of the used probability spaces and will also introduce some new notations.

For each  $k = 0, \dots, K - 1$ , we denote by  $\rho_k$  the measure on  $\Omega = \mathcal{S} \times \mathbb{R}$  which is jointly induced by  $(X_k, Z_{\tau_{k+1}^A})$ . The pair  $(\Omega, \rho_k)$  will be referred to as  $\Omega_k$ . Furthermore we get for the marginal probability measure as in equation (3.11)

$$(\rho_k)_\mathcal{S} = \rho_k$$

where  $\rho_k$  is the measure on  $\mathcal{S}$  induced by  $X_k$ . Again, for each  $k = 0, \dots, K - 1$ , we get a measure  $\mathcal{P}_k$  on  $\Omega \times \dots \times \Omega$  ( $(K - k)$ -times) induced by the sequence  $(X_j, Z_{\tau_{j+1}^A})_{j=k, \dots, K-1}$ . We will write  $\mathbf{\Omega}_k$  for the pair  $(\Omega \times \dots \times \Omega, \rho_k)$ . For a fixed sample path  $i = 1, \dots, N$  and fixed  $k$ , we denote the vector with varying time  $j = k, \dots, K - 1$  by

$$\omega_{ki} = \left( ({}^i x_k, {}^i z_{\tau_{(k+1)}^A}), \dots, ({}^i x_K, {}^i z_{\tau_{(K)}^A}) \right) \in \mathbf{\Omega}_k.$$

In case  $k = 0$  we use the notation  $\mathbf{\Omega} = \mathbf{\Omega}_k$  and  $\omega_{0i} = \omega_0$ . To generate  $N$  sample paths independently, starting at time  $k$ , we need to draw from the measure  $\mathbb{P}_k = \mathcal{P}_k \times \dots \times \mathcal{P}_k$  ( $N$ -times) on the space  $\Omega \times \dots \times \Omega$  ( $N(K - k)$ -times) and denote the pair with  $\mathbf{\Omega}_k^N = (\Omega \times \dots \times \Omega, \mathbb{P}_k)$ . Again set  $\mathbf{\Omega}^N = \mathbf{\Omega}_0^N$  and  $\mathbb{P} = \mathbb{P}_0$ . For a fixed time  $k$  we write for the vector of  $N$  independent samples

$$\omega^{kN} = \left( ({}^1 x_k, {}^1 z_{\tau_{(k+1)}^A}), \dots, ({}^N x_k, {}^N z_{\tau_{(k+1)}^A}) \right) \in \mathbf{\Omega}_k^N$$

and  $P_k = \rho_k \otimes \dots \otimes \rho_k$  ( $N$ -times) for the measure on  $\mathbf{\Omega}_k^N$ . Thus, for any time step  $k$  we have two possibilities to refer to a sample from  $\mathbf{\Omega}_k^N$ : by the  $N$ -tuple  $(\omega_{k1}, \dots, \omega_{kN})$  or by the  $(K - k)$ -tuple  $(\omega^{kN}, \dots, \omega^{(K-1)N})$ .

### 5.1 The two main results

The following theorem is the first of our two main results. The bound itself depends on how well the expected payoff, using the approximated exercising rule, can be reproduced by the approximation spaces. Note that the bound is not exponentially dependent on the number of possible exercising points  $K$ , what of course comes handy if this quantity is large. The proof uses all the tools from the previous chapter. We will show how to express the Longstaff-Schwartz algorithm within the terminology of statistical learning and use Pollard's inequality to bound the error of this modelling process. The previously found majorants for covering numbers will then be applied, what introduces the VC-dimension into the estimates.

**Theorem 5.1.** Set  $R = \max\{1, \|Z_1\|_\infty, \dots, \|Z_K\|_\infty\}$  and assume that  $R < \infty$ . For each  $k = 1, \dots, K-1$  we denote the approximation space by  $\mathcal{H}_k$ . It is an arbitrary subspace of  $L^2(\mathcal{S}, \rho_k)$ , with  $\dim_{VC}(\mathcal{H}_k) \leq t < \infty$  and bounded, in the sense that  $\sup\{\|f\|_\infty : f \in \mathcal{H}_k\} \leq L < \infty$ . In case  $k = 0$  we define the singleton set  $\mathcal{H}_0 = \{\mathbb{E}[Z_{\tau_1^A}]\}$ . Then for any number of simulations  $N$ , any  $\epsilon$  satisfying  $1 \geq \epsilon > 0$  and fulfilling the assumption that we can find an  $\epsilon_0$  (as in definition 3.6) such that  $0 \leq \epsilon_0 \leq 6\epsilon^2$ , we get for all  $k = 0, \dots, K-1$

$$\begin{aligned} & \mathbb{P}_k \left[ \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k+1}^A} | X_k] \right) - \mathbb{E}[Z_{\tau_{k+1}} | X_k] \right\|_{L_k^2} \right. \\ & \quad \leq 8(K-k)\epsilon + 2\sqrt{2} \sum_{j=k}^{K-1} \inf_{f \in \mathcal{H}_j} \left\| f - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2} \left. \right] \\ & \geq 1 - 8e(K-k)(t+1) \left( \frac{128eL(L+R)}{\epsilon^2} \right)^t \exp \left( -\frac{N\epsilon^4}{512(L+R)^4} \right) \end{aligned} \quad (5.41)$$

where we used  $L_k^2 := L^2(\mathcal{S}, \rho_k)$ .

*Proof.* Our first step will be to apply Pollard's inequality, Theorem 3.12. Fix a  $k = 0, \dots, K-1$ , then for each  $j = k, \dots, K-1$ ,

$$\mathcal{G} = \{g : g(\omega) = (f(s) - r)^2 : f \in \mathcal{H}_j\}$$

is a set of functions mapping  $\Omega = \mathcal{S} \times [-R, R] \mapsto \mathbb{R}$ , with  $\omega = (s, r), s \in \mathcal{S}, r \in [-R, R]$ . Since  $\mathcal{H}_j$  is bounded by  $L$ ,  $\mathcal{G}$  is bounded by  $(L+R)^2$  as is  $\mathcal{G}(\omega^{jN})$ . Recall that from (3.16) we have with this notation

$$\mathcal{G}(\omega^{jN}) = \left\{ \left( \begin{array}{c} \left( f(1x_j) - {}^1z_{\tau_{j+1}^A} \right)^2 \\ \vdots \\ \left( f(Nx_j) - {}^Nz_{\tau_{j+1}^A} \right)^2 \end{array} \right) : f \in \mathcal{H}_j \right\}.$$

Thus we get by Pollard's inequality for  $\epsilon^2 > 0$

$$\begin{aligned} & P_j \left[ \sup_{f \in \mathcal{H}_j} \left| \frac{1}{N} \sum_{i=1}^N \left( f(ix_j) - {}^iz_{\tau_{j+1}^A} \right)^2 - \mathbb{E} \left[ \left( f(X_j) - Z_{\tau_{j+1}^A} \right)^2 \right] \right| > \epsilon^2 \right] \\ & \leq 8\mathbb{E} \left[ \mathcal{N} \left( \frac{\epsilon^2}{8}, \mathcal{G}(\omega^{jN}) \right) \right] \exp \left( -\frac{N\epsilon^4}{512(L+R)^4} \right). \end{aligned} \quad (5.42)$$

From now on we will write  $\delta$  for the right hand side of equation (5.42).

Note that if we identify the measure  $\rho$  in definition 3.1 with  $\rho_j$ , we get

$$\text{MSE}(f) = \mathbb{E} \left[ \left( f(X_j) - Z_{\tau_{j+1}^A} \right)^2 \right]$$

and identifying  $\omega$  with  $\omega^{jN}$  gives

$$\text{EMSE}_{\omega^{jN}}(f) = \frac{1}{N} \sum_{i=1}^N \left( f(i x_j) - z_{\tau_{j+1}^A}^i \right)^2.$$

With this notation and inverting the probability, inequality (5.42) becomes

$$P_j \left[ \sup_{f \in \mathcal{H}_j} |\text{EMSE}_{\omega^{jN}}(f) - \text{MSE}(f)| \leq \epsilon^2 \right] \geq 1 - \delta. \quad (5.43)$$

The regression function  $f_{\rho_j}$  of definition 3.4 can be identified with  $\mathbb{E}[Z_{\tau_{j+1}^A} | X_j]$ .

If we look at the LS-algorithm,  $\mathcal{A} \left( \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right)$  is an approximator, as is  $f_{\omega^{jN}, \epsilon_0}$  from definition 3.6. Thus equation (5.43) fulfils the requirements (3.14) of Proposition 3.7, which gives us

$$\begin{aligned} P_j & \left[ \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right) - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2}^2 \right. \\ & \leq 2\epsilon^2 + \epsilon_0 + \inf_{f \in \mathcal{H}_j} \left\| f - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2}^2 \left. \right] \\ & \geq 1 - \delta \end{aligned}$$

Because  $0 \leq \epsilon_0 \leq 6\epsilon^2$ , taking the root gives

$$\begin{aligned} P_j & \left[ \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right) - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2} \right. \\ & \leq \sqrt{8\epsilon^2 + \inf_{f \in \mathcal{H}_j} \left\| f - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2}^2} \left. \right] \\ & \geq 1 - \delta \end{aligned}$$

and finally using the inequality  $\sqrt{a^2 + b^2} \leq \sqrt{2}(a + b)$  for  $a, b \geq 0$  leads to

$$\begin{aligned} P_j & \left[ \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right) - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2} \right. \\ & \leq \underbrace{4\epsilon + \sqrt{2} \inf_{f \in \mathcal{H}_j} \left\| f - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2}}_{=: \gamma_j} \left. \right] \quad (5.44) \\ & \geq 1 - \delta \end{aligned}$$

The next step is to use Haussler's inequality, Theorem 3.22, to get a bound on the covering number. To do so we will separate  $\mathcal{G}$  as follows:

$$\begin{aligned} \mathcal{G}_1 & = \{g_1 : g_1(\omega) = g_1(s, r) = f(s) : f \in \mathcal{H}_j\} \\ \mathcal{G}_2 & = \{g_2 : g_2(\omega) = g_2(s, r) = r\} \end{aligned}$$

Since  $\mathcal{G}_2$  consists only of the projection of  $\Omega$  onto  $\mathbb{R}$ , we get  $\mathcal{G}_2(\omega^{jN}) = \{(^1z_{\tau_{j+1}^A}, \dots, ^Nz_{\tau_{j+1}^A})\}$  and hence  $\mathcal{N}(\epsilon, \mathcal{G}_2(\omega^{jN})) = 1$ . With the notation  $\vec{s} = (^1x_j, \dots, ^Nx_j)$  we can rewrite  $\mathcal{G}_1$  as

$$\mathcal{G}_1(\omega^{jN}) = \left\{ \begin{pmatrix} f(^1x_j) \\ \vdots \\ f(^Nx_j) \end{pmatrix} : f \in \mathcal{H}_j \right\} = \mathcal{H}_j(\vec{s})$$

which of course leads to the equivalence of the covering numbers:

$$\mathcal{N}(\epsilon, \mathcal{G}_1(\omega^{jN})) = \mathcal{N}(\epsilon, \mathcal{H}_j(\vec{s})).$$

We can construct  $\mathcal{G}$  from  $\mathcal{G}_1$  and  $\mathcal{G}_2$  by  $\mathcal{G} = (\mathcal{G}_1 - \mathcal{G}_2)^2$ . Thus, applying the properties for covering numbers from Proposition 3.9, we get

$$\begin{aligned} \mathcal{N}\left(\frac{\epsilon^2}{8}, \mathcal{G}(\omega^{jN})\right) &\leq \mathcal{N}\left(\frac{\epsilon^2}{16(L+R)}, (\mathcal{G}_1 - \mathcal{G}_2)(\omega^{jN})\right) \\ &\leq \mathcal{N}\left(\frac{\epsilon^2}{32(L+R)}, \mathcal{H}_j(\vec{s})\right) \cdot 1 \end{aligned}$$

Applying Haussler's inequality and Lemma 3.14 gives us the following bound on the covering number:

$$\begin{aligned} \mathcal{N}\left(\frac{\epsilon^2}{32(L+R)}, \mathcal{H}_j(\vec{s})\right) \\ \leq e(\dim_{VC}(\mathcal{H}_j) + 1) \left(\frac{32(L+R)}{\epsilon^2} 4eL\right)^{\dim_{VC}(\mathcal{H}_j)} \end{aligned}$$

If we use this bound for the covering number in equation (5.44), change to the complementary event and recall that  $\dim_{VC}(\mathcal{H}_j) \leq t$ , we establish

$$\begin{aligned} P_j \left[ \left\| \mathcal{A}\left(\mathbb{E}[Z_{\tau_{j+1}^A} | X_j]\right) - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2} > \gamma_j \right] \\ \leq \underbrace{8e(t+1) \left(\frac{128eL(L+R)}{\epsilon^2}\right)^t}_{=:\theta} \exp\left(-\frac{N\epsilon^4}{512(L+R)^4}\right) \end{aligned} \quad (5.45)$$

By subadditivity and De Morgan's laws we have for any events  $(E_j)_{j=k, \dots, K-1}$

$$\begin{aligned} \mathbb{P}_k \left[ \bigcap_{j=k}^{K-1} E_j \right] &= 1 - \mathbb{P}_k \left[ \left( \bigcap_{j=k}^{K-1} E_j \right)^c \right] \\ &= 1 - \mathbb{P}_k \left[ \bigcup_{j=k}^{K-1} E_j^c \right] \\ &\geq 1 - \sum_{j=k}^{K-1} \mathbb{P}_k \left[ E_j^c \right] \end{aligned}$$

and thus

$$\begin{aligned} &\mathbb{P}_k \left[ \bigcap_{j=k}^{K-1} \left[ \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right) - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2} \leq \gamma_j \right] \right] \\ &\geq 1 - \sum_{j=k}^{K-1} \mathbb{P}_k \left[ \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right) - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2} > \gamma_j \right] \quad (5.46) \\ &\geq 1 - (K - k)\theta \end{aligned}$$

where we used (5.45) in the last step. Since all  $\gamma_j$  are positive we can conclude that

$$\begin{aligned} &\bigcap_{j=k}^{K-1} \left[ \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right) - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2} \leq \gamma_j \right] \\ &\subseteq \left[ \sum_{j=k}^{K-1} \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right) - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2} \leq \sum_{j=k}^{K-1} \gamma_j \right] \\ &= \left[ 2 \sum_{j=k}^{K-1} \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right) - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2} \leq 2 \sum_{j=k}^{K-1} \gamma_j \right]. \end{aligned}$$

and hence we can use the lower bound from (5.46) to get

$$\begin{aligned} &\mathbb{P}_k \left[ 2 \sum_{j=k}^{K-1} \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right) - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2} \leq 2 \sum_{j=k}^{K-1} \gamma_j \right] \\ &\geq 1 - (K - k)\theta. \end{aligned}$$

Now use Proposition 4.3 to substitute the left hand side of the above event and write out in full the  $\gamma_j$ . Doing so, we include the exact continuation

value and obtain

$$\begin{aligned}
& \mathbb{P}_k \left[ \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k+1}^A} | X_k] \right) - \mathbb{E}[Z_{\tau_{k+1}} | X_k] \right\|_{L_k^2} \right. \\
& \quad \left. \leq 8(K-k)\epsilon + 2\sqrt{2} \sum_{j=k}^{K-1} \inf_{f \in \mathcal{H}_j} \left\| f - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2} \right] \\
& \geq 1 - (K-k)\theta
\end{aligned}$$

□

**Corollary 5.2.** With the same notation as in Theorem 5.1, we have

$$\begin{aligned}
& \mathbb{P} \left[ \left| \mathcal{A} \left( \mathbb{E}[Z_{\tau_0^A}] \right) - \mathbb{E}[Z_{\tau_0}] \right| \right. \\
& \quad \left. \leq 8K\epsilon + 2\sqrt{2} \sum_{j=1}^{K-1} \inf_{f \in \mathcal{H}_j} \left\| f - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2} \right] \\
& \geq 1 - 8eK(t+1) \left( \frac{128eL(L+R)}{\epsilon^2} \right)^t \exp \left( -\frac{N\epsilon^4}{512(L+R)^4} \right)
\end{aligned}$$

*Proof.* Set  $k = 0$  in (5.41), recall that we have

$$\mathbb{E}[Z_{\tau_0}] = \max(Z_0, \mathbb{E}[Z_{\tau_1}]) \text{ and } \mathcal{A} \left( \mathbb{E}[Z_{\tau_0^A}] \right) = \max(Z_0, \mathcal{A} \left( \mathbb{E}[Z_{\tau_1^A}] \right)).$$

Apply the inequality  $|\max(a, b) - \max(a, c)| \leq |b - c|$  and

$$\inf_{f \in \mathcal{H}_0} \left\| f - \mathbb{E}[Z_{\tau_1^A}] \right\|_{L_0^2} = 0.$$

□

The next corollary is a version of Theorem 5.1, if the approximation space is chosen to be a subspace spanned by elements of a orthonormal basis of the Hilbert space  $L^2(\mathcal{S}, \rho_k)$ .

**Corollary 5.3.** For all  $k = 1, \dots, K-1$  we denote by  $\{\phi_{ki}\}_{i=1}^\infty$  a orthonormal basis for  $L^2(\mathcal{S}, \rho_k)$ . Assume that it fulfils  $\|\phi_{ki}\|_\infty \leq L_{ki} < \infty$  and that  $R = \max\{1, \|Z_1\|_\infty, \dots, \|Z_K\|_\infty\} < \infty$ . Let  $T < \infty$  be such that it dominates  $\sum_{j=1}^K \|Z_j\|_{L_j^2}$  and let  $N$ , the number of simulations, be any natural number. Then for all  $k = 1, \dots, K-1$ ,  $1 \geq \epsilon > 0$  and  $\epsilon_0$  any number such that  $0 \leq \epsilon_0 \leq 6\epsilon^2$ , we can find positive integers  $M_k$ , depending on  $\epsilon$ , and approximation spaces

$$\mathcal{H}_k = \left\{ \sum_{i=1}^{M_k} c_i \phi_{ki} : \forall i = 1, \dots, M_k : c_i \in \mathbb{R}, |c_i| \leq T \right\}$$



which allow us to bound the error for all  $k = 0, \dots, K - 1$  as

$$\begin{aligned} & \mathbb{P}_k \left[ \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k+1}^A} | X_k] \right) - \mathbb{E}[Z_{\tau_{k+1}} | X_k] \right\|_{L_k^2} \leq \epsilon \right] \\ & \geq 1 - 8e(K - k)^{2M+1} (M + 1) \left( \frac{12^2 \cdot 128\epsilon L(L + R)}{\epsilon^2} \right)^M \\ & \quad \cdot \exp \left( -\frac{N\epsilon^4}{12^4 \cdot 512(K - k)^4(L + R)^4} \right) \end{aligned} \quad (5.47)$$

We used the notation  $M = \max\{M_k + 1 : k = 1, \dots, K - 1\}$  and  $L \geq 1$  is any suitable number with  $L \geq \max\{M_k \cdot L_{ki} \cdot T : k = 1, \dots, K - 1; i = 1, \dots, M_k\}$ .

*Proof.* From the definition of  $L$  we get that it uniformly bounds all the approximation spaces  $\mathcal{H}_k$ , i.e. we have  $\max\{\sup\{\|f\|_\infty : f \in \mathcal{H}_k\} : k = 1, \dots, K - 1\} \leq L$ . Lemma 3.20 assures that  $\dim_{VC}(\mathcal{H}_k) \leq M$  for all  $k = 1, \dots, K - 1$ . As showed in Proposition 4.2,  $\mathbb{E}[Z_{\tau_{k+1}^A} | X_k]$  lies in  $L^2(\mathcal{S}, \rho_k)$  and this allows us, by Fourier decomposition [6], to find a  $M_k = M_k(\epsilon)$ , such that  $\forall \delta > 0$

$$\left\| \sum_{i=1}^{M_k} c_i \phi_{ki} - E[Z_{\tau_{k+1}^A} | X_k] \right\|_{L_k^2} \leq \sqrt{2}\delta,$$

where

$$c_{ki} = \int_{\mathcal{S}} E[Z_{\tau_{k+1}^A} | X_k] \phi_{ki} d\rho_k.$$

Note that we get by Hölder's inequality, the fact that  $\phi_{ki}$  belongs to a orthonormal system and by equation (4.29), which is also true for the approximated counterpart,

$$\begin{aligned} |c_{ki}| & \leq \int_{\mathcal{S}} \left| E[Z_{\tau_{k+1}^A} | X_k] \phi_{ki} \right| d\rho_k \\ & \leq \left\| E[Z_{\tau_{k+1}^A} | X_k] \right\|_{L_k^2} \|\phi_{ki}\|_{L_k^2} = \left\| E[Z_{\tau_{k+1}^A} | X_k] \right\|_{L_k^2} \cdot 1 \\ & \leq \sum_{j=1}^K \|Z_j\|_{L_j^2} \leq T \end{aligned}$$

This assures that the approximation  $\sum_{i=1}^{M_k} c_i \phi_{ki}$  belongs to  $\mathcal{H}_k$ , and therefore we get

$$\inf_{f \in \mathcal{H}_k} \left\| f - E[Z_{\tau_{k+1}^A} | X_k] \right\| \leq \sqrt{2}\delta.$$

Thus by Theorem 5.1, we get

$$\begin{aligned}
& \mathbb{P}_k \left[ \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k+1}^A} | X_k] \right) - \mathbb{E}[Z_{\tau_{k+1}} | X_k] \right\|_{L_k^2} \right. \\
& \quad \left. \leq 8(K-k)\delta + 2\sqrt{2} \sum_{j=k}^{K-1} \sqrt{2}\delta \right] \\
& \geq 1 - 8e(K-k)(M+1) \left( \frac{128eL(L+R)}{\delta^2} \right)^M \exp \left( -\frac{N\delta^4}{512(L+R)^4} \right).
\end{aligned}$$

Now define  $\epsilon = (K-k)12\delta$  to get (5.47).  $\square$

**Corollary 5.4.** With the same notation as in corollary 5.3, we have

$$\begin{aligned}
& \mathbb{P} \left[ \left| \mathcal{A} \left( \mathbb{E}[Z_{\tau_0^A}] \right) - \mathbb{E}[Z_{\tau_0}] \right| \leq \epsilon \right] \\
& \geq 1 - 8eK^{2M+1}(M+1) \left( \frac{12^2 \cdot 128eL(L+R)}{\epsilon^2} \right)^M \\
& \quad \cdot \exp \left( -\frac{N\epsilon^4}{12^4 \cdot 512K^4(L+R)^4} \right)
\end{aligned} \tag{5.48}$$

*Proof.* The proof is the same as for corollary 5.2.  $\square$

The next theorem is again an error estimate. This time it involves the true, exact continuation value rather than its approximately stopped counterpart, as did theorem 5.1.

**Theorem 5.5.** Let, for all  $k = 1, \dots, K-1$ , the approximation space  $\mathcal{H}_k$  be of finite VC-dimension less or equal to  $t$  and an arbitrary subset of  $L^2(\mathcal{S}, \rho_k)$ . Assume that it is uniformly bounded by  $1 \leq L < \infty$  and that  $R = \max\{1, \|Z_1\|_\infty, \dots, \|Z_K\|_\infty\} < \infty$ . Then for any number  $N > 0$  of simulations, any  $0 < \epsilon \leq 1$  and  $0 \leq \epsilon_0 \leq 6\epsilon^2$ , we have, invoking the notation  $\mathcal{H}_0 = \{\mathbb{E}[Z_{\tau_1^A}]\}$ , for all  $k = 0, \dots, K-1$ ,

$$\begin{aligned}
& \mathbb{P}_k \left[ \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k+1}^A} | X_k] \right) - \mathbb{E}[Z_{\tau_{k+1}} | X_k] \right\|_{L_k^2} \right. \\
& \quad \left. \leq 5^{(K-k)} \left( 2\epsilon + \max_{j=k, \dots, K-1} \left( \inf_{f \in \mathcal{H}_j} \|f - \mathbb{E}[Z_{\tau_{j+1}} | X_j]\|_{L_j^2} \right) \right) \right] \\
& \geq 1 - 8e(K-k)(t+1) \left( \frac{128eL(L+R)}{\epsilon^2} \right)^t \exp \left( -\frac{N\epsilon^4}{512(L+R)^4} \right)
\end{aligned} \tag{5.49}$$

And for the algorithm's overall error

$$\begin{aligned}
& \mathbb{P} \left[ \left| \mathcal{A} \left( \mathbb{E}[Z_{\tau_0^A}] \right) - \mathbb{E}[Z_{\tau_0}] \right| \right] \\
& \leq 5^K \left( 2\epsilon + \max_{j=1, \dots, K-1} \left( \inf_{f \in \mathcal{H}_j} \|f - \mathbb{E}[Z_{\tau_{j+1}} | X_j]\|_{L_j^2} \right) \right) \\
& \geq 1 - 8eK(t+1) \left( \frac{128eL(L+R)}{\epsilon^2} \right)^t \exp \left( -\frac{N\epsilon^4}{512(L+R)^4} \right)
\end{aligned} \tag{5.50}$$

*Proof.* Fix a  $k = 0, \dots, K-1$ , and denote by  $[A_k]$  the following event, which occurred already in the proof of theorem 5.1:

$$\begin{aligned}
[A_k] & := \bigcap_{j=k}^{K-1} \left[ \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right) - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2} \right. \\
& \leq 4\epsilon + \sqrt{2} \inf_{f \in \mathcal{H}_j} \left. \left\| f - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2} \right]
\end{aligned} \tag{5.51}$$

From there, see equation (5.46), we also know, that

$$\mathbb{P}_k[A_k] \geq 1 - 8e(K-k)(t+1) \left( \frac{128eL(L+R)}{\epsilon^2} \right)^t \exp \left( -\frac{N\epsilon^4}{512(L+R)^4} \right) \tag{5.52}$$

For all  $j = k, \dots, K-1$  we will use the notation

$$[C_j] := \left[ \underbrace{8(K-j)\epsilon + 2\sqrt{2} \sum_{i=j}^{K-1} \inf_{f \in \mathcal{H}_i} \left\| f - \mathbb{E}[Z_{\tau_{i+1}^A} | X_i] \right\|_{L_i^2}}_{c_j :=} \leq 5^{K-j}(2\epsilon + M(j)) \right]$$

where

$$M(j) = \max_{i=j, \dots, K-1} \left( \inf_{f \in \mathcal{H}_i} \left\| f - \mathbb{E}[Z_{\tau_{i+1}^A} | X_i] \right\|_{L_i^2} \right).$$

The next step is to show, by backwards induction, that whenever the event  $[A_k]$  occurs, so does  $[C_k]$ , i.e.  $[A_k] \subseteq [C_k]$ . First note, that, using the triangle inequality, we have

$$\begin{aligned}
& 8\epsilon + 2\sqrt{2} \inf_{f \in \mathcal{H}_{k_0}} \left\| f - \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right\|_{L_{k_0}^2} \\
& \leq 8\epsilon + 2\sqrt{2} \left( \inf_{f \in \mathcal{H}_{k_0}} \left\| f - \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}] \right\|_{L_{k_0}^2} + \right. \\
& \quad \left. \left\| \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}] - \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right\|_{L_{k_0}^2} \right).
\end{aligned} \tag{5.53}$$

It is easily seen that  $C_{K-1}$  is always true, since  $\tau_K^A = \tau_K$ . Now assume that we already showed the statement for  $j = k_0 + 1$  and write  $c_{k_0}$  as

$$\begin{aligned}
c_{k_0} &= 8\epsilon + 2\sqrt{2} \inf_{f \in \mathcal{H}_{k_0}} \left\| f - \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right\|_{L_{k_0}^2} \\
&\quad + 8(K - k_0 - 1)\epsilon + 2\sqrt{2} \sum_{i=k_0+1}^{K-1} \inf_{f \in \mathcal{H}_i} \left\| f - \mathbb{E}[Z_{\tau_{i+1}^A} | X_i] \right\|_{L_i^2} \\
&\leq 8\epsilon + 2\sqrt{2} \inf_{f \in \mathcal{H}_{k_0}} \left\| f - \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}] \right\|_{L_{k_0}^2} \\
&\quad + 2\sqrt{2} \left\| \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}] - \mathbb{E}[Z_{\tau_{k_0+1}^A} | X_{k_0}] \right\|_{L_{k_0}^2} \\
&\quad + 8(K - k_0 - 1)\epsilon + 2\sqrt{2} \sum_{i=k_0+1}^{K-1} \inf_{f \in \mathcal{H}_i} \left\| f - \mathbb{E}[Z_{\tau_{i+1}^A} | X_i] \right\|_{L_i^2}
\end{aligned}$$

where we used (5.53). Applying (4.31) to the second term leaves us with

$$\begin{aligned}
c_{k_0} &\leq 8\epsilon + 2\sqrt{2} \inf_{f \in \mathcal{H}_{k_0}} \left\| f - \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}] \right\|_{L_{k_0}^2} \\
&\quad + 4\sqrt{2} \sum_{i=k_0+1}^{K-1} \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{i+1}^A} | X_i] \right) - \mathbb{E}[Z_{\tau_{i+1}} | X_i] \right\|_{L_i^2} \\
&\quad + 8(K - k_0 - 1)\epsilon + 2\sqrt{2} \sum_{i=k_0+1}^{K-1} \inf_{f \in \mathcal{H}_i} \left\| f - \mathbb{E}[Z_{\tau_i^A} | X_i] \right\|_{L_i^2}
\end{aligned}$$

Now conditioned on the event  $[A_{k_0+1}]$  from (5.51), we can again bound the second term in the above with

$$\begin{aligned}
c_{k_0} &\leq 8\epsilon + 2\sqrt{2} \inf_{f \in \mathcal{H}_{k_0}} \left\| f - \mathbb{E}[Z_{\tau_{k_0+1}} | X_{k_0}] \right\|_{L_{k_0}^2} \\
&\quad + 2\sqrt{2} \sum_{i=k_0+1}^{K-1} \left( 8\epsilon + 2\sqrt{2} \inf_{f \in \mathcal{H}_i} \left\| f - \mathbb{E}[Z_{\tau_i^A} | X_i] \right\|_{L_i^2} \right) \\
&\quad + 8(K - k_0 - 1)\epsilon + 2\sqrt{2} \sum_{i=k_0+1}^{K-1} \inf_{f \in \mathcal{H}_i} \left\| f - \mathbb{E}[Z_{\tau_i^A} | X_i] \right\|_{L_i^2}
\end{aligned}$$

Using the induction hypothesis, we get

$$\begin{aligned}
c_{k_0} &\leq 8\epsilon + 2\sqrt{2}M(k_0) \\
&\quad + 2\sqrt{2}5^{K-(k_0+1)}(2\epsilon + M(k_0 + 1)) \\
&\quad + 5^{K-(k_0+1)}(2\epsilon + M(k_0 + 1)) \\
&\leq 2\epsilon(4 + 2\sqrt{2}5^{K-(k_0+1)} + 5^{K-(k_0+1)}) \\
&\quad + M(k_0)(2\sqrt{2} + 2\sqrt{2}5^{K-(k_0+1)} + 5^{K-(k_0+1)}) \\
&\leq 2\epsilon(5^{K-(k_0+1)} + 2\sqrt{2}5^{K-(k_0+1)} + 5^{K-(k_0+1)}) \\
&\quad + M(k_0)(5^{K-(k_0+1)} + 2\sqrt{2}5^{K-(k_0+1)} + 5^{K-(k_0+1)}) \\
&= 2\epsilon 5^{K-(k_0+1)}(1 + 2\sqrt{2} + 1) + M(k_0)5^{K-(k_0+1)}(1 + 2\sqrt{2} + 1) \\
&\leq 5^{K-k_0}(2\epsilon + M(k_0))
\end{aligned}$$

This showed  $[A_k] \subseteq [C_k]$ . By the proof of theorem 5.1, we already know that the event

$$\begin{aligned}
[D_k] &:= \left[ \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k+1}^A} | X_k] \right) - \mathbb{E}[Z_{\tau_{k+1}} | X_k] \right\|_{L_k^2} \right. \\
&\quad \left. \leq 8(K-k)\epsilon + 2\sqrt{2} \sum_{j=k}^{K-1} \inf_{f \in \mathcal{H}_j} \left\| f - \mathbb{E}[Z_{\tau_{j+1}^A} | X_j] \right\|_{L_j^2} \right]
\end{aligned}$$

is a superset of  $[A_k]$ . We want to bound the probability of the event in (5.49), which can now be written as  $[C_k] \cap [D_k]$ . Since  $[A_k] \subseteq [C_k] \cap [D_k]$  and because of (5.52) the proof of (5.49) is finished. To obtain the expression (5.50) involving the algorithm's overall error, set  $k = 0$  in (5.49) and proceed as in the proof of corollary 4.4  $\square$

## 6 Polynomial Approximations

In this section we are going to approximate the exact continuation value by polynomials. This will enable us, whenever certain regularity conditions are met, to bound the error with arbitrarily high probability. We require the state space  $\mathcal{S}$  to be bounded, only then we are able to give bounds on the norm of polynomial approximation spaces. Since a model of a underlying might well be unbounded, it can be necessary to truncate the process, see [17] for details.

**Definition 6.1.** Let  $\mathcal{S} \subseteq \mathbb{R}^d$  and denote by  $\mathcal{R}_m^d(\mathcal{S})$  for  $m \geq 1$  the space of all real polynomials on  $\mathcal{S}$  with degree less or equal to  $m - 1$ :

$$\mathcal{R}_m^d(\mathcal{S}) = \text{span} \left\{ x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \sum_{i=1}^d \alpha_i \leq m - 1, \alpha_i \geq 0 \right\}$$

We will need to bound the VC-dimension of this polynomial space in order to use theorem 5.5. It is obviously a linear space, thus once we know the ordinary vector dimension we can apply Lemma 3.20. To do so, we present, without proofs, two basic results from combinatorics. The first one is sometimes called the hockey stick identity [22].

**Lemma 6.2.** For all integers  $n$  and  $r$ , we have

$$\sum_{l=0}^r \binom{n+l}{l} = \binom{n+r+1}{r}$$

The second one gives us the number of weak compositions of an integer  $n$  into  $d$  parts. That is, in how many ways can one write  $\alpha_1 + \cdots + \alpha_d = n$ , if  $\alpha_i \geq 0$  and the order matters.

**Lemma 6.3.** There are  $\binom{n+d-1}{d-1}$  weak compositions of  $n$  into  $d$  terms [4].

The dimension of  $\mathcal{R}_m^d(\mathcal{S})$  is now in easy consequence of the two preceding lemmas.

**Lemma 6.4.** With the notation of definition 6.1, we get

$$\dim(\mathcal{R}_m^d(\mathcal{S})) = \binom{m+d-1}{d}$$

*Proof.* By identifying a base polynomial  $x_1^{\alpha_1} \cdots x_d^{\alpha_d}$  of degree at most  $m - 1$  with the sequence  $(\alpha_1, \dots, \alpha_d)$ , we know that there must be  $\binom{n+d-1}{d-1}$  base polynomials. Since  $\mathcal{R}_m^d(\mathcal{S})$  consists of all polynomials up to a degree of  $m - 1$ , it must be true that

$$\begin{aligned} \dim(\mathcal{R}_m^d(\mathcal{S})) &= \sum_{n=0}^{m-1} \binom{n+d-1}{d-1} = \sum_{n=0}^{m-1} \binom{n+d-1}{n} \\ &= \binom{d+m-1}{m-1} = \binom{d+m-1}{d} \end{aligned}$$

□

For our next corollary we need the definition of a Sobolev space.

**Definition 6.5.** On  $L^\infty(\mathcal{S}, \rho_k)$  the Sobolev space  $W_k^n$  of order  $n$  is defined to be

$$W_k^n = \{f \in L_k^\infty : D^\alpha f \in L_k^\infty : \forall |\alpha| \leq n\}$$

where  $\alpha = (\alpha_1, \dots, \alpha_d)$  is a multi index,  $|\alpha| := \sum_{i=1}^d \alpha_i$  and  $D^\alpha = \frac{\partial^{|\alpha|}}{\partial \alpha_1 \dots \partial \alpha_d}$  is the partial differential operator. We use the following (semi-)norm on  $W_k^n$ :

$$\|f\|_{W_k^n} = \sum_{|\alpha| \leq n} \|D^\alpha f\|_{L_k^\infty}$$

This norm is often used as a measure of smoothness. In what is to come we will assume that the exact continuation values are sufficiently smooth to lie in a Sobolev space. Furthermore we will use the following lemma from [10] about polynomial approximation in those spaces.

**Lemma 6.6.** Let  $\mathcal{S}$  be the interior of a cube around the origin in  $\mathbb{R}^d$  and  $f \in W_k^n$ . Then we can find a  $m > n$  and  $C_S \geq 1$ , such that there exists a polynomial  $p \in \mathcal{R}_d^m(\mathcal{S})$

$$\|p - f\|_{L_k^\infty} \leq C_S m^{-n} \|f\|_{W_k^n}$$

We can use this approximation error bound within the context of theorem 5.5. This will give us the following corollary.

**Corollary 6.7.** Let  $\mathcal{S}$  be the interior of a cube around the origin in  $\mathbb{R}^d$ . With the notation of theorem 5.5 assume that  $R < \infty$ ,  $N > 0$  and that for all  $k = 1, \dots, K-1$  the exact continuation value  $\mathbb{E}[Z_{\tau_{k+1}} | X_k]$  lies in  $W_k^n$ . Let  $m$  be large enough, such that  $C_S/m^n \leq 1$ ,  $C_S$  as in lemma 6.6. Define the approximation spaces as

$$\mathcal{H}_k = \left\{ p \in \mathcal{R}_d^m(\mathcal{S}) : \|p\|_{L_k^\infty} \leq 2 \|\mathbb{E}[Z_{\tau_{k+1}} | X_k]\|_{W_k^n} \right\}$$

Then for any  $0 < \epsilon \leq 1$  we have, invoking the notation  $\mathcal{H}_0 = \{\mathbb{E}[Z_{\tau_1^A}]\}$ , for all  $k = 0, \dots, K-1$ ,

$$\begin{aligned} \mathbb{P}_k \left[ \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k+1}^A} | X_k] \right) - \mathbb{E}[Z_{\tau_{k+1}} | X_k] \right\|_{L_k^2} \leq 5^{(K-k)} (2\epsilon + C_1 m^{-n}) \right] \\ \geq 1 - 8\epsilon(K-k)(t+1) \left( \frac{256\epsilon C_1 (2C_1 + R)}{\epsilon^2} \right)^t \exp \left( -\frac{N\epsilon^4}{512(2C_1 + R)^4} \right) \end{aligned} \quad (6.54)$$

where  $t = \binom{m+d-1}{d} + 1$  and  $C_1 \geq 1$  is any constant such that  $C_1 \geq C_S \|\mathbb{E}[Z_{\tau_{k+1}} | X_k]\|_{W_k^n}$ .

*Proof.* We basically apply Theorem 5.5, thus we have to check that all the assumptions are met. We know that  $\mathbb{R}_d^m(\mathcal{S})$  is a linear space and that  $\mathcal{H}_k \subseteq \mathbb{R}_d^m(\mathcal{S})$ , thus we get by lemma 3.20 and 6.4, for all  $k = 1, \dots, K-1$

$$\dim_{VC}(\mathcal{H}_k) \leq \dim_{VC}(\mathbb{R}_d^m(\mathcal{S})) \leq \dim(\mathbb{R}_d^m(\mathcal{S})) + 1 = t \quad (6.55)$$

Since  $C_S \geq 1$ , the constant  $2C_1$  uniformly bounds all the approximations spaces  $\mathcal{H}_k$ . By lemma 6.6, we can find, for all  $j = k, \dots, K-1$ , a  $p_j \in \mathcal{R}_d^m(\mathcal{S})$ , such that

$$\begin{aligned} \inf_{p \in \mathcal{H}_j} \|p - \mathbb{E}[Z_{\tau_{j+1}}|X_j]\|_{L_j^2} &\leq \|p_j - \mathbb{E}[Z_{\tau_{j+1}}|X_j]\|_{L_j^2} \\ &\leq \|p_j - \mathbb{E}[Z_{\tau_{j+1}}|X_j]\|_{L_j^\infty} \leq C_1 m^{-n} \end{aligned}$$

Note that the polynomial  $p_j$  actually lies in  $\mathcal{H}_j$ , since

$$\begin{aligned} \|p_j\|_{L_j^\infty} &\leq \|p_j - \mathbb{E}[Z_{\tau_{j+1}}|X_j]\|_{L_j^\infty} + \|\mathbb{E}[Z_{\tau_{j+1}}|X_j]\|_{L_j^\infty} \\ &\leq 2 \|\mathbb{E}[Z_{\tau_{j+1}}|X_j]\|_{W_j^n} \end{aligned}$$

□

For our next corollary we will assume that the exact continuation values are Lipschitz continuous. Denote by  $Q_d(\sigma)$  the closed cube in  $\mathbb{R}^d$ , centred around the origin with side lengths of  $2\sigma$ . Write  $\text{Lip}(Q_d(\sigma))$  for all functions  $f : Q_d(\sigma) \mapsto \mathbb{R}$ , such that there exists a Lipschitz constant  $C_L$ ,  $0 \leq C_L < \infty$ , such that  $\forall f \in \text{Lip}(Q_d(\sigma))$ ,

$$|f(x_1) - f(x_2)| \leq C_L |x_1 - x_2|$$

is satisfied for all  $x_1, x_2 \in Q_d(\sigma)$ . We are also going to use the following lemma from [12] about the approximation of Lipschitz functions with polynomials.

**Lemma 6.8.** For all  $f \in \text{Lip}(Q_d(\sigma))$  we have, over the cube  $Q_d(\sigma)$ ,

$$\inf_{p \in \mathcal{R}_d^m} (\|f - p\|_\infty) \leq \frac{88C_L\sigma d}{d + m + 1}$$

We are now in a position to state our last corollary, which assumes the exact continuation value to be Lipschitz continuous, instead of  $n$ -times differentiable.

**Corollary 6.9.** Set  $\mathcal{S} = Q_d(\sigma)$ . With the notation of theorem 5.5 assume that  $R < \infty$ ,  $N > 0$  is an integer and that  $\forall k = 1, \dots, K-1$  the exact continuation value  $\mathbb{E}[Z_{\tau_{k+1}}|X_k]$  is Lipschitz continuous with constant  $C_L^k$ .



Furthermore define  $C_L = \max_{k=1, \dots, K-1} C_L^k$  and choose the approximation spaces

$$\mathcal{H}_k = \left\{ p \in R_d^m(Q_d(\sigma)); \|p\|_{L_k^\infty} \leq 2 \|\mathbb{E}[Z_{\tau_{k+1}} | X_k]\|_{L_k^\infty} \right\}$$

where  $m$  is any integer large enough, such that

$$\frac{88C_L\sigma d}{d+m+1} \leq \|\mathbb{E}[Z_{\tau_{k+1}} | X_k]\|_{L_k^\infty}$$

Then we have for all  $0 < \epsilon \leq 1$

$$\begin{aligned} \mathbb{P}_k \left[ \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k+1}^A} | X_k] \right) - \mathbb{E}[Z_{\tau_{k+1}} | X_k] \right\|_{L_k^2} \leq 5^{(K-k)} \left( 2\epsilon + \frac{88C_L\sigma d}{d+m+1} \right) \right] \\ \geq 1 - 8\epsilon(K-k)(t+1) \left( \frac{256\epsilon C_1(2C_1+R)}{\epsilon^2} \right)^t \exp \left( -\frac{N\epsilon^4}{512(2C_1+R)^4} \right) \end{aligned} \quad (6.56)$$

where  $t = \binom{m+d-1}{d} + 1$ ,  $C_1 \geq 1$  is any constant such that for all  $k$ :  $C_1 \geq \|\mathbb{E}[Z_{\tau_{k+1}} | X_k]\|_{L_k^\infty}$ .

*Proof.* We have to show that all the assumptions from theorem 5.5 are met. We justify our choice of  $t$  with the same argumentation as in (6.55). We also note, that  $2C_1$  uniformly bounds all the approximations spaces. By lemma 6.8 we have  $\forall j = k, \dots, K-1$

$$\begin{aligned} \inf_{p \in \mathcal{H}_j} \|p - \mathbb{E}[Z_{\tau_{j+1}} | X_j]\|_{L_j^2} &\leq \inf_{p \in \mathcal{H}_j} \|p - \mathbb{E}[Z_{\tau_{j+1}} | X_j]\|_{L_j^\infty} \\ &= \inf_{p \in \mathcal{R}_d^m(Q_d(\sigma))} \|p - \mathbb{E}[Z_{\tau_{j+1}} | X_j]\|_{L_j^\infty} \leq \frac{88C_L\sigma d}{d+m+1} \end{aligned}$$

Note that the polynomial  $p_j := \arg \inf_{p \in \mathcal{R}_d^m(Q_d(\sigma))} \|p - \mathbb{E}[Z_{\tau_{j+1}} | X_j]\|_{L_j^\infty}$  actually lies in  $\mathcal{H}_j$ , since

$$\begin{aligned} \|p_j\|_{L_j^\infty} &\leq \|p_j - \mathbb{E}[Z_{\tau_{j+1}} | X_j]\|_{L_j^\infty} + \|\mathbb{E}[Z_{\tau_{j+1}} | X_j]\|_{L_j^\infty} \\ &\leq \frac{88C_L\sigma d}{d+m+1} + \|\mathbb{E}[Z_{\tau_{j+1}} | X_j]\|_{L_j^\infty} \leq 2 \|\mathbb{E}[Z_{\tau_{j+1}} | X_j]\|_{L_j^\infty} \end{aligned}$$

□

## 7 Approximation with Artificial Neural Networks

The aim of this sections is to use Artificial Neural Networks (ANN) as an approximation space in the Longstaff-Schwartz algorithm. After a small introduction, we will establish bounds on the approximation error and the VC-dimension of so called feedforward ANNs. This will give us all the necessary tools to apply Theorem 5.5.

### 7.1 Introduction to ANN

The human brain is very good in pattern recognition. With only little effort, we are able to make sense of a huge amount of information we constantly receive through our sensory system. Research on ANNs was always motivated by understanding how the brain works and using those insights to build powerful computational systems that perform well in analysing unstructured data [21]. An ANN is always a labelled and directed graph, where the nodes are called processing units or neurons. They receive an input and perform a simple calculation to generate an output. The links between the neurons are assigned with a weight, which dictates if the signal is either amplified or diminished. Those weights play an important role in how the network behaves and are, next to the structure of the net itself, the key factor in modelling ANNs.

The first mathematical model is due to McCulloch and Pitts. They used a binary thresholding function as a neuron, that is a function  $\sigma : \mathbb{R} \mapsto \{0, 1\}$  with  $\sigma(x) = \mathbb{I}_{[b, \infty)}(x)$  where  $b \in \mathbb{R}$  is the threshold. The neuron receives an input  $x \in \mathbb{R}^d$ , which is adjusted by a weight vector  $\omega \in \mathbb{R}^d$ . If the input  $\omega^T x$  is grater or equal to the threshold  $b$ , the neuron is activated and outputs a 1. This neuron learns by adjusting the threshold and the weights. Note that we can add a 1 to the beginning of the input vector, and then treat the threshold as a weight of the link from the 1 to the neuron, see figure 3.

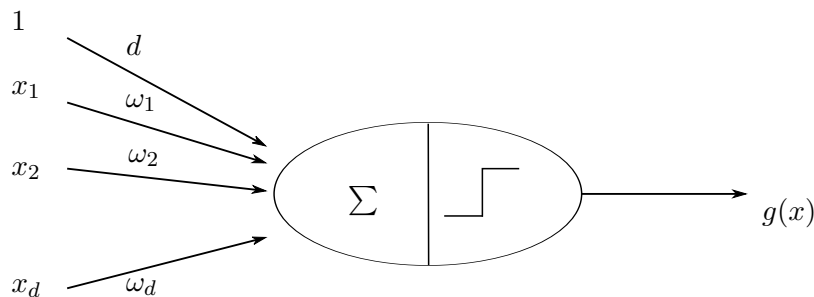


Figure 3: The McCulloch-Pitts neuron

We can replace the threshold function by other functions to modify the

ANN. In general an artificial neuron is a function on  $\mathbb{R}^d$  with real valued output, given by

$$g(x) = \sigma(\omega^T x + b)$$

If  $\sigma$  maps to the interval  $[0, 1]$ , we call it a sigmoid function [14].

**Definition 7.1.** A sigmoid function  $\sigma$  that is non decreasing with

$$\lim_{x \rightarrow -\infty} \sigma(x) = 0 \text{ and } \lim_{x \rightarrow \infty} \sigma(x) = 1$$

is called a squashing function.

We already introduced the threshold squasher, which is used in the McCulloch-Pitts neuron. Later in this chapter we will use the logistic squasher

$$\sigma(x) = \frac{1}{1 + \exp(-x)} \quad (7.57)$$

to establish error estimates when using ANNs for approximation. We require our networks to have a special structure: The information must always pass to the next layer of neurons, that is the underlying graph must be acyclic. We call those systems feedforward networks. It is thus possible to label the units in such a way, that whenever there is a connection from unit  $i$  to unit  $j$ , we have  $i < j$  [2]. Thus a feedforward network with one hidden layer consists of one layer of inputs, one layer of  $k$  neurons and one output unit, see figure 4. If we choose as the output unit a linear neuron, our neural network is a real valued function on  $\mathbb{R}^d$  of the form

$$f(x) = \sum_{i=1}^k c_i \sigma(\omega_i^T x + b_i) + c_0$$

where  $b_1, \dots, b_k, c_0, c_1, \dots, c_k \in \mathbb{R}$  and  $\omega_1, \dots, \omega_k \in \mathbb{R}^d$  are the parameters of the network [14].

## 7.2 Neural Network Error Estimates

Before we can derive the error estimate we will need the following proposition about the rate of approximation of convex combinations in  $L^2(\mathcal{S}, \mu)$ , see [14] and [17]. Let  $S \subseteq \mathcal{S}$  and introduce the notation  $\|f\|_S^2 := \int_S f^2(x) d\mu(x)$ .

**Proposition 7.2.** Denote by  $S_r \subseteq \mathcal{S} \subseteq \mathbb{R}^d$  the ball with radius  $r$  centred around the origin. Define

$$\Phi = \{\phi_\omega : \forall \omega \in W : \phi_\omega \in L^2(S_r, \mu), \|\phi_\omega\|_{S_r}^2 \leq B^2\}$$

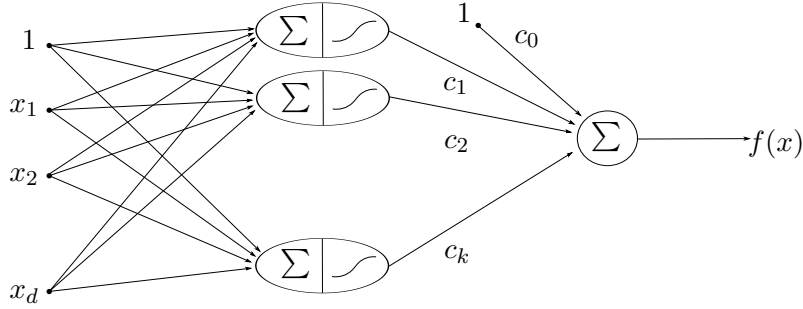


Figure 4: A feedforward neural network with one hidden layer

where  $W \subset \mathbb{R}^m$  is a set of parameters of the real valued function  $\phi$ . Furthermore let  $h : W \mapsto \mathbb{R}$  be any measurable mapping with  $\int_W |h(\omega)| d\lambda(\omega) \in (0, \infty)$ , such that for any  $c \in \mathbb{R}$  and for all  $x \in S_r$

$$f(x) = \int_W \phi_\omega(x) h(\omega) d\lambda(\omega) + c$$

is a valid representation. Then we can find for all  $k \in \mathbb{N}$ , a function  $f_k$ ,

$$f_k(x) = \sum_{j=1}^k c_j \phi_{\omega_j}(x) + c$$

such that

$$\|f - f_k\|_{S_r}^2 < \frac{B^2}{k} \left( \int_W |h(\omega)| d\lambda(\omega) \right)^2$$

where the coefficients  $c_j$  can be chosen such that

$$\sum_{j=1}^k |c_j| \leq \int_W |h(\omega)| d\lambda(\omega). \quad (7.58)$$

*Proof.* For  $x \in S_r$ , we have

$$\begin{aligned} f(x) &= \int_W \phi_\omega(x) h(\omega) d\lambda(\omega) + c \\ &= \int_W \operatorname{sgn}(h(\omega)) \phi_\omega(x) |h(\omega)| d\lambda(\omega) + c \end{aligned} \quad (7.59)$$

Introduce the new probability measure  $Q$  on  $W$  by

$$dQ(\omega) = \frac{|h(\omega)|}{\int_W |h(\omega)| d\lambda(\omega)} d\lambda(\omega)$$

and write  $D = \int_W |h(\omega)| d\lambda(\omega)$  for the total variation of the signed measure  $h(\omega)d\lambda(\omega)$ . With the new measure (7.59) becomes

$$\begin{aligned} f(x) &= \int_W \operatorname{sgn}(h(\omega)) \phi_\omega(x) D dQ(\omega) + c \\ &= D \mathbb{E}_Q[\operatorname{sgn}(\Omega) \phi_\Omega(x)] + c \end{aligned} \quad (7.60)$$

where  $\Omega$  is a random variable, taking values in  $W$  with distribution  $Q$ . Let  $\Omega_1, \dots, \Omega_k$  be random variables, identically distributed according to  $Q$  and independent of  $\Omega$  and each other. Write

$$f_k(x) = \frac{D}{k} \sum_{j=1}^k \operatorname{sgn}(h(\Omega_j)) \phi_{\Omega_j}(x) + c$$

Now we use the Fubini's theorem and (7.60) to calculate

$$\begin{aligned} &\mathbb{E}_Q[\|f - f_k\|_{S_r}^2] \\ &= \mathbb{E}_Q \left[ \int_{S_r} (f(x) - f_k(x))^2 d\mu(x) \right] \\ &= \int_{S_r} \mathbb{E}_Q [(f(x) - f_k(x))^2] d\mu(x) \\ &= \int_{S_r} \mathbb{E}_Q [(D \mathbb{E}_Q[\operatorname{sgn}(\Omega) \phi_\Omega(x)] + c \\ &\quad - \left( \frac{D}{k} \sum_{j=1}^k \operatorname{sgn}(h(\Omega_j)) \phi_{\Omega_j}(x) + c \right))^2] d\mu(x) \end{aligned} \quad (7.61)$$

Define the random variable  $X$  by

$$X = \frac{1}{k} \sum_{j=1}^k \operatorname{sgn}(h(\Omega_j)) \phi_{\Omega_j}(x) + c$$

then  $\mathbb{E}_Q[X] = \mathbb{E}_Q[\operatorname{sgn}(\Omega) \phi_\Omega(x)] + c$  and (7.61) becomes

$$\begin{aligned} &\mathbb{E}_Q[\|f - f_k\|_{S_r}^2] \\ &= D^2 \int_{S_r} (\mathbb{E}_Q[X] - X)^2 d\mu(x) \\ &= D^2 \int_{S_r} \mathbb{V}_Q[X] d\mu(x) \\ &= \frac{D^2}{k^2} \int_{S_r} k \mathbb{V}_Q[\operatorname{sgn}(\Omega) \phi_\Omega(x)] d\mu(x) \end{aligned} \quad (7.62)$$

From (7.60) we know that

$$\mathbb{E}_Q[\text{sgn}(\Omega)\phi_\Omega(x)] = \frac{1}{D}(f(x) - c).$$

Now choose  $c$  such that  $\mathbb{E}_Q[\text{sgn}(\Omega)\phi_\Omega(x)] > 0$ , on a set  $S \subseteq \mathcal{S}$  which satisfies  $\mu(S \cap \mathcal{S}_r) > 0$ . For every random variable  $Y$  it is true that  $\mathbb{E}_Q[Y^2] = \mathbb{V}_Q[Y] + \mathbb{E}[Y]^2$ , and thus we can establish on the set  $S$  the inequality

$$\mathbb{E}_Q[\phi_\Omega(x)^2] = \mathbb{E}_Q[(\text{sgn}(\Omega)\phi_\Omega(x))^2] > \mathbb{V}_Q[\text{sgn}(\Omega)\phi_\Omega(x)]$$

If we now substitute the variance in (7.62) we get

$$\begin{aligned} \mathbb{E}_Q[\|f - f_k\|_{\mathcal{S}_r}^2] &= \frac{D^2}{k} \int_{\mathcal{S}_r} \mathbb{V}_Q[\text{sgn}(\Omega)\phi_\Omega(x)] d\mu(x) \\ &< \frac{D^2}{k} \int_{\mathcal{S}_r} \mathbb{E}_Q[\phi_\Omega(x)^2] d\mu(x) \\ &= \frac{D^2}{k} \mathbb{E}_Q\left[\int_{\mathcal{S}_r} \phi_\Omega(x)^2 d\mu(x)\right] \\ &= \frac{D^2}{k} \mathbb{E}_Q[\|\phi_\Omega(x)\|_{\mathcal{S}_r}^2] \leq \frac{D^2}{k} B^2 \end{aligned} \tag{7.63}$$

Since the expectation of  $\|f - f_k\|_{\mathcal{S}_r}^2$  is less than  $\frac{D^2}{k} B^2$ , we can find  $\omega_1, \dots, \omega_k \in W$ , such that

$$\|f - f_k\|_{\mathcal{S}_r}^2 < \frac{B^2 D^2}{k} = \frac{B^2}{k} \left( \int_W |h(\omega)| d\lambda(\omega) \right)^2.$$

For the last statement of the theorem (7.58), note that the coefficients  $c_j$  of  $f_k$  are  $c_j = \frac{D}{k} \text{sgn}(h(\omega))$  and thus they sum up to at most  $D = \int_W |h(\omega)| d\lambda(\omega)$ .  $\square$

For our next Lemma we need to review some basic properties of the Fourier transform. For a function  $f \in L^1(\mathbb{R}^d, \lambda)$  it is defined as

$$\hat{f}(\omega) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \exp(-i\omega^T x) f(x) d\lambda(x)$$

If the Fourier transform is itself a  $L^1$  function, we can apply the inverse formula

$$f(x) = \int_{\mathbb{R}^d} \exp(i\omega^T x) \hat{f}(\omega) d\lambda(\omega) \tag{7.64}$$

Set  $x = 0$  in (7.64) and we get

$$f(0) = \int_{\mathbb{R}^d} \hat{f}(\omega) d\lambda(\omega)$$

and thus (7.64) is equivalent to

$$\begin{aligned}
f(x) &= \int_{\mathbb{R}^d} (\exp(i\omega^T x) + 1 - 1) \hat{f}(\omega) d\lambda(\omega) \\
&= \int_{\mathbb{R}^d} \hat{f}(\omega) d\lambda(\omega) + \int_{\mathbb{R}^d} (\exp(i\omega^T x) - 1) \hat{f}(\omega) d\lambda(\omega) \\
&= f(0) + \int_{\mathbb{R}^d} (\exp(i\omega^T x) - 1) \hat{f}(\omega) d\lambda(\omega)
\end{aligned} \tag{7.65}$$

Denote with  $\mathcal{F}_C$  the class of functions with a Fourier transform that lies in  $L^1$ , such that it additionally has a bounded first absolute moment. That is

$$\mathcal{F}_C = \left\{ f : \mathbb{R}^d \mapsto \mathbb{R} : \hat{f} \in L^1, \int_{\mathbb{R}^d} \|\omega\| |\hat{f}(\omega)| d\lambda(\omega) \right\}$$

where  $\|\cdot\|$  is the euclidean norm and  $|\hat{f}(\omega)| = \sqrt{\Re^2(\hat{f}(\omega)) + \Im^2(\hat{f}(\omega))}$ . We can now state our lemma about the rate of approximation for feedforward neural networks with squashing functions as neurons.

**Lemma 7.3.** Let  $\mu$  be any probability measure on  $\mathcal{S}$ ,  $f \in \mathcal{F}_C$  and  $k \geq 1$ . Furthermore let  $\sigma$  be a squashing function and  $S_r \subseteq \mathcal{S}$  the ball around the origin in  $\mathbb{R}^d$  with radius  $r$ . Then there exists a neural network  $f_k$  in

$$\mathcal{F}_k = \left\{ \sum_{i=1}^k c_i \sigma(\omega_i^T x + b_i) + c_0 : \omega_i \in \mathbb{R}^d, b_i, c_i \in \mathbb{R} : \sum_{i=0}^k |c_i| \leq 3rC + |f(0)| \right\}$$

such that

$$\|f - f_k\|_{S_r}^2 \leq \frac{(2rC)^2}{k}$$

*Proof.* For a class of functions  $\mathcal{F} \subset L^2(\mathcal{S}_r, \mu)$  denote the closure with respect to the  $L^2$ -norm of the convex hull of  $\mathcal{F}$  by  $\overline{\mathcal{F}}$ . Define the set

$$G_\sigma = \left\{ c\sigma(\omega^T x + b) : \omega \in \mathbb{R}^d, b, c \in \mathbb{R} : |c| \leq 2rC \right\}$$

We will now establish that for every  $f \in \mathcal{F}_C$ , the function  $f(x) - f(0)$  lies in  $\overline{G_\sigma}$ . Write  $\rho(\omega)$  for the phase angle of  $\hat{f}(\omega)$ , that is

$$\rho(\omega) = \arctan \frac{\Im(\hat{f}(\omega))}{\Re(\hat{f}(\omega))}$$

Thus we get

$$\begin{aligned}
\hat{f}(\omega) &= |\hat{f}(\omega)| (\cos \rho(\omega) + i \sin \rho(\omega)) \\
&= |\hat{f}(\omega)| \exp(i\rho(\omega))
\end{aligned} \tag{7.66}$$

We introduce  $W = \{\omega \in \mathbb{R}^d : \omega \neq 0\}$  and use the inverse transform (7.65) and (7.66) to calculate for the real valued  $f$  on  $S_r$

$$\begin{aligned}
f(x) - f(0) &= \Re(f(x) - f(0)) \\
&= \Re\left(\int_{\mathbb{R}^d} (\exp(i\omega^T x) - 1) \hat{f}(\omega) d\lambda(\omega)\right) \\
&= \Re\left(\int_W (\exp(i\omega^T x) - 1) |\hat{f}(\omega)| \exp(i\rho(\omega)) d\lambda(\omega)\right) \\
&= \Re\left(\int_W \left(\cos(\omega^T x + \rho(\omega)) + i \sin(\omega^T x + \rho(\omega))\right. \right. \\
&\quad \left. \left. - (\cos \rho(\omega) + i \sin \rho(\omega))\right) |\hat{f}(\omega)| d\lambda(\omega)\right) \\
&= \int_W (\cos(\omega^T x + \rho(\omega)) - \cos \rho(\omega)) |\hat{f}(\omega)| d\lambda(\omega)
\end{aligned} \tag{7.67}$$

Define the function  $g$  on  $\mathcal{S} \times \mathbb{R}^d$  as

$$g(x, \omega) = \frac{\cos(\omega^T x + \rho(\omega)) - \cos \rho(\omega)}{\|\omega\|}$$

Hence, (7.67) becomes

$$f(x) - f(0) = \int_W g(x, \omega) \|\omega\| |\hat{f}(\omega)| d\lambda(\omega) \tag{7.68}$$

□

Next we introduce a modified step function  $\mathbb{I}^*$ , which at the point  $t = 0$  equals the sigmoid function used in the neural network. More formally we have

$$\mathbb{I}^*(t) = \begin{cases} 0 & \text{if } t < 0 \\ \sigma(0) & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases}$$

This function is  $\lambda$ -a.s. equal to  $\mathbb{I}_{(0, \infty)}$ . Recall that  $x \in S_r$ , and thus Schwarz's inequality states, with the notation  $\Delta := \frac{\omega^T x}{\|\omega\|}$ , that  $|\Delta| \leq \|x\| \leq r$ . Now consider two cases (we are going to skip the trivial case  $\Delta = 0$ ).

CASE 1:  $\Delta > 0$  We use the rule  $\int \sin(ax + b) dx = -\frac{1}{a} \cos(ax + b) + \text{const}$  to compute

$$\begin{aligned}
&\int_{[0, \Delta]} \sin(\|\omega\|t + \rho(\omega)) d\lambda(t) \\
&= -\frac{1}{\|\omega\|} \cos(\|\omega\|\Delta + \rho(\omega)) + \frac{1}{\|\omega\|} \cos \rho(\omega) \\
&= -g(x, \omega)
\end{aligned}$$



In this case we get for the indicator function of the set  $(0, \Delta)$

$$\mathbb{I}_{(0,\Delta)}(t) = \mathbb{I}_{(0,\infty)}(\Delta - t)\mathbb{I}_{(0,\infty)}(t) \quad (7.69)$$

CASE 2:  $\Delta < 0$  We find that

$$\int_{[\Delta,0]} \sin(\|\omega\|t + \rho(\omega))d\lambda(t) = g(x, \omega)$$

and we can expand the indicator function of the interval  $(\Delta, 0)$  to

$$\begin{aligned} \mathbb{I}_{(\Delta,0)}(t) &= \mathbb{I}_{(-\infty,0)}(t)\mathbb{I}_{(0,\infty)}(t - \Delta) \\ &= \mathbb{I}_{(-\infty,0)}(t) (1 - \mathbb{I}_{(0,\infty)}(\Delta - t)) \end{aligned} \quad (7.70)$$

In either case we can write, using the convention that if  $a > b : (a, b) = \emptyset$ ,  $\mathbb{I}_{\emptyset} = 0$  and the expressions (7.69) and (7.70), the function  $g$  as

$$\begin{aligned} g(x, \omega) &= \int_{[-r,r]} (\mathbb{I}_{(\Delta,0)}(t) - \mathbb{I}_{(0,\Delta)}(t)) \sin(\|\omega\|t + \rho(\omega))d\lambda(t) \\ &= \int_{[-r,0]} (1 - \mathbb{I}_{(0,\infty)}(\Delta - t)) \sin(\|\omega\|t + \rho(\omega))d\lambda(t) \\ &\quad - \int_{[0,r]} \mathbb{I}_{(0,\infty)}(\Delta - t) \sin(\|\omega\|t + \rho(\omega))d\lambda(t) \\ &= \int_{[-r,0]} (1 - \mathbb{I}^*(\Delta - t)) \sin(\|\omega\|t + \rho(\omega))d\lambda(t) \\ &\quad - \int_{[0,r]} \mathbb{I}^*(\Delta - t) \sin(\|\omega\|t + \rho(\omega))d\lambda(t) \\ &= \int_{[-r,0]} \sin(\|\omega\|t + \rho(\omega))d\lambda(t) \\ &\quad - \int_{[-r,r]} \mathbb{I}^*(\Delta - t) \sin(\|\omega\|t + \rho(\omega))d\lambda(t) \end{aligned}$$

Now we substitute this expression for  $g$  into equation (7.68). To do so we will use the abbreviations

$$c = \int_W \int_{[-r,0]} \sin(\|\omega\|t + \rho(\omega))d\lambda(t)\|\omega\|\|\hat{f}(\omega)\|d\lambda(\omega)$$

and

$$\nu(t, \omega) = \sin(\|\omega\|t + \rho(\omega))\|\omega\|\|\hat{f}(\omega)\|$$

Thus (7.68) becomes

$$f(x) - f(0) = c - \int_{[-r,r] \times W} \mathbb{I}^*(\Delta - t)\nu(t, \omega)d\lambda(t, \omega) \quad (7.71)$$

We will now apply proposition 7.2. To do so we write

$$\phi_{(t,a)}(x) = \mathbb{I}^* (a^T x - \|a\|t) = \mathbb{I}^* \left( \frac{a^T x}{\|a\|} - t \right)$$

and identify the set  $\Phi$  from proposition 7.2 with the set

$$G_{\text{step}} = \left\{ \phi_{(t,\omega)} : a \in \mathbb{R}^d, t \in [-r, r] \right\}$$

Since  $\mu$  is a probability measure and  $\phi_{(t,\omega)}$  a step function, we get  $\|\phi_{(t,\omega)}\|_{S_r}^2 \leq 1$ . We can bound the total variation of the signed measure  $\nu(t, \omega)d\lambda(t, \omega)$  by

$$\begin{aligned} & \int_{[-r,r] \times \mathbb{R}^d} |\nu(t, \omega)| d\lambda(t, \omega) \\ &= \int_{\mathbb{R}^d} \int_{[-r,r]} |\sin(\|\omega\|t + \rho(\omega))| d\lambda(t) \|\omega\| |\hat{f}(\omega)| d\lambda(\omega) \\ &\leq 2r \int_{\mathbb{R}^d} \|\omega\| |\hat{f}(\omega)| d\lambda(\omega) \leq 2rC \end{aligned}$$

Thus we can find a function  $f_k = \sum_{j=1}^k c_j \phi_{(t_j, \omega_j)}(x) + c_0$ , which is an affine combination of functions from  $G_{\text{step}}$ , as an approximation for  $f(x)$  in  $L^2(S_r, \mu)$  with an error of at most

$$\|f - f_k\|_{S_r}^2 \leq \frac{4r^2 C^2}{k}.$$

It remains to show that we can approximate the function  $f_k$ , which uses the modified step function  $\mathbb{I}^*$ , with linear combinations of sigmoid functions. To show this we will apply the dominated convergence theorem. If we use the properties of sigmoid functions, we easily see that we have pointwise convergence, as

$$\begin{aligned} \phi_{(t_j, \omega_j)}(x) &= \mathbb{I}^* \left( \frac{\omega^T x}{\|\omega\|} - t \right) \\ &= \lim_{L \rightarrow \infty} \sigma \left( L \left( \frac{\omega^T x}{\|\omega\|} - t \right) \right). \end{aligned} \tag{7.72}$$

Since  $\sigma(t) \leq 1$  and  $\mu$  is a probability measure, this convergence also holds in  $L^2$  sense. Therefore the error over  $S_r$ , using linear combinations of sigmoid

functions to approximate  $f(x)$ , can be bounded by

$$\begin{aligned}
& \limsup_{L \rightarrow \infty} \left\| f(x) - \sum_{j=1}^k \sigma \left( L \left( \frac{\omega_j^T x}{\|\omega_j\|} - t \right) \right) + c_0 \right\|_{S_r} \\
&= \limsup_{L \rightarrow \infty} \left\| f(x) - \left( \sum_{j=1}^k c_j \phi_{(t_j, \omega_j)}(x) + c_0 \right) + \left( \sum_{j=1}^k c_j \phi_{(t_j, \omega_j)}(x) + c_0 \right) \right. \\
&\quad \left. - \sum_{j=1}^k \sigma \left( L \left( \frac{\omega_j^T x}{\|\omega_j\|} - t \right) \right) + c_0 \right\|_{S_r} \\
&\leq \left\| f(x) - \left( \sum_{j=1}^k c_j \phi_{(t_j, \omega_j)}(x) + c_0 \right) \right\|_{S_r} \\
&\quad + \sum_{j=1}^k |c_j| \limsup_{L \rightarrow \infty} \left\| \phi_{(t_j, \omega_j)}(x) - \sigma \left( L \left( \frac{\omega_j^T x}{\|\omega_j\|} - t \right) \right) \right\|_{S_r} \\
&= \|f - f_k\|_{S_r} + 0 \leq \frac{2rC}{\sqrt{k}}
\end{aligned}$$

We can calculate the constant  $c_0$  with the help of (7.71) and then bound it for  $f \in \mathcal{F}_C$  by

$$\begin{aligned}
|c_0| &= |f(0) + c| \\
&\leq |f(0)| + \int_W \int_{[-r, 0]} |\sin(\|\omega\|t + \rho(\omega))| d\lambda(t) \|\omega\| |\hat{f}(\omega)| d\lambda(\omega) \\
&\leq |f(0)| + rC
\end{aligned}$$

Proposition 7.2 states for the coefficients

$$\sum_{j=1}^k |c_j| \leq \int_{[-r, r] \times W} |\nu(t, \omega)| d\lambda(t, \omega) \leq 2rC$$

And thus

$$\sum_{j=0}^k |c_j| \leq 3rC + |f(0)|$$

### 7.3 The VC-dimension of sigmoid networks

If it comes to bounding the VC-dimension of neural networks, the chosen activation function of the neurons play an important role. Even simple network architectures may have an infinite VC-dimension for some functions. Fortunately that is not the case if we choose the logistic squasher (7.57). If

we speak of the VC-dimension of a network, we mean the VC-dimension of the set of functions that can be realized by varying the parameters of the network. The following lemma is from [1] and stated without proof.

**Lemma 7.4.** Denote by  $\mathcal{F}$  a feed-forward network with logistic squasher neurons and a linear threshold output unit. Let the dimension of the input be  $d$  and the number of parameters in this network (weights and thresholds) be  $W$ . If the total number of neurons is  $l$ , then

$$\dim_{VC}(\mathcal{F}) \leq (Wl)^2 + 11Wl \log_2(18Wl^2)$$

Note that the network in the above lemma has a binary valued linear threshold output unit. To derive a bound for the VC-dimension of networks with a real valued linear output neuron, we will embed it in a binary valued network, as shown in [2].

**Theorem 7.5.** Denote by  $\mathcal{F}$  a feed-forward network with logistic squasher neurons and a real valued linear output unit. Let the number of parameters in this network (weights and thresholds) be  $W$ . If the total number of neurons is  $l$ , then

$$\dim_{VC}(\mathcal{F}) \leq ((W+2)(l+1))^2 + 11(W+2)(l+1) \log_2(18(W+2)(l+1)^2)$$

*Proof.* We will embed the network  $\mathcal{F}$  in a binary output network. To build this network  $\mathcal{G}$  add a linear threshold neuron after the output of  $\mathcal{F}$ . This neuron also gets an additional input  $r \in \mathbb{R}$ , with weight parameter  $a$ , see figure 5. Write  $\gamma$  for the parameters of  $\mathcal{F}$ , thus

$$\mathcal{F} = \left\{ f_\gamma(x) = \sum_{i=1}^k c_k \sigma(\omega_i^T x + b_i) + c_0 : \gamma \in \mathbb{R}^{(d+2)k+1} \right\}$$

Then the network  $\mathcal{G}$  has the form

$$\mathcal{G} = \{g(x, r)_{\gamma, a, b} = \text{sgn}(f_\gamma(x) + ar - b), f_\gamma \in \mathcal{F}, a, b \in \mathbb{R}\}$$

But if we restrict to  $b = 0$  and  $a = -1$ , we get

$$\begin{aligned} \dim_{VC}(\mathcal{F}) &= \dim_{VC} \{G_{\mathcal{F}}\} = \dim_{VC} \{G_f : f \in \mathcal{F}\} \\ &= \dim_{VC} \{ \{(x, r) \in \mathcal{S} \times \mathbb{R} : r \leq f(x)\} : f \in \mathcal{F} \} \\ &= \dim_{VC} \{ \mathbb{I}_{[r \leq f(x)]}(x, r) : f \in \mathcal{F} \} \\ &= \dim_{VC} \{ \text{sgn}(f(x) - r) : f \in \mathcal{F} \} \\ &= \dim_{VC} \{g(x, r) : g \in \mathcal{G} : a = -1, b = 0\} \leq \dim_{VC}(\mathcal{G}) \end{aligned}$$

Since the extended network has one more neuron and two more parameters, the statement follows from lemma 7.4.  $\square$

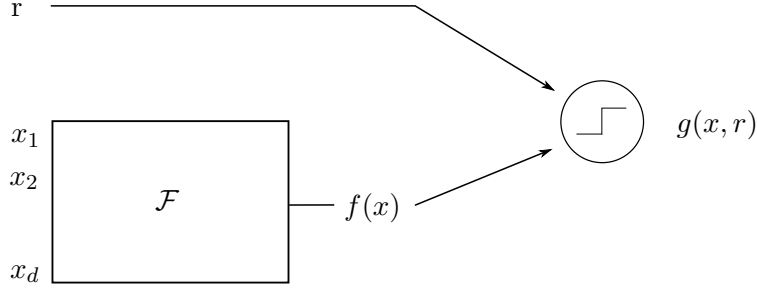


Figure 5: Embedding of the real valued Network  $\mathcal{F}$  in a boolean one

#### 7.4 Using Neural Networks in the LS-Algorithm

We established a bound on the VC-dimension and an error estimate for neural networks in the preceding sections. This gives us all that is necessary to formulate the following version of theorem 5.5. [26]

**Theorem 7.6.** Let  $\mathcal{S} = S_r$ , where  $S_r$  is the (closed) ball in  $\mathbb{R}^d$ , centred around the origin, with radius  $r$ . Assume that the continuation values  $\mathbb{E}[Z_{\tau_{k+1}}|X_k]$  can be extended to functions  $F_k$  that lie in  $\mathcal{F}_C$  for  $C > 0$ . Define  $L = \max_{k=1, \dots, K-1} 3rC + |F_k(0)|$ ,  $R = \max\{1, \|Z_1\|_\infty, \dots, \|Z_K\|_\infty\}$  and assume that  $R < \infty$ . For any integer  $\kappa > 0$ , we choose, as the approximation structure in every time step  $k = 1, \dots, K-1$ , a feed-forward network with real valued output and  $\kappa$  logistic squasher neurons in the hidden layer. Thus, for all  $k = 1, \dots, K-1$

$$\mathcal{H}_k = \left\{ \sum_{i=1}^{\kappa} c_i \sigma(\omega_i^T x + b_i) + c_0 : \omega_i \in \mathbb{R}^d, b_i, c_i \in \mathbb{R} : \sum_{i=0}^{\kappa} |c_i| \leq L \right\}$$

Then for any number  $N > 0$  of simulations, any  $0 < \epsilon \leq 1$  and  $0 \leq \epsilon_0 \leq 6\epsilon^2$ , we have, invoking the notation  $\mathcal{H}_0 = \{\mathbb{E}[Z_{\tau_1^A}]\}$ , for all  $k = 0, \dots, K-1$ ,

$$\begin{aligned} \mathbb{P}_k \left[ \left\| \mathcal{A} \left( \mathbb{E}[Z_{\tau_{k+1}^A} | X_k] \right) - \mathbb{E}[Z_{\tau_{k+1}} | X_k] \right\|_{L_k^2} \leq 5^{(K-k)} \left( 2\epsilon + \frac{2rC}{\sqrt{\kappa}} \right) \right] \\ \geq 1 - 8e(K-k)(t+1) \left( \frac{128eL(L+R)}{\epsilon^2} \right)^t \exp \left( -\frac{N\epsilon^4}{512(L+R)^4} \right) \end{aligned}$$

where  $t = C_0^2 + 11C_0 \log_2(18C_0(\kappa+2))$  with  $C_0 = ((2+d)\kappa+3)(\kappa+2)$ .

*Proof.* We need to check that all the assumptions of theorem 5.5 are fulfilled. We start with the VC-dimension of  $\mathcal{H}_k$ . Denote by  $W$  the number of parameters and with  $l$  the number of neurons. The network we use in the approximation has  $l = \kappa + 1$  of them and  $W = d\kappa + 2\kappa + 1$  parameters. The

constant  $C_0$  is thus  $C_0 = (W + 2)(l + 2)$  and by theorem 7.5 we get that  $t$  is an upper bound for  $\dim_{VC}(\mathcal{H}_k)$ . Lemma 7.3 ensures that

$$\max_{j=k, \dots, K-1} \left( \inf_{f \in \mathcal{H}_j} \|f - \mathbb{E}[Z_{\tau_{j+1}} | X_j]\|_{L_j^2} \right) \leq \frac{2rC}{\sqrt{\kappa}}$$

Clearly  $L$  uniformly bounds the spaces  $\mathcal{H}_k$  as it imposes restrictions on the network's parameters.  $\square$

## References

- [1] M. ANTHONY, *Discrete Mathematics of Neural Networks: Selected Topics*, SIAM e-books, Society for Industrial and Applied Mathematics, 2001.
- [2] M. ANTHONY AND P. BARTLETT, *Neural Network Learning: Theoretical Foundations*, Cambridge University Press, 2009.
- [3] P. BINEV, A. COHEN, W. DAHMEN, R. A. DEVORE, AND V. N. TEMLYAKOV, *Universal Algorithms for Learning Theory Part 1: Piecewise Constant Functions*, in *Journal of Machine Learning Research*, 2005, pp. 1297–1321.
- [4] M. BONA, *Combinatorics of Permutations*, Discrete Mathematics and Its Applications, Taylor & Francis, 2012.
- [5] E. CLEMENT, D. LAMBERTON, AND P. PROTTER, *An Analysis of a Least Squares Regression Method for American Option Pricing*, *Finance and Stochastics*, 6 (2002), pp. 449–471.
- [6] C. CONSTANTINESCU, *Hilbert Spaces*, no. Bd. 4 in North-Holland Mathematical Library, Elsevier Science, 2001.
- [7] F. CUCKER AND S. SMALE, *On the Mathematical Foundations of Learning*, *Bulletin of the American Mathematical Society*, 39 (2002), pp. 1–49.
- [8] L. DEVROYE, L. GYÖRFI, AND G. LUGOSI, *A Probabilistic Theory of Pattern Recognition*, Applications of Mathematics: Stochastic Modelling and Applied Probability, Springer, 1996.
- [9] S. S. DRAGOMIR, *A Generalization of Grüss’s Inequality in Inner Product Spaces and Applications*, *Journal of Mathematical Analysis and Applications*, 237 (1999), pp. 74 – 82.
- [10] D. EGLOFF, *Monte Carlo Algorithms for Optimal Stopping and Statistical Learning*, *The Annals of Applied Probability*, 15 (2005), pp. 1396–1432.
- [11] D. EGLOFF AND M. MINOO, *Convergence of Monte Carlo Algorithms for Pricing American Options*, 2002.
- [12] R. FEINERMAN AND D. NEWMAN, *Polynomial Approximation*, Williams & Wilkins, 1973.
- [13] G. GAVIN, *Generalization Performances of Perceptrons*, in *Proceedings of the International Conference on Artificial Neural Networks, ICANN ’01*, London, UK, 2001, Springer-Verlag, pp. 259–264.

- [14] L. GYÖRFI, M. KOHLER, A. KRZYŻAK, AND H. WALK, *A Distribution-Free Theory of Nonparametric Regression*, Springer Series in Statistics, Springer, New York, Berlin, Paris, 2002.
- [15] D. HAUSSLER, *Sphere Packing Numbers for Subsets of the Boolean  $n$ -Cube with Bounded Vapnik-Chervonenkis Dimension*, Journal of Combinatorial Theory, Series A, 69 (1995), pp. 217 – 232.
- [16] W. HOEFFDING, *Probability Inequalities for Sums of Bounded Random Variables*, Journal of the American Statistical Association, 58 (1963), pp. 13–30.
- [17] M. KOHLER, A. KRZYŻAK, AND N. TODOROVIC, *Pricing of High-Dimensional American Options by Neural Networks*, Mathematical Finance, 20 (2010), pp. 383–410.
- [18] V. KOLTCHINSKII, *Oracle Inequalities in Empirical Risk Minimization and Sparse Recovery Problems*, Ecole d’été de probabilités de Saint-Flour, Springer, 2011.
- [19] J. V. KUJALA, *A Remark on the Assumptions of Bayes’ Theorem*, ArXiv e-prints, (2011).
- [20] F. A. LONGSTAFF AND E. S. SCHWARTZ, *Valuing American Options by Simulation: A Simple Least-Squares Approach*, Review of Financial Studies, 14 (2001), pp. 113–147.
- [21] K. MEHROTRA, C. MOHAN, AND S. RANKA, *Elements of Artificial Neural Networks*, A Bradford book, MIT Press, 1997.
- [22] K. ROSEN, *Discrete Mathematics and Its Applications*, McGraw-Hill, 2011.
- [23] J. TSITSIKLIS AND B. VAN ROY, *Regression Methods for Pricing Complex American-Style Options*, Neural Networks, IEEE Transactions on, 12 (2001), pp. 694–703.
- [24] M. VIDYASAGAR, *Learning and Generalisation: With Applications to Neural Networks*, Communications and Control Engineering, Springer, 2003.
- [25] D. Z. ZANGER, *Convergence of a Least-Squares Monte Carlo Algorithm for Bounded Approximating Sets*, Applied Mathematical Finance, Taylor & Francis Journals, 16 (2009), pp. 123–150.
- [26] —, *Quantitative Error Estimates for a Least-Squares Monte Carlo Algorithm for American Option Pricing*, Finance and Stochastics, 17 (2013), pp. 503–534.