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Analysis of the α-hypergeometric stochastic volatility model

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Abstract

Major spoiler alert. We investigate the α-hypergeometric stochastic volatility model. We present results including the martingale property of the forward and calculate certain transforms of the forward as well as the volatility itself, which enable us to perform plain vanilla pricing and pricing of volatility derivatives. Furthermore we derive certain large deviation problems associated with the α-hypergeometric stochastic volatility model as well as other asymptotics.

Danksagung

Ich möchte an dieser Stelle die Möglichkeit nutzen meinen außerordentlichen Dank auszudrücken. Großer Dank gilt meinen Eltern, die mir all ihre Liebe angedeihen ließen, zu derer sie fähig waren. Ebenso danke ich meinen alten Schulfreunden, die mir in Trauer wieder aufgeholfen haben, in Freude diese noch vermehrt haben und trotz meines absurden Zeitmanagements immer an meiner Seite standen. Meinen Studienkollegen, von denen viele überaus gute Freunde geworden sind, möchte ich danken. Nicht nur, dass sie mir privat immer nahe standen, sie teilen mit mir auch dieselbe Leidenschaft - die Mathematik. Ich danke ihnen für diverse Anregungen und interessante Gespräche. Weiters danke ich Stefan Gerhold, ohne den diese Arbeit nicht ihr heutiges Format und ihre heutige Fülle hätte. Stefan stand mir bei jeglichen Fragen und Ideen gerne und motiviert zur Seite. Ebenso gilt mein herzlicher Dank allen Professoren der TU Wien, die mich im Rahmen meines Studiums gefordert und gefördert haben.

I went to the woods because I wished to live deliberately, to front only the essential facts of life, and see if I could not learn what it had to teach, and not, when I came to die, discover that I had not lived. I did not wish to live what was not life, living is so dear; nor did I wish to practise resignation, unless it was quite necessary. I wanted to live deep and suck out all the marrow of life, to live so sturdily and Spartan-like as to put to rout all that was not life, to cut a broad swath and shave close, to drive life into a corner, and reduce it to its lowest terms.

Henry David Thoreau

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1 Introduction

Stochastic volatility models are nowadays an important tool in financial mathematics. The most common model is the Heston model, which however when calibrated to real world data often does not satisfy the Feller condition. This in turn results in the volatility hitting zero in finite time. Volatility models where the volatility itself is modelled using an Ohrnstein-Uhlenbeck process however immediately lacks the property of positivity. Therefore one essential property is the strict positivity of the volatility at all time. Of course another important property one would like to have is that there are certain closed form expressions for example of the call option price. This is however only possible for a small amount of models. One way to handle this is to integral transform the call price like in the Heston model.

This master thesis analyses a new stochastic volatility model called the α-hypergeometric stochastic volatility model introduced by [Da Fonseca and Martini, 2014]. By construction this model satisfies the assumption of strict positivity of the volatility process at all time. Furthermore it has the property of being a non affine model. Finally it is tractable in the sense that we can express the Laplace transform in time of the Mellin transform of the call option price in terms of standard functions and hypergeometric series.

This thesis is structured as follows: First we introduce the model and investigate certain dependencies on model parameters. Then we investigate the martingality of the forward where we follow the ideas of [Da Fonseca and Martini, 2014]. An original result of this thesis is lemma 2.4 which enables us to perform a crucial measure change afterwards. Another original result of this thesis is an alternative proof theorem 2.6. Using results of [Lions and Musiela, 2007] we were able to derive theorem 2.8 which analyses the integrability of certain powers of the forward. At the end of section 2 we consider the dual market. In section 3 we give a quick reminder of the concept of generalized hypergeometric functions and related functions. Section 4 is dedicated to the asymptotic analysis of the model. First we consider the short and long term behaviour of the expected variance and the variance swap. Proposition 4.2 is an original generalization of proposition 3 in [Da Fonseca and Martini, 2014]. We furthermore investigate the case of deterministic volatility as an heuristic approximation for the model with small volatility of volatility. In this case we were able to express the price of a call option in terms of hypergeometric functions, see Proposition 4.11. At the end of section 4 we derive large deviation problems associated with the model and derive a small time behaviour of the implied volatility. This is done following ideas of [Forde and Jacquier, 2011]. The theorems 4.14 and 4.20 are original results of this thesis. Section 5 deals with certain transforms of the volatility as well as the forward itself which are then necessary to perform pricing of plain vanilla options, which in turn is done in section 6. Both sections follow [Da Fonseca and Martini, 2014]. We were however able to simplify certain hypergeometric series, see the proof of proposition 5.2.

2 The model

2.1 Definition of the model

The risk neutral dynamics under the pricing measure $\mathbb{P}$ of the forward S and the instantaneous log volatility v in the α-hypergeometric model are given by

$$dS_t = S_t \mathrm{e}^{v_t} dW_t, \tag{1}$$

$$dv_t = \left(a - b\mathrm{e}^{\alpha v_t}\right) dt + \sigma dB_t, \tag{2}$$

$$dW_t dB_t = \rho dt, \tag{3}$$

with deterministic initial data $S_0 = 1$ and $v_0 \in \mathbb{R}$, where W and B are correlated Brownian motions with correlation $\rho \in (-1, 1)$ and constants $\alpha > 0$, $a \in \mathbb{R}$, $b > 0$ and $\sigma > 0$. W.l.o.g. we assume that the interest rate is equal to zero. In the α-hypergeometric model the instantaneous variance is therefore given by $V_t = \mathrm{e}^{2v_t}$, $t \geq 0$.

Since the coefficients of the SDE in (1) and (2) are locally Lipschitz continuous strong uniqueness holds. In fact the SDE (2) has a unique strong solution given by

$$v_t = v_0 + at + \sigma B_t - \frac{1}{\alpha} \ln \left(1 + \alpha b \int_0^t \exp\left(\alpha \left(v_0 + as + \sigma B_s\right)\right) ds \right), \tag{4}$$

see for example section 2.1.1 in [Da Fonseca and Martini, 2014] for a direct derivation and section 4.4 equation (4.53) in [Kloeden and Platen, 1992] for a much more general approach. The dynamics of S and V are given by

$$dS_t = S_t \sqrt{V_t} dW_t,$$

$$dV_t = \left(\left(2a + 2\sigma^2\right) V_t - 2bV_t^{1+\frac{\alpha}{2}} \right) dt + 2\sigma V_t dB_t,$$

which shows the non-affinity of the model in question.

Remark 2.2. From the dynamics (1) and (2) one can easily verify that

$$\alpha v_{v_0,\alpha,a,b,\sigma} = v_{\alpha v_0,1,\alpha a,\alpha b,\alpha\sigma},$$

where $v_{v_0,\alpha,a,b,\sigma}$ denotes the solution of (2) with the corresponding parameters.

2.3 Martingality and moments of S

In order to prove certain martingale properties of S we are going to need the following lemma, which is an original result of this thesis and will be necessary to perform a crucial measure change hereafter.

Lemma 2.4. *For $a \in \mathbb{R}, b \geq 0$ and $c \geq 0$, a Brownian motion B and*

$$L_t := \int_0^t a - b \exp\left(cB_s\right) dB_s$$

for $t \geq 0$, the corresponding stochastic exponential $\mathcal{E}(L)$ is a martingale.

Proof. By exercise 5.38 in [Karatzas and Shreve, 1991] we know that $\mathcal{E}(L)$ is a martingale if and only if the function

$$\varphi : x \mapsto \int_0^x \int_0^y \exp\left(-2\int_z^y a - b\mathrm{e}^{cu} du\right) dz dy$$

satisfies $\lim_{x\to\pm\infty} \varphi(x) = +\infty$. For $c \neq 0$ and $x \geq 0$ elementary calculations show that

$$\varphi(x) = \int_0^x \underbrace{\exp\left(-2ay + \frac{2b}{c}\mathrm{e}^{cy}\right)}_{=:f(y)} \underbrace{\int_0^y \exp\left(2az - \frac{2b}{c}\mathrm{e}^{cz}\right) dz}_{=:g_+(y)} dy.$$

A closer look at the function f shows that

$$\lim_{y\to\infty} f(y) = \begin{cases} +\infty & a < 0 \\ 1 & a = 0, b = 0 \\ +\infty & a = 0, b > 0 \\ 0 & a > 0, b = 0 \\ +\infty & a > 0, b > 0. \end{cases}$$

Except for the case $a > 0$ and $b = 0$ we can immediately conclude $\varphi(+\infty) = +\infty$, since g_+ is trivially bounded away from zero at infinity. For the case $a > 0$ and $b = 0$ we have

$$\begin{aligned} \varphi(x) &= \int_0^x \exp(-2ay) \int_0^y \exp(2az)\, dzdy \\ &= \int_0^x \frac{1 - \exp(-2ay)}{2a} dy \end{aligned}$$

and we can conclude as above. The case $c = 0$ is handled analogously. Therefore we obtain $\varphi(+\infty) = +\infty$ for all $a \in \mathbb{R}, b \geq 0$ and $c \geq 0$.

For $c \neq 0$ and $x \leq 0$ we have

$$\varphi(x) = \int_x^0 \underbrace{\exp\left(-2ay + \frac{2b}{c}\mathrm{e}^{cy}\right)}_{=f(y)} \underbrace{\int_y^0 \exp\left(2az - \frac{2b}{c}\mathrm{e}^{cz}\right) dz}_{=:g_-(y)} dy.$$

Again a closer look at the function f shows that

$$\lim_{y\to-\infty} f(y) = \begin{cases} 0 & a < 0 \\ 1 & a = 0 \\ +\infty & a > 0 \end{cases}$$

and we can immediately conclude that $\varphi(-\infty) = +\infty$ except for the case $a < 0$. But since the inequalities

$$f(y) = \exp\left(-2ay + \frac{2b}{c}\mathrm{e}^{cy}\right) \geq \exp(-2ay),$$
$$g_-(y) = \int_y^0 \exp\left(2az - \frac{2b}{c}\mathrm{e}^{cz}\right) dz \geq \int_y^0 \exp\left(2az - \frac{2b}{c}\right) dz$$

hold for $y \leq 0$ we have

$$\begin{aligned}\varphi(x) &\geq \int_x^0 \exp(-2ay) \int_y^0 \exp\left(2az - \frac{2b}{c}\right) dzdy \\ &= \mathrm{e}^{-\frac{2b}{c}} \int_x^0 \exp(-2ay) \int_y^0 \exp(2az)\, dzdy \\ &= \mathrm{e}^{-\frac{2b}{c}} \int_x^0 \frac{\exp(-2ay) - 1}{2a} dy.\end{aligned}$$

Therefore we can conclude $\varphi(-\infty) = +\infty$. As in the case $x \to \infty$ one can easily verify $\varphi(-\infty) = +\infty$ in the case $c = 0$. This concludes the proof. □

Remark 2.5. Lemma 2.4 guaranties the validity of an essential measure change in the proof of the following theorem. Note however that the Novikov condition does not hold in general. Consider for example the case $a = 0$ and $b, c > 0$. Using the Jensen inequality for the convex function $x \mapsto (b\mathrm{e}^{cx})^2$ and the fact that $\int_0^t B_s ds$ is Gaussian we arrive at

$$\begin{aligned}\mathbb{E}\left(\exp\left(\frac{1}{2}\left\langle \int_0^\cdot -b\exp(cB_s)\, dB_s \right\rangle_t\right)\right) &= \mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^t (-b\exp(cB_s))^2 ds\right)\right) \\ &\geq \mathbb{E}\left(\exp\left(\frac{b^2 t}{2}\exp\left(\frac{2c}{t}\int_0^t B_s ds\right)\right)\right) = \infty.\end{aligned}$$

Theorem 2.6. *The forward S in the α-hypergeometric model is a martingale if and only if $\alpha \geq 2$ or $\alpha < 2$ and one of the following conditions is fulfilled:*

- $\rho \leq 0$,
- $\alpha > 1$,
- $\alpha = 1$ *and* $b \geq \rho\sigma$.

Proof. We follow the reasoning of proposition 6 in [Da Fonseca and Martini, 2014]. First we consider the case $\alpha = 2$. Note that by integrating (2) and with (4) we arrive at

$$\begin{aligned}\int_0^t \exp(\alpha v_s)\, ds &= -\frac{1}{b}(v_t - v_0 - at - \sigma B_t) \\ &= \frac{1}{\alpha b}\ln\left(1 + \alpha b\int_0^t \exp(\alpha(v_0 + as + \sigma B_s))\, ds\right).\end{aligned}$$

Using the above equation we get

$$\begin{aligned}\exp\left(\frac{1}{2}\left\langle\int_0^{\cdot}\exp(v_s)\,dW_s\right\rangle_t\right) &= \exp\left(\frac{1}{2}\int_0^t\exp(2v_s)\,ds\right)\\ &= \exp\left(\frac{1}{4b}\ln\left(1+2b\int_0^t\exp(2(v_0+as+\sigma B_s))\,ds\right)\right)\\ &= \left(1+2b\int_0^t\exp(2(v_0+as+\sigma B_s))\,ds\right)^{\frac{1}{4b}}\\ &\leq (1+2bt\exp(2(v_0+|a|\,t+\sigma B_t^*)))^{\frac{1}{4b}}\\ &\leq (1+2bt\exp(2(v_0+|a|\,t)))^{\frac{1}{4b}}\exp\left(\frac{2\sigma B_t^*}{4b}\right),\end{aligned}$$

where B^* denotes the running maximum of B , i.e. $B_t^* = \max_{0\leq s\leq t} B_t$. Therefore, by the Novikov condition, we conclude that S is a martingale.

We now consider $\alpha > 2$. By Hölder's inequality with $p = \frac{\alpha}{\alpha-2}$ and $q = \frac{\alpha}{2}$ we get the inequality

$$\int_0^t \exp(2v_s)\,ds \leq t^{\frac{\alpha-2}{\alpha}}\left(\int_0^t\exp(\alpha v_s)\,ds\right)^{\frac{2}{\alpha}}.$$

Now as above and again with the Novikov condition we conclude that S is a martingale.

Now we consider the case $\alpha < 2$. Since S is given by a stochastic exponential it is a martingale if and only if its expectation is constant, i.e.

$$\mathbb{E}(S_t) \equiv 1.$$

Note that by introducing the standard Brownian motion $\left(B_t, B_t^{\perp}\right)_{t\geq 0}$, $\overline{\rho} = \sqrt{1-\rho^2}$ and the sigma algebra $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$ we have

$$\begin{aligned}\mathbb{E}(S_t) &= \mathbb{E}\left(\mathcal{E}\left(\int_0^{\cdot}\mathrm{e}^{v_s}dW_s\right)_t\right)\\ &= \mathbb{E}\left(\mathcal{E}\left(\rho\int_0^{\cdot}\mathrm{e}^{v_s}dB_s+\overline{\rho}\int_0^{\cdot}\mathrm{e}^{v_s}dB_s^{\perp}\right)_t\right)\\ &= \mathbb{E}\left(\mathcal{E}\left(\rho\int_0^{\cdot}\mathrm{e}^{v_s}dB_s\right)_t\mathbb{E}\left(\mathcal{E}\left(\overline{\rho}\int_0^{\cdot}\mathrm{e}^{v_s}dB_s^{\perp}\right)_t\middle|\mathcal{F}_t\right)\right)\\ &= \mathbb{E}\left(\mathcal{E}\left(\rho\int_0^{\cdot}\mathrm{e}^{v_s}dB_s\right)_t\right).\end{aligned}$$

Let

$$\tilde{B}_s := B_s - \int_0^s \frac{a - b\mathrm{e}^{\alpha v_0}\exp(\alpha\sigma B_u)}{\sigma}du$$

and let $\tilde{\mathbb{P}}^t$, given by the Girsanov theorem, which can be applied to the above process by lemma 2.4, denote the probability measure under which $\left(\tilde{B}_s\right)_{0\leq s\leq t}$ is a Brownian motion.

The density is therefore given by

$$\begin{aligned}\frac{d\tilde{\mathbb{P}}^t}{d\mathbb{P}} &= \mathcal{E}\left(\int_0^{\cdot} \frac{a - b\mathrm{e}^{\alpha v_0} \exp\left(\alpha\sigma B_s\right)}{\sigma} dB_s\right)_t \\ &= \exp\left(\int_0^t \frac{a - b\mathrm{e}^{\alpha v_0} \exp\left(\alpha\sigma B_s\right)}{\sigma} dB_s - \frac{1}{2}\int_0^t \left(\frac{a - b\mathrm{e}^{\alpha v_0} \exp\left(\alpha\sigma B_s\right)}{\sigma}\right)^2 ds\right).\end{aligned}$$

The law of $(v_s, B_s)_{0\le s\le t}$ under $\mathbb{P}$ is now the same as the one of the process $\left(v_0 + \sigma B_s, \tilde{B}_s\right)_{0\le s\le t}$ under $\tilde{\mathbb{P}}^t$. This can be easily seen by

$$\begin{aligned} dv_s &= \left(a - b\mathrm{e}^{\alpha v_s}\right) dt + \sigma dB_s && \text{under } \mathbb{P}, \\ d\left(v_0 + \sigma B_s\right) &= \left(a - b\mathrm{e}^{\alpha\left(\left(v_0 + \sigma B_s\right)\right)}\right) ds + \sigma d\tilde{B}_s && \text{under } \tilde{\mathbb{P}}^t,\end{aligned}$$

and by the Yamada Watanabe theorem, since we have already shown pathwise uniqueness. This enables us to conclude that

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}\left(\mathcal{E}\left(\rho\int_0^{\cdot} \mathrm{e}^{v_s} dB_s\right)_t\right) \\ &= \mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\mathcal{E}\left(\rho \mathrm{e}^{v_0}\int_0^{\cdot} \exp\left(\sigma B_s\right) d\tilde{B}_s\right)_t\right) \\ &= \mathbb{E}^{\mathbb{P}}\left(\mathcal{E}\left(\rho \mathrm{e}^{v_0}\int_0^{\cdot} \exp\left(\sigma B_s\right)\left(dB_s - \frac{a - b\mathrm{e}^{\alpha v_0}\exp\left(\alpha\sigma B_s\right)}{\sigma} ds\right)\right)_t \right.\\ &\qquad \left. \cdot\mathcal{E}\left(\int_0^{\cdot} \frac{a - b\mathrm{e}^{\alpha v_0}\exp\left(\alpha\sigma B_s\right)}{\sigma} dB_s\right)_t\right) \\ &= \mathbb{E}^{\mathbb{P}}\left(\mathcal{E}\left(\int_0^{\cdot} \rho \mathrm{e}^{v_0}\exp\left(\sigma B_s\right) + \frac{a - b\mathrm{e}^{\alpha v_0}\exp\left(\alpha\sigma B_s\right)}{\sigma} dB_s\right)_t\right) \\ &= \mathbb{E}^{\mathbb{P}}\left(\mathcal{E}\left(\int_0^{\cdot} b\left(B_s\right) dB_s\right)_t\right),\end{aligned}$$

with

$$b\left(x\right) := \rho \mathrm{e}^{v_0}\exp\left(\sigma x\right) + \frac{a}{\sigma} - \frac{b\mathrm{e}^{\alpha v_0}}{\sigma}\exp\left(\alpha\sigma x\right).$$

As in the proof of lemma 2.4 we are therefore left with the computations of the behaviour of the function

$$\varphi : x \mapsto \int_0^x \int_0^y \exp\left(-2\int_z^y b\left(u\right) du\right) dz dy$$

at $\pm\infty$. For $x \ge 0$ straightforward calculations show

$$\begin{aligned}&\varphi\left(x\right) = \\ &\int_0^x \underbrace{\exp\left(-\frac{2\rho \mathrm{e}^{v_0}}{\sigma}\mathrm{e}^{\sigma y} - \frac{2a}{\sigma}y + \frac{2b\mathrm{e}^{\alpha v_0}}{\alpha\sigma^2}\mathrm{e}^{\alpha\sigma y}\right)}_{=:f(y)} \underbrace{\int_0^y \exp\left(\frac{2\rho \mathrm{e}^{v_0}}{\sigma}\mathrm{e}^{\sigma z} + \frac{2a}{\sigma}z - \frac{2b\mathrm{e}^{\alpha v_0}}{\alpha\sigma^2}\mathrm{e}^{\alpha\sigma z}\right) dz}_{=:g_+(y)}\, dy\end{aligned}$$

and for $x < 0$ we have

$$\varphi(x) = \int_x^0 \underbrace{\exp\left(-\frac{2\rho e^{v_0}}{\sigma} e^{\sigma y} - \frac{2a}{\sigma} y + \frac{2b e^{\alpha v_0}}{\alpha \sigma^2} e^{\alpha \sigma y}\right)}_{=f(y)} \underbrace{\int_y^0 \exp\left(\frac{2\rho e^{v_0}}{\sigma} e^{\sigma z} + \frac{2a}{\sigma} z - \frac{2b e^{\alpha v_0}}{\alpha \sigma^2} e^{\alpha \sigma z}\right) dz}_{=:g_-(y)} \, dy.$$

We first take a look at the behaviour of φ for $x \to -\infty$. With $C_1 := \max\left(\frac{2|\rho| e^{v_0}}{\sigma}, \frac{2b e^{\alpha v_0}}{\alpha \sigma^2}\right)$ and for $y \le 0$ we have

$$\begin{aligned} f(y) &\ge e^{C_1} \exp\left(-\frac{2a}{\sigma} y\right) \\ g_-(y) &\ge e^{C_1} \int_y^0 \exp\left(\frac{2a}{\sigma} z\right) dz. \end{aligned}$$

Therefore we have

$$\begin{aligned} \varphi(x) &= \int_x^0 f(y)\, g_-(y)\, dy \\ &\ge e^{2C_1} \int_x^0 \exp\left(-\frac{2a}{\sigma} y\right) \int_y^0 \exp\left(\frac{2a}{\sigma} z\right) dz dy \end{aligned}$$

which diverges for all $a \in \mathbb{R}$ for $x \to -\infty$. We now consider the case $x \to \infty$. A closer look at the function f shows

$$\lim_{y \to \infty} f(y) = \begin{cases} +\infty & \rho \le 0, \\ +\infty & \alpha > 1, \\ +\infty & \alpha = 1 \text{ and } b > \rho\sigma, \\ 0 & \alpha = 1 \text{ and } b < \rho\sigma, \\ 0 & \alpha < 1 \text{ and } \rho > 0. \end{cases}$$

Note that we excluded the case $\alpha = 1$ and $b = \rho\sigma$, since straightforward calculations show $\lim_{x \to \infty} \varphi(x) = +\infty$. Therefore we have $\lim_{x \to \infty} \varphi(x) = +\infty$ if $\rho \le 0$, $\alpha > 1$ or $\alpha = 1$ and $b \ge \rho\sigma$.

For the other cases it can be shown by some lengthy computations that the $\varphi(x)$ converges as $x \to \infty$. We refer to proposition 6 in [Da Fonseca and Martini, 2014]. □

We have also derived a different approach to show the martingale property using results of the paper [Lions and Musiela, 2007]. They considered the following stochastic volatility models. Using their notation we have:

$$\begin{aligned} dF_t &= \sigma_t F_t dW_t, \\ d\sigma_t &= b(\sigma_t)\, dt + \mu(\sigma_t)\, dZ_t, \\ dW_t dB_t &= \rho dt, \end{aligned}$$

with deterministic initial data $F_0 > 0$ and $\sigma_0 > 0$, where W and Z are correlated Brownian motions with correlation $\rho \in [-1, 1]$ and μ and b are smooth functions on $[0, \infty)$ such that

$$\mu(0) = 0, \qquad b(0) \geq 0, \tag{5}$$
$$\mu(\xi) > 0 \text{ for } \xi > 0, \qquad \mu \text{ is Lipschitz on } [0, \infty) \tag{6}$$
$$b(\xi) \leq C(1 + \xi) \text{ on } [0, \infty), \text{ for some } C > 0. \tag{7}$$

They derived the following

Theorem 2.7.

- *If the following condition holds*

$$\limsup_{\xi \to \infty} \frac{\rho\mu(\xi)\xi + b(\xi)}{\xi} < \infty,$$

then $\mathbb{E}(F_t |\ln F_t|) < \infty$, $\mathbb{E}\left(\sup_{0 \leq s \leq t} F_s\right) < \infty$ *for all* $t \geq 0$ *and* F *is a nonnegative martingale.*

- *If the following condition holds*

$$\liminf_{\xi \to \infty} \frac{\rho\mu(\xi)\xi + b(\xi)}{\varphi(\xi)} > 0,$$

for some smooth, positive, increasing function φ *such that* $\int^{\infty} \frac{1}{\varphi(\xi)} d\xi < \infty$, *then* F *is not a martingale and we have:*

$$\mathbb{E}(F_t) < F_0 \text{ for all } t > 0.$$

Proof. See theorem 2.4 in [Lions and Musiela, 2007]. □

In the α-hypergeometric model we then have with a slight abuse of notation

$$F_t = S_t, \qquad \sigma_t = \mathrm{e}^{v_t}, \qquad b(\xi) = \left(a + \frac{\sigma^2}{2}\right)\xi - b\xi^{1+\alpha}, \qquad \mu(\xi) = \sigma\xi.$$

In the above the left hand side is the notation of [Lions and Musiela, 2007] and the right hand side is our notation. Note that the α-hypergeometric model obviously satisfies all the conditions proposed by [Lions and Musiela, 2007]. We were able to derive an original proof of theorem 2.6.

Alternative proof of theorem 2.6. In view of theorem 2.7 lets calculate

$$\limsup_{\xi \to \infty} \frac{\rho\mu(\xi)\xi + b(\xi)}{\xi} = \lim_{\xi \to \infty} \rho\sigma\xi + \left(a + \frac{\sigma^2}{2}\right) - b\xi^{\alpha},$$

which is less than ∞ if and only if $\alpha \geq 2$ or $\alpha < 2$ and one of the following conditions is fulfilled:

- $\rho \leq 0$
- $\alpha > 1$
- $\alpha = 1$ and $b \geq \rho\sigma$.

We conclude that in these cases S is a martingale.

Choosing $\varphi(\xi) = \xi^2$, which obviously satisfies the assumptions of theorem 2.7 and considering all the other parameter cases now, which are $\alpha < 1, \rho > 0$ or $\alpha = 1, b < \rho\sigma$, we calculate

$$\liminf_{\xi \to \infty} \frac{\rho\mu(\xi)\xi + b(\xi)}{\varphi(\xi)} = \lim_{\xi \to \infty} \rho\sigma + \frac{a + \frac{\sigma^2}{2}}{\xi} - \frac{b}{\xi^{1-\alpha}} > 0.$$

Therefore with theorem 2.7 we conclude that in these cases S is not a martingale. $\square$

Aside from this general approach [Lions and Musiela, 2007] also considered the following stochastic volatility model:

$$\begin{aligned} dF_t &= \sigma_t^\delta F_t^\beta dW_t, \\ d\sigma_t &= b(\sigma_t)\, dt + \alpha\sigma_t^\gamma dZ_t, \\ dW_t dB_t &= \rho dt, \end{aligned}$$

with deterministic initial data $F_0 > 0$ and $\sigma_0 > 0$, where W and Z are correlated Brownian motions with correlation $\rho \in [-1, 1]$, $\alpha, \beta, \gamma, \delta > 0, b(0) \geq 0$, b locally Lipschitz on $[0, \infty)$ and b satisfies (7). Again the α-hypergeometric model fits into this model and again with a slight abuse of notation we have

$$F_t = S_t, \quad \sigma_t = \mathrm{e}^{v_t}, \quad b(\xi) = \left(a + \frac{\sigma^2}{2}\right)\xi - b\xi^{1+\alpha}, \quad \alpha = \sigma, \quad \beta = 1, \quad \gamma = 1, \quad \delta = 1.$$

Again in the above the left hand side is the notation of [Lions and Musiela, 2007] and the right hand side is our notation. Note that these parameters now correspond to the cases (v) and (iii) in theorem 3.2 and theorem 3.3 respectively in [Lions and Musiela, 2007]. Together with the remark after theorem 3.2 we are now able to quantify the behaviour of the moments of S, which is an original result of this thesis.

Theorem 2.8. *Let S be a martingale in the α-hypergeometric model. Then $S_t \in L^\theta$ or equivalently*

$$\mathbb{E}\left(\sup_{0 \leq s \leq t} S_s^\theta\right) < \infty \tag{8}$$

holds for all $t > 0$ in the cases

- $\alpha < 1$, $\rho < 0$ *and* $1 < \theta \leq \frac{1}{1-\rho^2}$,
- $\alpha = 1$, $b > \rho\sigma$ *and* $1 < \theta \leq \frac{\sigma - 2b\rho + \sqrt{(\sigma - 2b\rho)^2 + 4b^2(1-\rho^2)}}{2\sigma(1-\rho^2)}$,

- *$\alpha > 1$ and $\theta > 1$.*

Conversely

$$\mathbb{E}\left(S_t^{\theta}\right) = \infty \tag{9}$$

holds for all $t > 0$ in the cases

- *$\alpha < 1$, $\rho = 0$ and $\theta > 1$,*
- *$\alpha < 1$, $\rho < 0$ and $\theta > \frac{1}{1-\rho^2}$,*
- *$\alpha = 1$, $b = \rho\sigma$ and $\theta > 1$.*

Proof. Since we can apply theorem 3.2., the remark after theorem 3.2. and theorem 3.3 in [Lions and Musiela, 2007] we first need to calculate the quantity b_∞. Straightforward computations lead to

$$b_\infty = \begin{cases} 0 & \text{if } \alpha < 1, \\ -b & \text{if } \alpha = 1, \\ -\infty & \text{if } \alpha > 1. \end{cases}$$

- Case $\alpha < 1$: To ensure martingality of S one needs $\rho \leq 0$. If $\rho \leq -\sqrt{\frac{\theta-1}{\theta}}$ we have that (8) holds. Therefore for all $1 < \theta \leq \frac{1}{1-\rho^2}$ and all $t > 0$ there holds $\mathbb{E}\left(\sup_{0\leq s\leq t} S_s^{\theta}\right) < \infty$ and for all $\theta > \frac{1}{1-\rho^2}$ and all $t > 0$ there holds $\mathbb{E}\left(S_t^{\theta}\right) = \infty$. If $\rho = 0$ then $\mathbb{E}\left(S_t^{\theta}\right) = \infty$ for all $\theta > 1$ and all $t > 0$.

- Case $\alpha = 1$: To ensure martingality of S one needs $b \geq \rho\sigma$. Consider first the case $b > \rho\sigma$ then if

  $$\rho \leq -\sqrt{\frac{\theta-1}{\theta}} + \frac{b}{\sigma\theta}$$

  (8) holds. One can simply calculate that the above equation is satisfied if $1 < \theta \leq \theta_+$ with

  $$\theta_+ = \frac{\sigma - 2b\rho + \sqrt{(\sigma - 2b\rho)^2 + 4b^2\left(1-\rho^2\right)}}{2\sigma\left(1-\rho^2\right)}.$$

  In the case $b = \rho\sigma$ one immediately gets that for all $\theta > 1$ the inequality

  $$\rho > -\sqrt{\frac{\theta-1}{\theta}} + \frac{b}{\sigma\theta}$$

  holds and therefore (9) holds.

- Case $\alpha > 1$: Since $b_\infty = -\infty$ we immediately get (8) for all $\theta > 1$.

$\square$

2.9 The dual market

If S is a martingale we can take a look at the dual market given by the process $S' := \frac{1}{S}$ under the probability measure $d\mathbb{P}' := S_T d\mathbb{P}$. The dynamics of S' are then given by

$$dS'_t = S'_t \mathrm{e}^{2v_t} dt - S'_t \mathrm{e}^{v_t} dW_t.$$

Under $\mathbb{P}'$ the process

$$\tilde{W}_t := W_t - \int_0^t \mathrm{e}^{v_s} ds$$

is a Brownian motion according to the Girsanov theorem. We can therefore rewrite the dynamics of S' as

$$dS'_t = S'_t \mathrm{e}^{2v_t} dt - S'_t \mathrm{e}^{v_t} dW_t = -S'_t \mathrm{e}^{v_t} d\tilde{W}_t.$$

Note that under $\mathbb{P}'$ the process

$$\tilde{B}_t := B_t - \rho \int_0^t \mathrm{e}^{v_s} ds$$

is also a Brownian motion. Therefore the dynamics of v are given by

$$dv_t = (a - b\mathrm{e}^{\alpha v_t})\, dt + \sigma dB_t = (a - b\mathrm{e}^{\alpha v_t} + \rho\sigma \mathrm{e}^{v_t})\, dt + \sigma d\tilde{B}_t.$$

We can therefore conclude that the dual model belongs to the same family of processes if and only if $\rho = 0$, in which case the models are the same, or $\alpha = 1$.

Remark 2.10. Note furthermore that in the special case $\alpha = 1$ and $b = \rho\sigma$ one gets $v_t = at + \sigma\tilde{B}_t$, which results in the Hull-White model, see for example Section 2.3 in [Gulisashvili, 2012].

3 Hypergeometric functions

In the following we need certain special functions which are known as hypergeometric functions. Our notation is the same as in [DLMF]. Therefore the reader should take a look at

- Chapter 13 in [DLMF] for the definition of the confluent hypergeometric functions and the Whittaker functions as well as
- Chapter 15 in [DLMF] for the definition of the generalized hypergeometric functions.

Certain properties of these functions are cited directly by the corresponding equation number in [DLMF].

Lemma 3.1. *For $a, b, c, d, f \in \mathbb{R}$ such that $a, c \neq 0$, $\frac{a}{f} + 1 \neq 0, -1, -2, \dots$ there holds*

$$\int \mathrm{e}^{at} \left(c + d\mathrm{e}^{ft}\right)^b dt = \frac{1}{a} \mathrm{e}^{at} c^b {}_2F_1 \left(\left[-b, \frac{a}{f} \right] \left[\frac{a}{f} + 1 \right], -\frac{d\mathrm{e}^{ft}}{c} \right).$$

Proof. This is an immediate consequence of equation 7.3.1.28 in [Prudnikov et al., 1998] which enables us to write the right hand side in terms of the incomplete beta function. The result follows immediately from derivation. □

4 Asymptotic analysis

4.1 Small and long term behaviour

In the following we want to analyse the short and long term behaviour of the instantaneous variance V_t given by

$$V_t = \frac{V_0 \mathrm{e}^{2at+2\sigma B_t}}{\left(1 + \alpha b V_0^{\frac{\alpha}{2}} \int_0^t \exp\left(\alpha\left(as + \sigma B_s\right)\right) ds\right)^{\frac{2}{\alpha}}}$$

and the variance swap given by

$$\mathrm{VS}\left(t\right) = \frac{1}{t}\int_0^t \mathbb{E}\left(\mathrm{e}^{2v_s}\right) ds = \frac{1}{t}\int_0^t \mathbb{E}\left(V_s\right) ds.$$

Before however turning to the short and long term behaviour we take a look at certain moments of V_t. First note that $Z_t := V_t^{-\frac{\alpha}{2}} = \mathrm{e}^{-\alpha v_t}$ and by the Ito formula we have

$$dZ_t = \left(\alpha b + \left(\frac{\alpha^2\sigma^2}{2} - \alpha a\right) Z_t\right) dt - \alpha\sigma Z_t dB_t.$$

These kind of processes are sometimes called Shiryaev processes or Wong processes and were intensively studied in [Donati-Martin et al., 2001] and [Peskir, 2006]. In [Donati-Martin et al., 2001] they also derived an expression for the corresponding resolvent of the process. Now let $M_t^{(l)}$ be the l-th moment of Z_t. Since

$$dZ_t^l = lZ_t^{l-1}\left(\left(\alpha b + \left(\frac{\alpha^2\sigma^2}{2} - \alpha a\right) Z_t\right) dt - \alpha\sigma dB_t\right) + \frac{l\left(l-1\right)}{2} Z_t^{l-2}\alpha^2\sigma^2 Z_t^2 dt$$

one immediately gets

$$dM_t^{(l)} = \left(\alpha b l M_t^{(l-1)} + \left(\frac{\alpha^2\sigma^2}{2} l - \alpha a l + \frac{l\left(l-1\right)}{2}\alpha^2\sigma^2\right) M_t^{(l)}\right) dt$$

which can now be recursively solved.

4.1.1 Short term behaviour

The following proposition generalizes proposition 3 in [Da Fonseca and Martini, 2014]. In order to get an explicit asymptotic expansion one still has to extract the corresponding coefficients.

Proposition 4.2. *For $\gamma > 0$ the short term behaviour of $\mathbb{E}\left(V_t^\gamma\right)$ is given by*

$$\mathbb{E}\left(V_t^\gamma\right) = V_0^\gamma \mathrm{e}^{2a\gamma t + 2\sigma^2\gamma^2 t} \sum_{n=0}^{N} \alpha^n b^n V_0^{\frac{\alpha n}{2}} \binom{-\frac{2\gamma}{\alpha}}{n} \mathrm{e}^{-\frac{\mu^2 t}{2}} \sum_{j=0}^{n} C\left(n, j; \mu\right) \exp\left(t\frac{\left(\mu + 2j\right)^2}{2}\right)$$
$$+ \mathcal{O}\left(t^{N+1}\right)$$

with $\mu = \frac{2\left(a+2\sigma^2\gamma\right)}{\alpha\sigma^2}$ *and* $N \in \mathbb{N}$ *and where* $C\left(0,0;\mu\right) = 1$ *and for* $j = 0, \ldots, n$

$$C\left(n,j;\mu\right) = 2^{-n}\left(-1\right)^{n-j}\binom{n}{j}\prod_{\substack{k\neq j\\0\leq k\leq n}}\left(\mu+j+k\right)^{-1}.$$

Proof. Let

$$\tilde{B}_t := B_t - 2\sigma\gamma t$$

and let $\tilde{\mathbb{P}}^t$, given by the Girsanov theorem, denote the probability measure under which $\left(\tilde{B}_s\right)_{0\leq s\leq t}$ is a Brownian motion. The density is therefore given by

$$\frac{d\tilde{\mathbb{P}}^t}{d\mathbb{P}} = \exp\left(2\sigma\gamma B_t - \frac{\left(2\sigma\gamma\right)^2}{2}t\right).$$

Therefore we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left(V_t^\gamma\right) &= \mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\frac{V_0^\gamma \mathrm{e}^{2a\gamma t+2\sigma\gamma B_t}}{\left(1+\alpha b V_0^{\frac{\alpha}{2}}\int_0^t\exp\left(\alpha\left(as+\sigma B_s\right)\right)ds\right)^{\frac{2\gamma}{\alpha}}}\mathrm{e}^{-2\sigma\gamma B_t+2\sigma^2\gamma^2 t}\right)\\ &= V_0^\gamma \mathrm{e}^{2a\gamma t+2\sigma^2\gamma^2 t}\mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\left(1+\alpha b V_0^{\frac{\alpha}{2}}\underbrace{\int_0^t\exp\left(\alpha\left(\left(a+2\sigma^2\gamma\right)s+\sigma\tilde{B}_s\right)\right)ds}_{=:I_t}\right)^{-\frac{2\gamma}{\alpha}}\right).\end{aligned}$$

Since we have

$$\left(1+x\right)^\beta = \sum_{n=0}^N\binom{\beta}{n}x^n + f\left(x\right)$$

with $f \in \mathcal{O}\left(x^{N+1}\right)$ as x tends to zero, we can conclude that

$$\mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\left(1+\alpha b V_0^{\frac{\alpha}{2}}I_t\right)^{-\frac{2\gamma}{\alpha}}\right) = \sum_{n=0}^N\alpha^n b^n V_0^{\frac{\alpha n}{2}}\binom{-\frac{2\gamma}{\alpha}}{n}\mathbb{E}^{\tilde{\mathbb{P}}^t}\left(I_t^n\right) + \mathbb{E}^{\tilde{\mathbb{P}}^t}\left(f\left(I_t\right)\right).$$

Note that I_t satisfies

$$\begin{aligned}I_t &= \int_0^t\exp\left(\alpha\left(\left(a+2\sigma^2\gamma\right)s+\sigma\tilde{B}_s\right)\right)ds\\ &\overset{d}{=} \frac{4}{\alpha^2\sigma^2}\int_0^{\frac{\alpha^2\sigma^2}{4}t}\exp\left(2\left(\frac{2\left(a+2\sigma^2\gamma\right)}{\alpha\sigma^2}u+\hat{B}_u\right)\right)du\\ &= \frac{4}{\alpha^2\sigma^2}A^{(\mu)}_{\frac{\alpha^2\sigma^2}{4}t},\end{aligned}$$

where we have used the notation of [Dufresne, 2000] with $\mu = \frac{2\left(a+2\sigma^2\gamma\right)}{\alpha\sigma^2}$,

$$A_t^{(\mu)} = \int_0^t \exp\left(2\mu s + 2\hat{B}_s\right) ds$$

and an appropriate Brownian motion $\hat{B}$. Now by theorem 5.2 in [Matsumoto and Yor, 2005a] we know that for $\mu \geq 0$ the moments of $A_t^{(\mu)}$ can be calculated via

$$\mathbb{E}\left(\left(A_t^{(\mu)}\right)^n\right) = \mathrm{e}^{-\frac{\mu^2 t}{2}} \sum_{j=0}^{n} C\left(n, j; \mu\right) \exp\left(t \frac{\left(\mu + 2j\right)^2}{2}\right),$$

where $C\left(0, 0; \mu\right) = 1$ and for $j = 0, \ldots, n$

$$C\left(n, j; \mu\right) = 2^{-n} \left(-1\right)^{n-j} \binom{n}{j} \prod_{\substack{k \neq j \\ 0 \leq k \leq n}} \left(\mu + j + k\right)^{-1}.$$

We are therefore left to verify that

$$\mathbb{E}^{\tilde{\mathbb{P}}^t}\left(f\left(I_t\right)\right) \in \mathcal{O}\left(t^{N+1}\right).$$

By definition we know that there exist $\delta_f, M_f > 0$ such that $\left|f\left(t\right)\right| \leq M_f t^{N+1}$ for all $t \leq \delta_f$. Therefore we have

$$\mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\left|f\left(I_t\right)\right|\right) = \mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\left|f\left(I_t\right)\right| \mathbb{1}_{\left\{I_t \leq \delta_f\right\}}\right) + \mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\left|f\left(I_t\right)\right| \mathbb{1}_{\left\{I_t > \delta_f\right\}}\right).$$

By construction and by the definition of I_t we have

$$\begin{aligned}
\mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\left|f\left(I_t\right)\right| \mathbb{1}_{\left\{I_t \leq \delta_f\right\}}\right) &\leq M_f \mathbb{E}^{\tilde{\mathbb{P}}^t}\left(I_t^{N+1} \mathbb{1}_{\left\{I_t \leq \delta_f\right\}}\right) \\
&\leq M_f \mathbb{E}^{\tilde{\mathbb{P}}^t}\left(I_t^{N+1}\right) \\
&\leq M_f \mathbb{E}^{\tilde{\mathbb{P}}^t}\left(t^{N+1} \exp\left(C_1 + C_2 \tilde{B}_1^*\right)\right) \in \mathcal{O}\left(t^{N+1}\right),
\end{aligned}$$

with constants $C_1, C_2 > 0$ for $t \in [0, 1]$. On the other hand we have

$$\begin{aligned}
\mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\left|f\left(I_t\right)\right| \mathbb{1}_{\left\{I_t > \delta_f\right\}}\right) &\leq \mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\left|f\left(I_t\right)\right|^2\right) \tilde{\mathbb{P}}^t\left(I_t > \delta_f\right) \\
&\leq C_3 \tilde{\mathbb{P}}^t\left(I_t > \delta_f\right),
\end{aligned}$$

with some constant $C_3 > 0$ for $t \in [0, 1]$. Note that the probability of I_t getting larger than an arbitrary small level gets exponentially small as t tends to 0. This can be easily seen since

$$
\begin{aligned}
&\tilde{\mathbb{P}}^t \left(\int_0^t \exp\left(\alpha \left(\left(a + 2\sigma^2 \gamma \right) s + \sigma \tilde{B}_s \right) \right) ds \geq \varepsilon \right) \\
&\leq \tilde{\mathbb{P}}^t \left(\int_0^t \exp\left(c_1 t + c_2 \sup_{0 \leq u \leq t} \tilde{B}_u \right) ds \geq \varepsilon \right) \\
&= \tilde{\mathbb{P}}^t \left(c_1 t + c_2 \sup_{0 \leq u \leq t} \tilde{B}_u \geq \ln\left(\frac{\varepsilon}{t} \right) \right) \\
&= \tilde{\mathbb{P}}^t \left(\sup_{0 \leq u \leq t} \tilde{B}_u \geq \frac{1}{c_2} \ln\left(\frac{\varepsilon}{t} \right) - \frac{c_1}{c_2} t \right) \\
&= 2\tilde{\mathbb{P}}^t \left(\tilde{B}_t \geq \frac{1}{c_2} \ln\left(\frac{\varepsilon}{t} \right) - \frac{c_1}{c_2} t \right) \\
&\leq \frac{2\sqrt{t}}{\sqrt{2\pi}} \frac{1}{\frac{1}{c_2} \ln\left(\frac{\varepsilon}{t} \right) - \frac{c_1}{c_2} t} \exp\left(- \frac{\left(\frac{1}{c_2} \ln\left(\frac{\varepsilon}{t} \right) - \frac{c_1}{c_2} t \right)^2}{2t} \right) \\
&\leq \exp\left(- \frac{1}{2t} \right),
\end{aligned}
$$

with positive constants c_1 and c_2, t sufficiently small and with the help of theorem 21.19 in [Klenke, 2007]. We have therefore proven the assertion. $\square$

In fact using proposition 4.2 with $N = 1$, $\gamma = 1$ and extracting the corresponding coefficients in the exponential one arrives at

Corollary 4.3. *The short term behaviour of $\mathbb{E}(V_t)$ and $\mathrm{VS}(t)$ is given by*

$$
\begin{aligned}
\mathbb{E}(V_t) &= V_0 \left(1 + \left(2a + 2\sigma^2 - 2bV_0^{\frac{\alpha}{2}} \right) t \right) + \mathcal{O}\left(t^2 \right), \\
\mathrm{VS}(t) &= V_0 \left(1 + \left(2a + 2\sigma^2 - 2bV_0^{\frac{\alpha}{2}} \right) \frac{t}{2} \right) + \mathcal{O}\left(t^2 \right)
\end{aligned}
$$

as $t \to 0$.

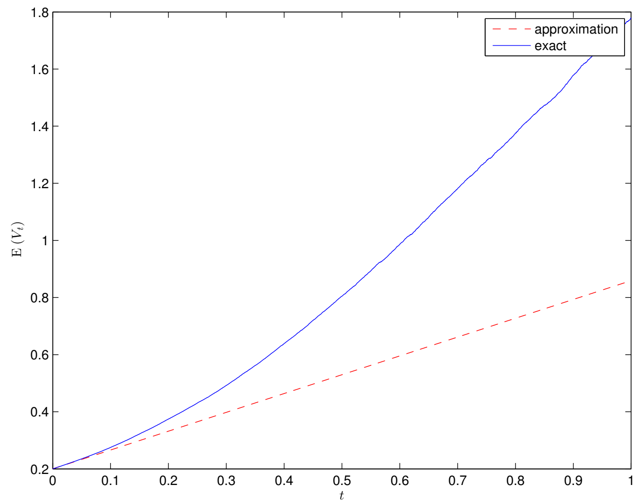

Figure 1: Short term behaviour of $t \mapsto \mathbb{E}(V_t)$ with initial instantaneous variance V_0 of 20% and model parameters given by $a = 0.8$, $b = 0.4$, $\alpha = 1.2$ and $\sigma = 1$. The exact expectation was calculated via simulation.

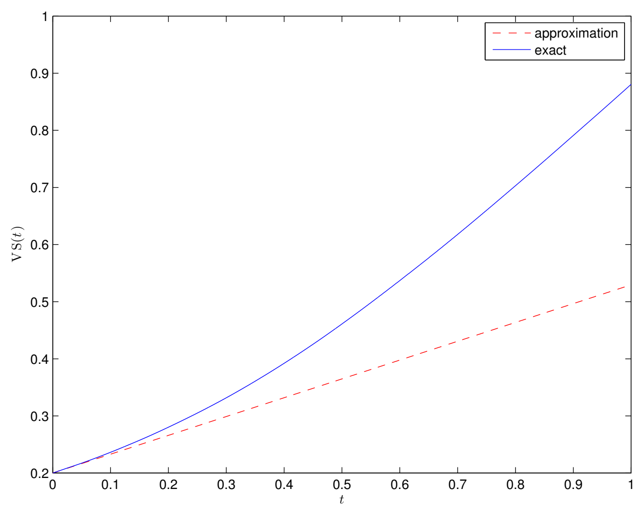

Figure 2: Short term behaviour of $t \mapsto \mathrm{VS}(t)$ with initial instantaneous variance V_0 of 20% and model parameters given by $a = 0.8$, $b = 0.4$, $\alpha = 1.2$ and $\sigma = 1$. The exact price was calculated via simulation.

4.3.1 Long term behaviour

We start with a quite useful lemma.

Lemma 4.4. *For $X \sim \Gamma(\alpha, \beta)$ and $\gamma \in (-\alpha, \infty)$ there holds*

$$\mathbb{E}(X^\gamma) = \frac{\Gamma(\alpha+\gamma)}{\beta^\gamma \Gamma(\alpha)}.$$

Proof. Straightforward calculations yield

$$\begin{aligned}
\mathbb{E}(X^\gamma) &= \int_0^\infty x^\gamma \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-\beta x} dx \\
&= \frac{\Gamma(\alpha+\gamma)}{\beta^\gamma \Gamma(\alpha)} \int_0^\infty \underbrace{\frac{\beta^{\alpha+\gamma}}{\Gamma(\alpha+\gamma)} x^{\alpha+\gamma-1} \mathrm{e}^{-\beta x}}_{\Gamma(\alpha+\gamma,\beta) \text{ density}} dx \\
&= \frac{\Gamma(\alpha+\gamma)}{\beta^\gamma \Gamma(\alpha)}.
\end{aligned}$$

□

Proposition 4.5. *For $a > 0$ and $Z \sim \Gamma(\mu, 1)$ with $\mu = \frac{2a}{\alpha\sigma^2}$ there holds*

$$V_t \xrightarrow{d} \left(\frac{2b}{\alpha\sigma^2}\right)^{-\frac{2}{\alpha}} Z^{\frac{2}{\alpha}},$$

as $t \to \infty$. Moreover there holds

$$\lim_{t\to\infty} \mathbb{E}(V_t) = \lim_{t\to\infty} \mathrm{VS}(t) = \left(\frac{2b}{\alpha\sigma^2}\right)^{-\frac{2}{\alpha}} \frac{\Gamma\left(\mu + \frac{2}{\alpha}\right)}{\Gamma(\mu)}.$$

Proof. We follow the ideas of section 2.1.9 in [Da Fonseca and Martini, 2014]. Note that by the law of large numbers we have

$$\lim_{t\to\infty} at + \sigma B_t = \infty$$

and since for all $\beta \in \mathbb{R}$ we have $(1+x)^\beta \sim x^\beta$ as $x \to \infty$, we can conclude that with

$$\begin{aligned}
V_t &= \frac{V_0 \mathrm{e}^{2at+2\sigma B_t}}{\left(1 + \alpha b V_0^{\frac{\alpha}{2}} \int_0^t \exp(\alpha(as + \sigma B_s))\, ds\right)^{\frac{2}{\alpha}}} \\
&= \underbrace{\frac{\left(\alpha b V_0^{\frac{\alpha}{2}} \int_0^t \exp(\alpha(as + \sigma B_s))\, ds\right)^{\frac{2}{\alpha}}}{\left(1 + \alpha b V_0^{\frac{\alpha}{2}} \int_0^t \exp(\alpha(as + \sigma B_s))\, ds\right)^{\frac{2}{\alpha}}}}_{=:X_t} \cdot \underbrace{\frac{V_0 \mathrm{e}^{2at+2\sigma B_t}}{\left(\alpha b V_0^{\frac{\alpha}{2}} \int_0^t \exp(\alpha(as + \sigma B_s))\, ds\right)^{\frac{2}{\alpha}}}}_{=:Y_t},
\end{aligned}$$

$X_t \xrightarrow{a.s.} 1$ as $t \to \infty$. On the other hand we can rewrite the process Y by time reversal and using an appropriate Brownian motion $\tilde{B}$ as

$$\begin{aligned}
Y_t &= \frac{V_0 \mathrm{e}^{2at+2\sigma B_t}}{\left(\alpha b V_0^{\frac{\alpha}{2}} \int_0^t \exp\left(\alpha\left(as + \sigma B_s\right)\right) ds\right)^{\frac{2}{\alpha}}} \\
&= \frac{1}{\left(\alpha b \int_0^t \exp\left(\alpha a\left(s - t\right) + \alpha\sigma\left(B_s - B_t\right)\right) ds\right)^{\frac{2}{\alpha}}} \\
&\stackrel{d}{=} \frac{1}{\left(\alpha b \int_0^t \exp\left(-\alpha a\left(t - s\right) - \alpha\sigma B_{t-s}\right) ds\right)^{\frac{2}{\alpha}}} \\
&\stackrel{d}{=} \frac{1}{\left(\alpha b \frac{4}{\alpha^2\sigma^2} \int_0^{\frac{\alpha^2\sigma^2}{4}t} \exp\left(2\left(-\frac{2a}{\alpha\sigma^2}u + \tilde{B}_u\right)\right) du\right)^{\frac{2}{\alpha}}} \\
&= \left(\frac{2b}{\alpha\sigma^2}\right)^{-\frac{2}{\alpha}} \left(\frac{1}{2A^{(-\mu)}_{\frac{\alpha^2\sigma^2}{4}t}}\right)^{\frac{2}{\alpha}},
\end{aligned}$$

where we again used the notation of [Dufresne, 2000] with $\mu = \frac{2a}{\alpha\sigma^2}$. From theorem C in [Dufresne, 2000] we know that for any $\mu > 0$, $\frac{1}{2A_\infty^{(-\mu)}} \sim \Gamma\left(\mu, 1\right)$. Since both X_t and Y_t converge in distribution and since $X_t \xrightarrow{a.s.} 1$, Slutsky's theorem and the continuous mapping theorem yield the first assertion.

In order to prove the second assertion first note that $X_t \leq 1$. Since $t \mapsto A_t^{(\mu)}$ is monotone we can apply the monotone convergence theorem in the following computations and together with lemma 4.4 we get

$$\mathbb{E}\left(V_t\right) = \mathbb{E}\left(X_t Y_t\right) \leq \mathbb{E}\left(Y_t\right) = \mathbb{E}\left(\left(\frac{2b}{\alpha\sigma^2}\right)^{-\frac{2}{\alpha}} \left(\frac{1}{2A^{(-\mu)}_{\frac{\alpha^2\sigma^2}{4}t}}\right)^{\frac{2}{\alpha}}\right) \xrightarrow{t\to\infty} \left(\frac{2b}{\alpha\sigma^2}\right)^{-\frac{2}{\alpha}} \frac{\Gamma\left(\mu + \frac{2}{\alpha}\right)}{\Gamma\left(\mu\right)}.$$

On the other hand by applying the reverse Hölder inequality for every $r > 1$ and again by the monotone convergence theorem we arrive at

$$\mathbb{E}\left(V_t\right) = \mathbb{E}\left(X_t Y_t\right) \geq \mathbb{E}\left(X_t^{-\frac{1}{r-1}}\right)^{-(r-1)} \mathbb{E}\left(Y_t^{\frac{1}{r}}\right)^r \xrightarrow{t\to\infty} \left(\frac{2b}{\alpha\sigma^2}\right)^{-\frac{2}{\alpha r}} \frac{\Gamma\left(\mu + \frac{2}{\alpha r}\right)^r}{\Gamma\left(\mu\right)}.$$

Since this holds for every $r > 1$ we have proven that

$$\lim_{t\to\infty} \mathbb{E}\left(V_t\right) = \left(\frac{2b}{\alpha\sigma^2}\right)^{-\frac{2}{\alpha}} \frac{\Gamma\left(\mu + \frac{2}{\alpha}\right)}{\Gamma\left(\mu\right)}.$$

The fact that

$$\lim_{t\to\infty} \mathrm{VS}\left(t\right) = \lim_{t\to\infty} \mathbb{E}\left(V_t\right)$$

is an immediate consequence of L'Hôpital's rule. $\square$

Remark 4.6. Note that the proof of proposition 4.5 stays valid if we replace X_t and Y_t with X_t^γ and Y_t^γ respectively for every $\gamma > 0$. We have therefore proven

Corollary 4.7. *For $a > 0$ and $\gamma > 0$ there holds*

$$\lim_{t\to\infty} \mathbb{E}\left(V_t^\gamma\right) = \left(\frac{2b}{\alpha\sigma^2}\right)^{-\frac{2\gamma}{\alpha}} \frac{\Gamma\left(\mu + \frac{2\gamma}{\alpha}\right)}{\Gamma\left(\mu\right)}$$

with $\mu = \frac{2a}{\alpha\sigma^2}$.

Remark 4.8. For simplicity note that with $\alpha = 1$ we have

$$\lim_{t\to\infty} \mathbb{E}\left(V_t\right) = \lim_{t\to\infty} \mathrm{VS}\left(t\right) = \left(\frac{2b}{\sigma^2}\right)^{-2} \frac{\Gamma\left(\mu+2\right)}{\Gamma\left(\mu\right)} = \left(\frac{\sigma^2}{2b}\right)^2 \frac{2a}{\sigma^2}\left(\frac{2a}{\sigma^2}+1\right),$$

and in the case $\alpha = 2$

$$\lim_{t\to\infty} \mathbb{E}\left(V_t\right) = \lim_{t\to\infty} \mathrm{VS}\left(t\right) = \left(\frac{b}{\sigma^2}\right)^{-1} \frac{\Gamma\left(\mu+1\right)}{\Gamma\left(\mu\right)} = \frac{a}{b}.$$

Proposition 4.9. *For $a < 0$ and $\gamma > 0$ there holds*

$$V_t^\gamma \xrightarrow{a.s.} 0,$$

as $t \to \infty$. If moreover $a < -\sigma^2\gamma$ then

$$\lim_{t\to\infty} \mathbb{E}\left(V_t^\gamma\right) = 0.$$

Proof. Note that by the law of large numbers we have

$$\lim_{t\to\infty} at + \sigma B_t = -\infty$$

therefore

$$V_0 \mathrm{e}^{2at+2\sigma B_t} \xrightarrow{a.s.} 0$$

and since

$$\frac{1}{\left(1 + \alpha b V_0^{\frac{\alpha}{2}} \int_0^t \exp\left(\alpha\left(as + \sigma B_s\right)\right) ds\right)^{\frac{2}{\alpha}}} \le 1,$$

we have proven the first assertion.

The second assertion follows immediately from the fact that

$$\begin{aligned} \mathbb{E}\left(V_t^\gamma\right) &\le \mathbb{E}\left(V_0^\gamma \mathrm{e}^{2a\gamma t + 2\sigma\gamma B_t}\right) \\ &= V_0^\gamma \mathrm{e}^{2a\gamma t + \frac{4\sigma^2\gamma^2}{2}t}. \end{aligned}$$

□

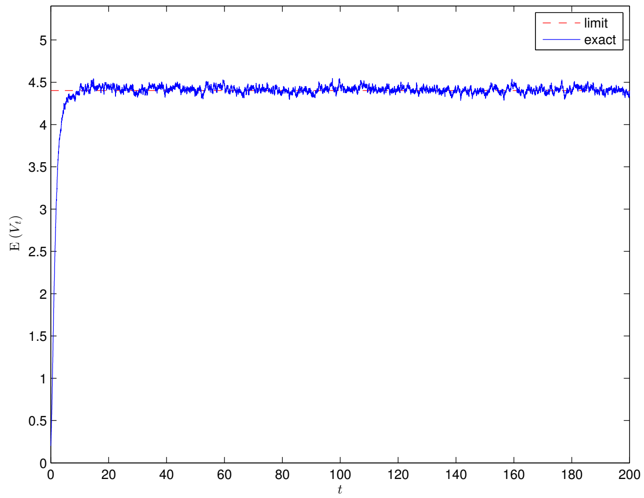

Figure 3: Long term behaviour of $t \mapsto \mathbb{E}(V_t)$ with initial instantaneous variance V_0 of 20% and model parameters given by $a = 0.8$, $b = 0.4$, $\alpha = 1.2$ and $\sigma = 1$. The exact expectation was calculated via simulation.

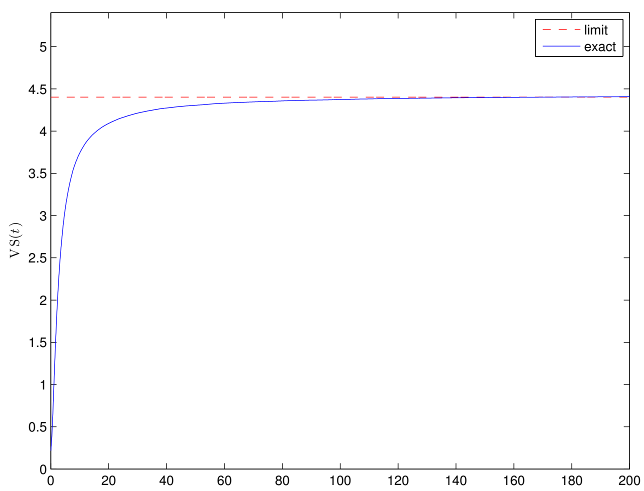

Figure 4: Long term behaviour of $t \mapsto \mathrm{VS}(t)$ with initial instantaneous variance V_0 of 20% and model parameters given by $a = 0.8$, $b = 0.4$, $\alpha = 1.2$ and $\sigma = 1$. The exact price was calculated via simulation.

4.10 Deterministic Volatility

An original result of this thesis concerning the case of deterministic volatility is the following

Proposition 4.11. *In the α-hypergeometric model with $\sigma = 0$ and $a \neq 0$ the log volatility v is given by*

$$v_t = v_0 + at - \frac{1}{\alpha} \ln \left(1 + \alpha b \int_0^t \exp\left(\alpha\left(v_0 + as\right)\right) ds \right)$$
$$= v_0 + at - \frac{1}{\alpha} \ln \left(1 + \frac{b\mathrm{e}^{\alpha v_0}}{a} \left(\mathrm{e}^{\alpha a t} - 1\right) \right).$$

Furthermore the value at time t of a European call with maturity T and strike K of the asset S is given by $\mathcal{BS}\left(S_t, \Sigma_t, t\right)$, where

$$\mathcal{BS}\left(x, \sigma, t\right) = x\Phi\left(d_+\left(\frac{x}{K}, T - t\right)\right) - K\Phi\left(d_-\left(\frac{x}{K}, T - t\right)\right),$$

with

$$d_\pm\left(y, u\right) = \frac{1}{\sqrt{\sigma^2 u}} \ln\left(y\right) \pm \frac{\sqrt{\sigma^2 u}}{2}$$

and

$$\begin{aligned}
{\Sigma_t}^2 &= \frac{1}{T-t} \int_t^T \mathrm{e}^{2v_s} ds \\
&= \frac{\mathrm{e}^{2v_0} \hat{c}^{\hat{b}}}{\left(T-t\right)\hat{a}} \left(\mathrm{e}^{\hat{a}T} {}_2F_1\left(\left[-\hat{b}, \frac{\hat{a}}{\hat{f}}\right] \left[\frac{\hat{a}}{\hat{f}} + 1\right], -\frac{\hat{d}\mathrm{e}^{\hat{f}T}}{\hat{c}} \right) - \mathrm{e}^{\hat{a}t} {}_2F_1\left(\left[-\hat{b}, \frac{\hat{a}}{\hat{f}}\right] \left[\frac{\hat{a}}{\hat{f}} + 1\right], -\frac{\hat{d}\mathrm{e}^{\hat{f}t}}{\hat{c}} \right) \right)
\end{aligned}$$

with constants $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ and $\hat{f}$ given by

$$\hat{a} = 2a \qquad \hat{b} = -\frac{2}{\alpha} \qquad \hat{c} = 1 - \frac{b\mathrm{e}^{\alpha v_0}}{a} \qquad \hat{d} = \frac{b\mathrm{e}^{\alpha v_0}}{a} \qquad \hat{f} = \alpha a.$$

Proof. The first assertion follows immediately from (4) with $\sigma = 0$ and elementary integration. The second assertion follows from theorem 7.1 and 7.3, therefore we are left with the explicit computation of Σ_t. Using lemma 3.1 we arrive at

$$\begin{aligned}
{\Sigma_t}^2 &= \frac{1}{T-t} \int_t^T \mathrm{e}^{2v_s} ds \\
&= \frac{\mathrm{e}^{2v_0}}{T-t} \int_t^T \mathrm{e}^{2at} \left(1 + \frac{b\mathrm{e}^{\alpha v_0}}{a}\left(\mathrm{e}^{\alpha a s} - 1\right)\right)^{-\frac{2}{\alpha}} ds \\
&= \frac{\mathrm{e}^{2v_0}}{T-t} \int_t^T \mathrm{e}^{2as} \left(1 - \frac{b\mathrm{e}^{\alpha v_0}}{a} + \frac{b\mathrm{e}^{\alpha v_0}}{a}\mathrm{e}^{\alpha a s}\right)^{-\frac{2}{\alpha}} ds \\
&= \frac{\mathrm{e}^{2v_0}}{T-t} \int_t^T \mathrm{e}^{\hat{a}s} \left(\hat{c} + \hat{d}\mathrm{e}^{\hat{f}s}\right)^{\hat{b}} ds \\
&= \frac{\mathrm{e}^{2v_0} \hat{c}^{\hat{b}}}{\left(T-t\right)\hat{a}} \left(\mathrm{e}^{\hat{a}T} {}_2F_1\left(\left[-\hat{b}, \frac{\hat{a}}{\hat{f}}\right] \left[\frac{\hat{a}}{\hat{f}} + 1\right], -\frac{\hat{d}\mathrm{e}^{\hat{f}T}}{\hat{c}} \right) - \mathrm{e}^{\hat{a}t} {}_2F_1\left(\left[-\hat{b}, \frac{\hat{a}}{\hat{f}}\right] \left[\frac{\hat{a}}{\hat{f}} + 1\right], -\frac{\hat{d}\mathrm{e}^{\hat{f}t}}{\hat{c}} \right) \right)
\end{aligned}$$

which concludes the proof. ☐

Remark 4.12. Proposition 4.11 enables us to explicitly calculate the value of a European call option in terms of hypergeometric functions. Note that for efficient computations it is however necessary to calculate ${}_2F_1$ in a fast and efficient manner. As an alternative solution one can still approximate the integral representations of Σ_t^2.

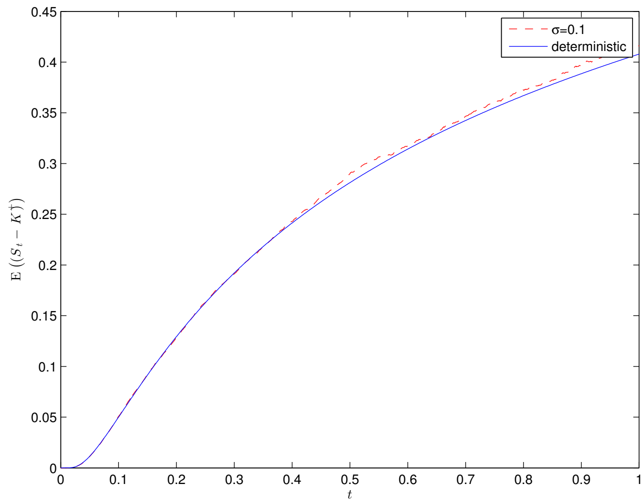

Figure 5: Plot of price of a European call option with $\sigma = 0.1$ in red and approximation with deterministic volatility. The exact expectation was calculated via simulation.

4.13 Large deviation problems

In the following we are going to derive certain large deviation problems associated with the α-hypergeometric volatility model. We follow the ideas of [Forde and Jacquier, 2011]. They derived a small-time behaviour of the log forward for quite general stochastic volatility models using Freidlin-Wentzell theory. Applying theorem 1.1 in [Forde and Jacquier, 2011] directly was however not possible but we were able to derive a small-time behaviour in the uncorrelated model by following the ideas of their proof. We also derived a result concerning the small-time behaviour of the log forward in the correlated model. We heavily use the results of section 2.2.1 in [Peithmann, 2007] in order to derive the necessary LDPs. These results are original results of this thesis.

Theorem 4.14. *In the α-hypergeometric model with $\rho = 0$ we have the small-time behaviour*

$$-\lim_{t\to 0} t \ln\left(\mathbb{P}\left(X_t \geq x_1\right)\right) = \frac{1}{2\sigma^2} \operatorname{arccosh}\left(\frac{\sqrt{\sigma^2 x_1^2 + y_0^2}}{y_0}\right)^2$$

for the log forward price $X_t = \ln\left(S_t\right)$ and for $x_1 > 0$ where $y_0 = \mathrm{e}^{v_0}$.

Proof. First we consider the dynamics of $X_t = \ln(S_t)$ and $Y_t = \mathrm{e}^{v_t}$ which are given by

$$\begin{aligned} dX_t &= -\frac{1}{2}Y_t^2 dt + Y_t dW_t, \\ dY_t &= Y_t\left(a + \frac{\sigma^2}{2} - bY_t^\alpha\right)dt + \sigma Y_t dB_t. \end{aligned}$$

Consider now the time changed processes $X_t^\varepsilon := X_{\varepsilon t}$ and $Y_t^\varepsilon := Y_{\varepsilon t}$ which then satisfy the SDEs

$$\begin{aligned} dX_t^\varepsilon &= -\varepsilon\frac{1}{2}Y_t^{\varepsilon 2} dt + \sqrt{\varepsilon} Y_t^\varepsilon dW_t, \\ dY_t^\varepsilon &= \varepsilon Y_t^\varepsilon\left(a + \frac{\sigma^2}{2} - bY_t^{\varepsilon\alpha}\right)dt + \sqrt{\varepsilon}\sigma Y_t^\varepsilon dB_t. \end{aligned}$$

Since the assumptions 2.6 in [Peithmann, 2007] are satisfied we know that $(X_t^\varepsilon, Y_t^\varepsilon)_{t\in[0,1]}$ satisfies a LDP on the pathspace $H^1_{(0,y_0)}([0,1];\mathbb{R}^2)$ with rate function I given by

$$I(x,y) = \frac{1}{2}\int_0^1 \frac{\dot{x}(t)}{y^2(t)} + \frac{\dot{y}(t)}{\sigma^2 y^2(t)} dt.$$

See theorem 2.9 in [Peithmann, 2007]. Now by the contraction principle applied to the point evaluation $(X_t^\varepsilon)_{t\in[0,1]} \mapsto X_1^\varepsilon$, we know that X_1^ε and therefore X_ε satisfies a small-time LDP with corresponding rate function given by

$$\iota(x_1) = \inf_{\substack{(x,y)\in H^1_{(0,y_0)}([0,1];\mathbb{R}^2), \\ x(1)=x_1}} \frac{1}{2}\int_0^1 \frac{\dot{x}}{y^2} + \frac{\dot{y}}{\sigma^2 y^2} dt.$$

Note that the above minimization problem is a well known problem in differential geometry. It is just the minimization of the energy functional on the Riemannian manifold $\mathbb{H}^2 := \{(x,y)\in\mathbb{R}^2 : y > 0\}$ with the Riemannian metric g given by

$$g = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{\sigma^2 y^2} \end{pmatrix}.$$

Since the space $\mathbb{H}^2$ is geodesically complete we just need to solve the geodesic equations. We introduce the functional $\hat{\iota}$ given by

$$\hat{\iota}(x_1, y_1) = \inf_{\substack{(x,y)\in H^1_{(0,y_0)}([0,1];\mathbb{R}^2), \\ x(1)=x_1, y(1)=y_1}} \frac{1}{2}\int_0^1 \frac{\dot{x}}{y^2} + \frac{\dot{y}}{\sigma^2 y^2} dt$$

We just have to find the geodesics connecting the points $(0, y_0), (x_1, y_1) \in \mathbb{H}^2$. In order to derive the geodesic equations we need the corresponding non-zero Christoffel symbols which are given by

$$\Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{y}, \qquad \Gamma^2_{11} = \frac{\sigma^2}{y}, \qquad \Gamma^2_{22} = -\frac{1}{y}.$$

The geodesic equations are hence given by

$$\ddot{x}y = 2\dot{x}\dot{y},$$
$$\ddot{y}y = \dot{y}^2 - \sigma^2\dot{x}^2.$$

One can easily verify that the general solutions are given by

$$x : t \mapsto a \tanh(bt + c) + d, \tag{10}$$
$$y : t \mapsto \frac{a\sigma}{\cosh(bt + c)}, \tag{11}$$

with $a, b, c, d \in \mathbb{R}$ chosen according to the initial and final conditions $x(0) = x_0, x(1) = x_1, y(0) = y_0$ and $y(1) = y_1$. Simple calculations then show that

$$\frac{1}{2}\int_0^1 \frac{\dot{x}}{y^2} + \frac{\dot{y}}{\sigma^2 y^2} dt = \frac{1}{2}\frac{b^2}{\sigma^2}$$

for x, y given by (10) and (11). Solving for the constants a, b, c and d one can calculate that b is given by

$$\ln\left(\frac{\sigma^2(x_0 - x_1)^2 + y_0^2 + y_1^2 + \sqrt{\left(\sigma^2(x_0 - x_1)^2 + (y_0 + y_1)^2\right)\left(\sigma^2(x_0 - x_1)^2 + (y_0 - y_1)^2\right)}}{2y_0y_1}\right),$$

which can be simplified, using the identity $\operatorname{arccosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$, to

$$b = \operatorname{arccosh}\left(1 + \frac{\sigma^2(x_0 - x_1)^2 + (y_0 - y_1)^2}{2y_0y_1}\right).$$

Hence we have

$$\hat{\imath}(x_1, y_1) = \frac{1}{2\sigma^2}\operatorname{arccosh}\left(1 + \frac{\sigma^2 x_1^2 + (y_0 - y_1)^2}{2y_0y_1}\right)^2.$$

In the case of $x_1 = 0$ we simply arrive at

$$\iota(x_1) = \inf_{y_1 > 0} \hat{\imath}(x_1, y_1) = 0.$$

In the case of $x_1 \neq 0$ simple calculations show that

$$\iota(x_1) = \inf_{y_1 > 0} \hat{\imath}(x_1, y_1) = \hat{\imath}\left(x_1, \sqrt{\sigma^2 x_1^2 + y_0^2}\right) = \frac{1}{2\sigma^2}\operatorname{arccosh}\left(\frac{\sqrt{\sigma^2 x_1^2 + y_0^2}}{y_0}\right)^2.$$

Since ι is continuous we have proven the assertion.

$\square$

Remark 4.15. The space $\mathbb{H}^2$ from the proof of theorem 4.14 is known as the hyperbolic half-plane or Poincarè half-plane.

Corollary 4.16. *In the uncorrelated α-hypergeometric model we have the following small-time behaviour for out-of-the-money put/call options on S_t*

$$-\lim_{t\to 0} t\ln\left(\mathbb{E}\left(S_t-K\right)^+\right)=\frac{1}{2\sigma^2}\operatorname{arccosh}\left(\frac{\sqrt{\sigma^2\ln\left(K\right)^2+y_0^2}}{y_0}\right)^2\qquad (K>1)\,,$$

$$-\lim_{t\to 0} t\ln\left(\mathbb{E}\left(K-S_t\right)^+\right)=\frac{1}{2\sigma^2}\operatorname{arccosh}\left(\frac{\sqrt{\sigma^2\ln\left(K\right)^2+y_0^2}}{y_0}\right)^2\qquad (K<1)\,,$$

with $y_0=\mathrm{e}^{v_0}$.

Proof. Simply apply corollary 1.2 in [Forde and Jacquier, 2011]. □

A simple use of the put-call parity immediately yields

Corollary 4.17. *In the uncorrelated α-hypergeometric model*[1] *for $K>0$ we have the following small-time behaviour*

$$\begin{aligned}-\lim_{t\to 0} t\ln\left(\mathbb{E}\left(S_t-K\right)^+-\left(1-K\right)^+\right)&=-\lim_{t\to 0} t\ln\left(\mathbb{E}\left(K-S_t\right)^+-\left(K-1\right)^+\right)\\&=\frac{1}{2\sigma^2}\operatorname{arccosh}\left(\frac{\sqrt{\sigma^2\ln\left(K\right)^2+y_0^2}}{y_0}\right)^2\end{aligned}$$

with $y_0=\mathrm{e}^{v_0}$.

Corollary 4.18. *In the uncorrelated α-hypergeometric model we have the following asymptotic behaviour for the implied volatility $\sigma_t\left(x\right)$ of a European call option on $S_t=\mathrm{e}^{X_t}$, with strike $K=\mathrm{e}^x$, as $t\to 0$*

$$I\left(x\right)=\lim_{t\to 0}\sigma_t\left(x\right)=\frac{x}{\frac{1}{\sigma}\operatorname{arccosh}\left(\frac{\sqrt{\sigma^2x^2+y_0^2}}{y_0}\right)}$$

with $y_0=\mathrm{e}^{v_0}$.

Proof. Simply apply corollary 1.4 in [Forde and Jacquier, 2011]. □

[1] note that $S_0=1$.

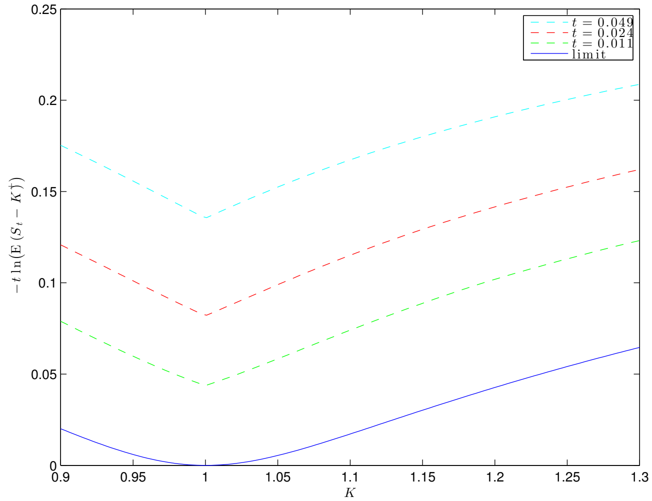

Figure 6: Plot of $K \mapsto -t\ln\left(\mathbb{E}\left(S_t - K\right)^+\right)$ for different values of t with initial instantaneous variance V_0 of 20% and model parameters given by $a = 3$, $b = 0.4$, $\alpha = 1$, $\sigma = 5$ and $\rho = 0$.

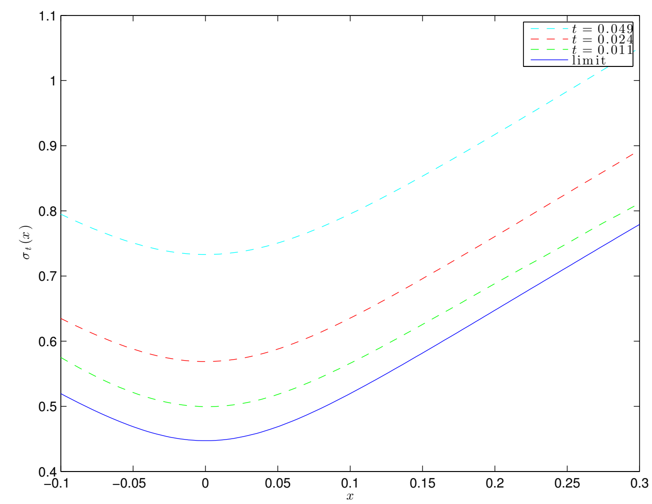

Figure 7: Plot of $x \mapsto \sigma_t\left(x\right)$ for different values of t with initial instantaneous variance V_0 of 20% and model parameters given by $a = 3$, $b = 0.4$, $\alpha = 1$, $\sigma = 5$ and $\rho = 0$.

Remark 4.19. A natural question to ask is whether theorem 4.14 can be generalized to the case $\rho \neq 0$. In order to follow the proof directly we first rewrite the SDEs as

$$\begin{aligned} dX_t &= -\frac{1}{2}Y_t^2 dt + \rho Y_t dB_t + \overline{\rho} Y_t dB_t^{\perp}, \\ dY_t &= Y_t\left(a + \frac{\sigma^2}{2} - bY_t^{\alpha}\right) dt + \sigma Y_t dB_t. \end{aligned}$$

by introducing the standard Brownian motion $\left(B_t, B_t^{\perp}\right)_{t\geq 0}$ and $\overline{\rho} = \sqrt{1-\rho^2}$. One then arrives at the Riemannian metric

$$g = \begin{pmatrix} \frac{1}{\overline{\rho}^2 y^2} & -\frac{\rho}{\sigma\overline{\rho}^2 y^2} \\ -\frac{\rho}{\sigma\overline{\rho}^2 y^2} & \frac{1}{\sigma^2\overline{\rho}^2 y^2} \end{pmatrix}.$$

Explicitly calculating the corresponding geodesics was however not possible.

Another approach for generalizing theorem 4.14 led to

Theorem 4.20. *In the α-hypergeometric model we have the small-time behaviour*

$$-\lim_{t\to 0} t \ln\left(\mathbb{P}\left(X_t \geq x_1\right)\right) = \frac{1}{2\sigma^2} \operatorname{arccosh}\left(1 + \frac{\frac{\sigma^2}{\overline{\rho}^2}\left(x_1 - \frac{\rho}{\sigma}\left(y_1^\star - y_0\right)\right)^2 + \left(y_0 - y_1^\star\right)^2}{2y_0 y_1^\star}\right)^2$$

where $y_1^\star$ is give by

$$y_1^\star = \sqrt{\left(\sigma x_1 + \rho y_0\right)^2 + \overline{\rho}^2 y_0^2},$$

for the log forward price $X_t = \ln\left(S_t\right)$ and for $x_1 > 0$ where $y_0 = \mathrm{e}^{v_0}$.

Proof. Analogously to the proof of theorem 4.14 and with the functions

$$\begin{aligned} \mu_1(y) &:= -\frac{1}{2}y^2, \\ \mu_2(y) &:= y\left(a + \frac{\sigma^2}{2} - by^{\alpha}\right) \end{aligned}$$

we write the SDEs as

$$\begin{aligned} dX_t &= \mu_1(Y_t)\, dt + \rho Y_t dB_t + \overline{\rho} Y_t dB_t^{\perp}, \\ dY_t &= \mu_2(Y_t)\, dt + \sigma Y_t dB_t, \end{aligned}$$

which we can also rewrite as

$$\begin{aligned} dX_t &= \mu_1(Y_t)\, dt + \rho Y_t dB_t + dZ_t, \\ dY_t &= \mu_2(Y_t)\, dt + \sigma Y_t dB_t, \\ dZ_t &= \overline{\rho} Y_t dB_t^{\perp}. \end{aligned}$$

Now the processes $X_t^\varepsilon := X_{\varepsilon t}$, $Y_t^\varepsilon := Y_{\varepsilon t}$ and $Z_t^\varepsilon := Z_{\varepsilon t}$ satisfy the SDEs

$$\begin{aligned} dX_t^\varepsilon &= \varepsilon\mu_1\left(Y_t^\varepsilon\right)dt + \sqrt{\varepsilon}\rho Y_t^\varepsilon dB_t + dZ_t^\varepsilon, \\ dY_t^\varepsilon &= \varepsilon\mu_2\left(Y_t^\varepsilon\right)dt + \sqrt{\varepsilon}\sigma Y_t^\varepsilon dB_t, \\ dZ_t^\varepsilon &= \sqrt{\varepsilon}\overline{\rho}Y_t^\varepsilon dB_t^\perp. \end{aligned}$$

Again by theorem 2.9 in [Peithmann, 2007] we know that $(Z_t^\varepsilon, Y_t^\varepsilon)_{t\in[0,1]}$ satisfies a LDP on the pathspace $H^1_{(0,y_0)}\left([0,1];\mathbb{R}^2\right)$ with rate function I given by

$$I\left(z,y\right) = \frac{1}{2}\int_0^1 \frac{\dot{z}\left(t\right)}{\overline{\rho}^2 y^2\left(t\right)} + \frac{\dot{y}\left(t\right)}{\sigma^2 y^2\left(t\right)}dt.$$

In fact we know that the processes $X^\varepsilon, Y^\varepsilon$ and Z^ε are exponentially equivalent to the processes $\tilde{X}^\varepsilon, \tilde{Y}^\varepsilon$ and $\tilde{Z}^\varepsilon$ given by

$$\begin{aligned} d\tilde{X}_t^\varepsilon &= \sqrt{\varepsilon}\rho\tilde{Y}_t^\varepsilon dB_t + d\tilde{Z}_t^\varepsilon, \\ d\tilde{Y}_t^\varepsilon &= \sqrt{\varepsilon}\sigma\tilde{Y}_t^\varepsilon dB_t, \\ d\tilde{Z}_t^\varepsilon &= \sqrt{\varepsilon}\overline{\rho}\tilde{Y}_t^\varepsilon dB_t^\perp, \end{aligned}$$

See theorem 2.7 and theorem 2.9 in [Peithmann, 2007]. Now note that

$$\begin{aligned} X_t^\varepsilon &= \sqrt{\varepsilon}\int_0^t \rho Y_t^\varepsilon dB_t + Z_t^\varepsilon \\ &= \frac{\rho}{\sigma}\left(Y_t^\varepsilon - y_0\right) + Z_t^\varepsilon. \end{aligned}$$

Now using the contraction principle with the continuous function

$$\left(Z_t^\varepsilon, Y_t^\varepsilon\right)_{t\in[0,1]} \mapsto \left(\frac{\rho}{\sigma}\left(Y_t^\varepsilon - y_0\right) + Z_t^\varepsilon\right)_{t\in[0,1]} = \left(\tilde{X}_t^\varepsilon\right)_{t\in[0,1]}$$

we know that $\left(\tilde{X}_t^\varepsilon\right)_{t\in[0,1]}$ satisfies a small-time LDP with corresponding rate function given by

$$\tilde{I}\left(x\right) = \inf_{\substack{(z,y)\in H^1_{(0,y_0)}\left([0,1];\mathbb{R}^2\right), \\ x=\frac{\rho}{\sigma}(y-y_0)+z}} I\left(z,y\right).$$

Analogously X_1^ε satisfies a LDP with rate function ι defined as

$$\begin{aligned} \iota\left(x_1\right) &= \inf_{\substack{x\in H^1_0([0,1];\mathbb{R}), \\ x(1)=x_1}} \tilde{I}\left(x\right) \\ &= \inf_{\substack{x\in H^1_0([0,1];\mathbb{R}), \\ x(1)=x_1}} \left\{ \inf_{\substack{(z,y)\in H^1_{(0,y_0)}\left([0,1];\mathbb{R}^2\right), \\ x=\frac{\rho}{\sigma}(y-y_0)+z}} I\left(z,y\right) \right\}. \end{aligned}$$

Now ι satisfies the following trivial inequalities

$$\begin{aligned}
\iota(x_1) &= \inf_{\substack{x\in H_0^1([0,1];\mathbb{R}),\\ x(1)=x_1}} \left\{ \inf_{\substack{(z,y)\in H^1_{(0,y_0)}\left([0,1];\mathbb{R}^2\right),\\ x=\frac{\rho}{\sigma}(y-y_0)+z}} I(z,y) \right\} \\
&\geq \inf_{\substack{x\in H_0^1([0,1];\mathbb{R}),\\ x(1)=x_1}} \left\{ \inf_{\substack{(z,y)\in H^1_{(0,y_0)}\left([0,1];\mathbb{R}^2\right),\\ x(1)=\frac{\rho}{\sigma}(y(1)-y_0)+z(1)}} I(z,y) \right\} \\
&\geq \inf_{\substack{(z,y)\in H^1_{(0,y_0)}\left([0,1];\mathbb{R}^2\right),\\ x_1=\frac{\rho}{\sigma}(y(1)-y_0)+z(1)}} I(z,y) \\
&= \inf_{\substack{z_1,y_1,\\ x_1=\frac{\rho}{\sigma}(y_1-y_0)+z_1}} \left\{ \inf_{\substack{(z,y)\in H^1_{(0,y_0)}\left([0,1];\mathbb{R}^2\right),\\ z(1)=z_1,y(1)=y_1}} I(z,y) \right\}.
\end{aligned} \tag{12}$$

Note that we are already able to calculate the right hand side as in the proof of theorem 4.14. We have

$$\inf_{\substack{(z,y)\in H^1_{(0,y_0)}\left([0,1];\mathbb{R}^2\right),\\ z(1)=z_1,y(1)=y_1}} I(z,y) = \frac{1}{2\sigma^2}\operatorname{arccosh}\left(1+\frac{\frac{\sigma^2}{\overline{\rho}^2}z_1^2+(y_0-y_1)^2}{2y_0y_1}\right)^2$$

and the infimum is attained at the corresponding geodesics. Therefore the inequalities in (12) are actually equalities, because we can just choose x to be the linear combination of these geodesics. We arrive at

$$\begin{aligned}
\iota(x_1) &= \inf_{\substack{z_1,y_1,\\ x_1=\frac{\rho}{\sigma}(y_1-y_0)+z_1}} \frac{1}{2\sigma^2}\operatorname{arccosh}\left(1+\frac{\frac{\sigma^2}{\overline{\rho}^2}z_1^2+(y_0-y_1)^2}{2y_0y_1}\right)^2 \\
&= \inf_{y_1\geq 0}\frac{1}{2\sigma^2}\operatorname{arccosh}\left(1+\frac{\frac{\sigma^2}{\overline{\rho}^2}\left(x_1-\frac{\rho}{\sigma}(y_1-y_0)\right)^2+(y_0-y_1)^2}{2y_0y_1}\right)^2.
\end{aligned}$$

An elementary calculation shows that the above infimum is attained at

$$y_1^\star := \sqrt{(\sigma x_1+\rho y_0)^2+\overline{\rho}^2y_0^2},$$

which proves the assertion. $\square$

Note that we can now derive the same results obtained in corollaries 4.16, 4.17 and 4.18 in the case of the correlated model. We therefore obtain the following corollaries.

Corollary 4.21. *In the α-hypergeometric model we have the following small-time behaviour for out-of-the-money put/call options on S_t*

$$-\lim_{t\to 0} t\ln\left(\mathbb{E}\left(S_t-K\right)^+\right)=\frac{1}{2\sigma^2}\operatorname{arccosh}\left(1+\frac{\frac{\sigma^2}{\overline{\rho}^2}\left(\ln(K)-\frac{\rho}{\sigma}\left(y_1^\star-y_0\right)\right)^2+\left(y_0-y_1^\star\right)^2}{2y_0y_1^\star}\right)^2,$$

$$-\lim_{t\to 0} t\ln\left(\mathbb{E}\left(K-S_t\right)^+\right)=\frac{1}{2\sigma^2}\operatorname{arccosh}\left(1+\frac{\frac{\sigma^2}{\overline{\rho}^2}\left(\ln(K)-\frac{\rho}{\sigma}\left(y_1^\star-y_0\right)\right)^2+\left(y_0-y_1^\star\right)^2}{2y_0y_1^\star}\right)^2,$$

where $y_1^\star$ is give by

$$y_1^\star=\sqrt{\left(\sigma\ln(K)+\rho y_0\right)^2+\overline{\rho}^2y_0^2},$$

with $y_0=\mathrm{e}^{v_0}$.

Corollary 4.22. *In the α-hypergeometric model[2] for $K>0$ we have the following small-time behaviour*

$$\begin{aligned}
&-\lim_{t\to 0} t\ln\left(\mathbb{E}\left(S_t-K\right)^+-\left(1-K\right)^+\right)\\
&\quad=-\lim_{t\to 0} t\ln\left(\mathbb{E}\left(K-S_t\right)^+-\left(K-1\right)^+\right)\\
&\quad=\frac{1}{2\sigma^2}\operatorname{arccosh}\left(1+\frac{\frac{\sigma^2}{\overline{\rho}^2}\left(\ln(K)-\frac{\rho}{\sigma}\left(y_1^\star-y_0\right)\right)^2+\left(y_0-y_1^\star\right)^2}{2y_0y_1^\star}\right)^2,
\end{aligned}$$

where $y_1^\star$ is give by

$$y_1^\star=\sqrt{\left(\sigma\ln(K)+\rho y_0\right)^2+\overline{\rho}^2y_0^2},$$

with $y_0=\mathrm{e}^{v_0}$.

Corollary 4.23. *In the α-hypergeometric model we have the following asymptotic behaviour for the implied volatility $\sigma_t(x)$ of a European call option on $S_t=\mathrm{e}^{X_t}$, with strike $K=\mathrm{e}^x$, as $t\to 0$*

$$I(x)=\lim_{t\to 0}\sigma_t(x)=\frac{x}{\frac{1}{\sigma}\operatorname{arccosh}\left(1+\frac{\frac{\sigma^2}{\overline{\rho}^2}\left(x-\frac{\rho}{\sigma}\left(y_1^\star-y_0\right)\right)^2+\left(y_0-y_1^\star\right)^2}{2y_0y_1^\star}\right)}$$

where $y_1^\star$ is give by

$$y_1^\star=\sqrt{\left(\sigma x+\rho y_0\right)^2+\overline{\rho}^2y_0^2},$$

with $y_0=\mathrm{e}^{v_0}$.

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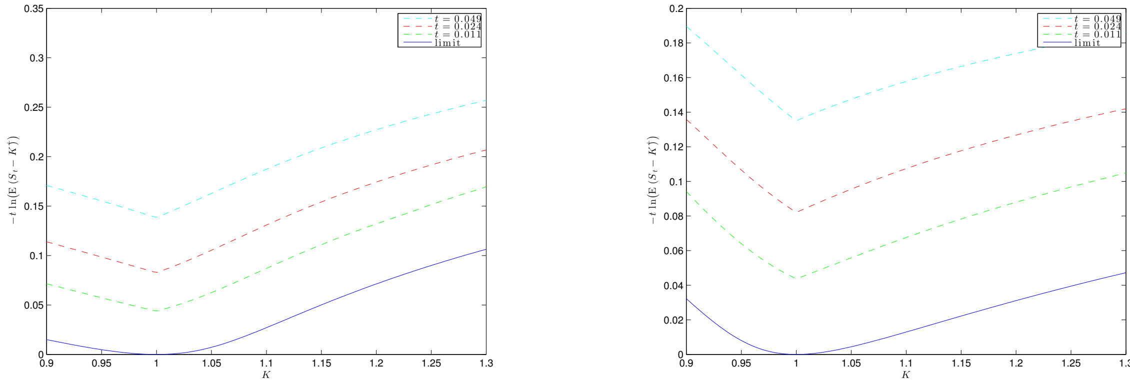
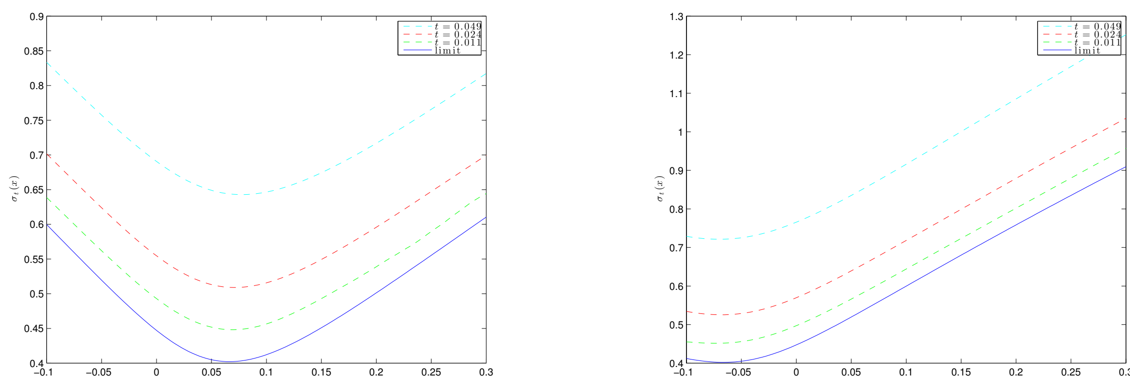
[2] note again that $S_0=1$.



(a) $\rho = -0.5$

(b) $\rho = 0.5$

Figure 8: Plot of $K \mapsto -t \ln \left(\mathbb{E}\left(S_t - K\right)^+\right)$ for different values of t with initial instantaneous variance V_0 of 20% and model parameters given by $a = 3$, $b = 0.4$, $\alpha = 1$, $\sigma = 5$ and different correlations ρ.



(a) $\rho = -0.5$

(b) $\rho = 0.5$

Figure 9: Plot of $x \mapsto \sigma_t\left(x\right)$ for different values of t with initial instantaneous variance V_0 of 20% and model parameters given by $a = 3$, $b = 0.4$, $\alpha = 1$, $\sigma = 5$ and different correlations ρ.

5 Transforms of the driving processes

Before we are able to start pricing plain vanilla options under the α-hypergeometric model we need to compute certain transforms of v. We follow the reasoning of [Da Fonseca and Martini, 2014].

5.1 Transforms of v

Proposition 5.2. *In the* 1*-hypergeometric model with* $\theta > 0$ *and* $\lambda > \frac{\theta^2}{2\sigma^2} + a\theta$ *the Laplace transform in time of the moment transform of* v *is given by*

$$\int_0^\infty \mathrm{e}^{-\lambda t}\mathbb{E}\left(\mathrm{e}^{\theta v_t}\right)dt = \frac{1}{\sigma^2}\exp\left(-\frac{a}{\sigma^2}v_0 + \frac{b}{\sigma^2}\mathrm{e}^{v_0}\right)(J_1 + J_2).$$

With

$$\begin{aligned}
J_1 &= 2\frac{\Gamma(a_1-1)}{\Gamma(b_1)}\mathrm{e}^{-\frac{z_0}{2}}z_0^\eta \mathrm{U}(a_1-1,b_1;z_0)(2\nu_2)^{-\theta-\frac{a}{\sigma^2}}I_1,\\
J_2 &= 2\frac{\Gamma(a_1-1)}{\Gamma(b_1)}\mathrm{e}^{-\frac{z_0}{2}}z_0^\eta \mathrm{M}(a_1-1,b_1;z_0)(2\nu_2)^{-\theta-\frac{a}{\sigma^2}}I_2,\\
I_1 &= \frac{z_0^{b_1-a_1+\theta}}{b_1-a_1+\theta}{}_2F_2\left([b_1-a_1+1,b_1-a_1+\theta][b_1-a_1+\theta+1,b_1],-z_0\right),\\
I_2 &= \frac{\Gamma(b_1-a_1+\theta)\Gamma(\theta-a_1+1)}{\Gamma(\theta)}\\
&\quad - z_0^{\theta-a_1+1}\frac{\Gamma(b_1-1)\,{}_2F_2\left([2-a_1,1+\theta-a_1][2-b_1,2+\theta-a_1],-z_0\right)}{\Gamma(a_1-1)(1+\theta-a_1)}\\
&\quad - z_0^{\theta-a_1+b_1}\frac{\Gamma(1-b_1)\,{}_2F_2\left([1-a_1+b_1,\theta-a_1+b_1][b_1,1+\theta-a_1+b_1],-z_0\right)}{\Gamma(a_1-b_1)(\theta-a_1+b_1)}
\end{aligned}$$

where $a_1 - 1 = \eta - \frac{a}{\sigma^2}$, $b_1 = 1 + 2\eta$, $\nu_2 = \frac{b}{\sigma^2}$, $z_0 = 2\nu_2\mathrm{e}^{v_0}$ *and* $\eta^2 = \frac{a^2}{\sigma^4} + \frac{2\lambda}{\sigma^2}$.

Proof. Let

$$\tilde{B}_s := B_s + \int_0^s \frac{a - b\mathrm{e}^{v_u}}{\sigma}du \tag{13}$$

and let $\tilde{\mathbb{P}}^t$, given by the Girsanov theorem, denote the probability measure under which $\left(\tilde{B}_s\right)_{0\le s\le t}$ is a Brownian motion. The density is therefore given by

$$\begin{aligned}
\frac{d\tilde{\mathbb{P}}^t}{d\mathbb{P}} &= \mathcal{E}\left(-\int_0^\cdot \frac{a-b\mathrm{e}^{v_u}}{\sigma}dB_u\right)_t\\
&= \exp\left(-\int_0^t \frac{a-b\mathrm{e}^{v_u}}{\sigma}dB_u - \frac{1}{2}\int_0^t\left(\frac{a-b\mathrm{e}^{v_u}}{\sigma}\right)^2du\right).
\end{aligned}$$

The new dynamics under $\tilde{\mathbb{P}}^t$ are therefore given by

$$\begin{aligned} dv_t &= \sigma d\tilde{B}_t, \\ d\mathrm{e}^{v_t} &= \frac{\sigma^2}{2}\mathrm{e}^{v_t}dt + \sigma \mathrm{e}^{v_t}d\tilde{B}_t \end{aligned}$$

and in integrated version

$$v_t - v_0 = \sigma \tilde{B}_t, \tag{14}$$

$$\mathrm{e}^{v_t} - \mathrm{e}^{v_0} = \frac{\sigma^2}{2}\int_0^t \mathrm{e}^{v_s}ds + \sigma \int_0^t \mathrm{e}^{v_s}d\tilde{B}_s. \tag{15}$$

Performing this measure change and using (13), (14) and (15) we arrive at

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\left(\mathrm{e}^{\theta v_t}\right) &= \mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\exp\left(\theta v_t + \int_0^t \frac{a - b\mathrm{e}^{v_s}}{\sigma}dB_s + \frac{1}{2}\int_0^t \left(\frac{a - b\mathrm{e}^{v_s}}{\sigma}\right)^2 ds\right)\right) \\ &= \mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\exp\left(\theta v_t + \int_0^t \frac{a - b\mathrm{e}^{v_s}}{\sigma}d\tilde{B}_s - \frac{1}{2}\int_0^t \left(\frac{a - b\mathrm{e}^{v_s}}{\sigma}\right)^2 ds\right)\right) \\ &= \mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\exp\left(\theta v_t + \frac{a}{\sigma}\tilde{B}_t - \frac{b}{\sigma}\int_0^t \mathrm{e}^{v_s}d\tilde{B}_s - \frac{1}{2\sigma^2}\int_0^t \left(a^2 - 2ab\mathrm{e}^{v_s} + b^2\mathrm{e}^{2v_s}\right)ds\right)\right) \\ &= \mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\exp\left(\theta v_t + \frac{a}{\sigma^2}v_t - \frac{a}{\sigma^2}v_0 - \frac{b}{\sigma^2}\left(\mathrm{e}^{v_s} - \mathrm{e}^{v_0} - \frac{\sigma^2}{2}\int_0^t \mathrm{e}^{v_s}ds\right)\right.\right. \\ &\qquad \left.\left. - \frac{a^2 t}{2\sigma^2} + \frac{ab}{\sigma^2}\int_0^t \mathrm{e}^{v_s}ds - \frac{b^2}{2\sigma^2}\int_0^t \mathrm{e}^{2v_s}ds\right)\right) \\ &= \exp\left(-\frac{a}{\sigma^2}v_0 + \frac{b}{\sigma^2}\mathrm{e}^{v_0} - \frac{a^2 t}{2\sigma^2}\right) \\ &\quad \cdot \mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\exp\left(-\int_0^t \frac{\beta_2^2}{2}\mathrm{e}^{2v_s} - \beta_1\mathrm{e}^{v_s}ds\right)\exp\left(\left(\theta + \frac{a}{\sigma^2}\right)v_t - \frac{b}{\sigma^2}\mathrm{e}^{v_t}\right)\right) \end{aligned}$$

with

$$\beta_1 = \frac{ab}{\sigma^2} + \frac{b}{2}, \qquad \beta_2^2 = \frac{b^2}{\sigma^2}.$$

With the Feynman-Kac formula we arrive at the following PDE for the above expectation

$$\begin{aligned} F_t &= \frac{\sigma^2}{2}F_{vv} - \frac{\beta_2^2}{2}\mathrm{e}^{2v}F + \beta_1\mathrm{e}^{v}F, \\ F(0, v) &= \exp\left(\left(\theta + \frac{a}{\sigma^2}\right)v - \frac{b}{\sigma^2}\mathrm{e}^{v}\right). \end{aligned} \tag{16}$$

This enables us to calculate the moment generating function via

$$\mathbb{E}^{\mathbb{P}}\left(\mathrm{e}^{\theta v_t}\right) = \exp\left(-\frac{a}{\sigma^2}v_0 + \frac{b}{\sigma^2}\mathrm{e}^{v_0} - \frac{a^2 t}{2\sigma^2}\right)F(t, v_0).$$

The function

$$g\left(t,v\right)=F\left(\frac{t}{\sigma^2},v\right)$$

then obviously satisfies the PDE

$$\begin{aligned} g_t &= -Hg, \\ g\left(0,v\right) &= \exp\left(\left(\theta+\frac{a}{\sigma^2}\right)v-\frac{b}{\sigma^2}\mathrm{e}^v\right), \end{aligned} \tag{17}$$

where, following the proof of theorem 8.1 in [Matsumoto and Yor, 2005a], H is the Schrödinger operator with Morse potential given by

$$H=-\frac{1}{2}\frac{d^2}{dv^2}+\frac{\nu_2^2}{2}\mathrm{e}^{2v}-\nu_1\mathrm{e}^v$$

with

$$\nu_1=\frac{\beta_1}{\sigma^2}, \qquad \nu_2^2=\frac{\beta_2^2}{\sigma^2}.$$

Let $q\left(t,v,y\right)$ be the transition density associated to the semigroup generated by H. We can calculate F via

$$F\left(t,v\right)=g\left(\sigma^2t,v\right)=\int_{-\infty}^{\infty}q\left(\sigma^2t,v_0,y\right)F\left(0,y\right)dy.$$

The associated Green function G satisfies, with $M_{\kappa,\mu}$ and $W_{\kappa,\mu}$ being the Whittaker functions,

$$\begin{aligned} G\left(v,y,\frac{\eta^2}{2}\right) &= \int_0^{\infty}\mathrm{e}^{-\frac{\eta^2}{2}t}q\left(t,v,y\right)dy \\ &= \frac{\Gamma\left(\eta-\frac{\nu_1}{\nu_2}+\frac{1}{2}\right)}{\nu_2\Gamma\left(1+2\eta\right)}\mathrm{e}^{-\frac{v+y}{2}}W_{\frac{\nu_1}{\nu_2},\eta}\left(2\nu_2\mathrm{e}^{\max(v,y)}\right)M_{\frac{\nu_1}{\nu_2},\eta}\left(2\nu_2\mathrm{e}^{\min(v,y)}\right), \end{aligned}$$

for $\eta\geq 0$, where if $\frac{\nu_1}{\nu_2}>0$ one has to ensure that $\eta>\frac{\nu_1}{\nu_2}-\frac{1}{2}$. Therefore by Laplace transforming the moment generating function in time we arrive at

$$\begin{aligned} \int_0^{\infty}\mathrm{e}^{-\lambda t}\mathbb{E}^{\mathbb{P}}\left(\mathrm{e}^{\theta v_t}\right)dt &= \exp\left(-\frac{a}{\sigma^2}v_0+\frac{b}{\sigma^2}\mathrm{e}^{v_0}\right)\int_0^{\infty}\exp\left(-\left(\frac{a^2}{\sigma^2}+2\lambda\right)\frac{t}{2}\right)F\left(t,v_0\right)dt \\ &= \exp\left(-\frac{a}{\sigma^2}v_0+\frac{b}{\sigma^2}\mathrm{e}^{v_0}\right) \\ &\quad\cdot\int_0^{\infty}\int_{-\infty}^{\infty}\exp\left(-\left(\frac{a^2}{\sigma^2}+2\lambda\right)\frac{t}{2}\right)q\left(\sigma^2t,v_0,y\right)F\left(0,y\right)dydt \\ &= \frac{1}{\sigma^2}\exp\left(-\frac{a}{\sigma^2}v_0+\frac{b}{\sigma^2}\mathrm{e}^{v_0}\right) \\ &\quad\cdot\int_0^{\infty}\int_{-\infty}^{\infty}\exp\left(-\left(\frac{a^2}{\sigma^4}+\frac{2\lambda}{\sigma^2}\right)\frac{t}{2}\right)q\left(t,v_0,y\right)F\left(0,y\right)dydt. \end{aligned}$$

Now let $\eta^2 := \frac{a^2}{\sigma^4} + \frac{2\lambda}{\sigma^2}$. Since $\frac{\nu_1}{\nu_2} = \frac{a}{\sigma^2} + \frac{1}{2}$ this choice is convenient in the sense that $\eta > \frac{\nu_1}{\nu_2} - \frac{1}{2}$ is always satisfied. With Fubini's theorem the above can be rewritten as

$$\int_0^\infty \mathrm{e}^{-\lambda t} \mathbb{E}^{\mathbb{P}} \left(\mathrm{e}^{\theta v_t} \right) dt = \frac{1}{\sigma^2} \exp \left(-\frac{a}{\sigma^2} v_0 + \frac{b}{\sigma^2} \mathrm{e}^{v_0} \right) \int_{-\infty}^\infty G \left(v_0, y, \frac{\eta^2}{2} \right) F(0, y) \, dy.$$

In order to calculate the integral on the right hand side we plug in the Green function and split up the integral such that

$$\int_{-\infty}^\infty G \left(v_0, y, \frac{\eta^2}{2} \right) F(0, y) \, dy = J_1 + J_2,$$

where

$$J_1 := \frac{\Gamma \left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2} \right)}{\nu_2 \Gamma (1 + 2\eta)} \mathrm{e}^{-\frac{v_0}{2}} W_{\frac{\nu_1}{\nu_2}, \eta} \left(2\nu_2 \mathrm{e}^{v_0} \right) \int_{-\infty}^{v_0} \mathrm{e}^{-\frac{y}{2}} M_{\frac{\nu_1}{\nu_2}, \eta} \left(2\nu_2 \mathrm{e}^{y} \right) F(0, y) \, dy,$$
$$J_2 := \frac{\Gamma \left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2} \right)}{\nu_2 \Gamma (1 + 2\eta)} \mathrm{e}^{-\frac{v_0}{2}} M_{\frac{\nu_1}{\nu_2}, \eta} \left(2\nu_2 \mathrm{e}^{v_0} \right) \int_{v_0}^{\infty} \mathrm{e}^{-\frac{y}{2}} W_{\frac{\nu_1}{\nu_2}, \eta} \left(2\nu_2 \mathrm{e}^{y} \right) F(0, y) \, dy.$$

As a reminder the Whittaker functions and the confluent hypergeometric functions are related via

$$M_{\kappa,\mu}(z) = \mathrm{e}^{-\frac{z}{2}} z^{\frac{1}{2}+\mu} \mathrm{M} \left(\frac{1}{2} + \mu - \kappa, 1 + 2\mu; z \right),$$
$$W_{\kappa,\mu}(z) = \mathrm{e}^{-\frac{z}{2}} z^{\frac{1}{2}+\mu} \mathrm{U} \left(\frac{1}{2} + \mu - \kappa, 1 + 2\mu; z \right).$$

Plugging in the definition of the Whittaker functions and the initial conditions and using the change of variables $z = 2\nu_2 \mathrm{e}^y$ we can write J_1 and J_2 as

$$J_1 = \frac{\Gamma \left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2} \right)}{\nu_2 \Gamma (1 + 2\eta)} \mathrm{e}^{-\frac{v_0}{2}} W_{\frac{\nu_1}{\nu_2}, \eta} \left(2\nu_2 \mathrm{e}^{v_0} \right) (2\nu_2)^{\frac{1}{2} - \theta - \frac{a}{\sigma^2}}$$
$$\cdot \int_0^{z_0} z^{\eta - 1 + \theta + \frac{a}{\sigma^2}} \mathrm{e}^{-z} \mathrm{M} \left(\eta - \frac{a}{\sigma^2}, 1 + 2\eta; z \right) dz,$$
$$J_2 = \frac{\Gamma \left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2} \right)}{\nu_2 \Gamma (1 + 2\eta)} \mathrm{e}^{-\frac{v_0}{2}} M_{\frac{\nu_1}{\nu_2}, \eta} \left(2\nu_2 \mathrm{e}^{v_0} \right) (2\nu_2)^{\frac{1}{2} - \theta - \frac{a}{\sigma^2}}$$
$$\cdot \int_{z_0}^{\infty} z^{\eta - 1 + \theta + \frac{a}{\sigma^2}} \mathrm{e}^{-z} \mathrm{U} \left(\eta - \frac{a}{\sigma^2}, 1 + 2\eta; z \right) dz.$$

Now let $a_1 - 1 = \eta - \frac{a}{\sigma^2}$ and $b_1 = 1 + 2\eta$. We can therefore rewrite J_1 and J_2 as

$$J_1 = 2\frac{\Gamma(a_1-1)}{\Gamma(b_1)} \mathrm{e}^{-\frac{z_0}{2}} z_0^{\eta} \mathrm{U}(a_1-1, b_1; z_0) (2\nu_2)^{-\theta-\frac{a}{\sigma^2}} \cdot \underbrace{\int_0^{z_0} z^{b_1-a_1+\theta-1} \mathrm{e}^{-z} \mathrm{M}(a_1-1, b_1; z)\, dz}_{=:I_1}$$

$$J_2 = 2\frac{\Gamma(a_1-1)}{\Gamma(b_1)} \mathrm{e}^{-\frac{z_0}{2}} z_0^{\eta} \mathrm{M}(a_1-1, b_1; z_0) (2\nu_2)^{-\theta-\frac{a}{\sigma^2}} \cdot \underbrace{\int_{z_0}^{\infty} z^{b_1-a_1+\theta-1} \mathrm{e}^{-z} \mathrm{U}(a_1-1, b_1; z)\, dz}_{=:I_2}.$$

Note that $b_1 - a_1 + \theta - 1 > -1$. Now for the integral I_1 there holds

$$\begin{aligned}
I_1 &= \int_0^{z_0} z^{b_1-a_1+\theta-1} \mathrm{e}^{-z} \mathrm{M}(a_1-1, b_1; z)\, dz \\
&\overset{(1)}{=} \int_0^{z_0} z^{b_1-a_1+\theta-1} \mathrm{M}(b_1-a_1+1, b_1; -z)\, dz \\
&\overset{(2)}{=} \sum_{n=0}^{\infty} \frac{(b_1-a_1+1)_n}{(b_1)_n\, n!} (-1)^n \int_0^{z_0} z^{b_1-a_1+\theta-1+n} dz \\
&= \sum_{n=0}^{\infty} \frac{(b_1-a_1+1)_n}{(b_1-a_1+\theta+n)(b_1)_n\, n!} (-1)^n z_0^{b_1-a_1+\theta+n} \\
&\overset{(3)}{=} \frac{z_0^{b_1-a_1+\theta}}{b_1-a_1+\theta} \sum_{n=0}^{\infty} \frac{(b_1-a_1+1)_n (b_1-a_1+\theta)_n}{(b_1-a_1+\theta+1)_n (b_1)_n\, n!} (-1)^n z_0^n \\
&\overset{(4)}{=} \frac{z_0^{b_1-a_1+\theta}}{b_1-a_1+\theta} {}_2F_2([b_1-a_1+1, b_1-a_1+\theta][b_1-a_1+\theta+1, b_1], -z_0).
\end{aligned}$$

In the above calculations equality (1) holds because of Kummer's transformation

$$\mathrm{e}^{-z} \mathrm{M}(a_1-1, b_1; z) = \mathrm{M}(b_1-a_1+1, b_1; -z),$$

see equation 13.2.39 in [DLMF]. Equality (2) is justified since M can be written as a powerseries with infinite radius of convergence and therefore it converges uniformly on compact sets, especially on the interval $[0, z_0]$. Therefore we can interchange the integral with the sum. Equality (3) follows from the fact that

$$b_1 - a_1 + \theta + n = \frac{(b_1-a_1+\theta+1)_n (b_1-a_1+\theta)}{(b_1-a_1+\theta)_n}$$

and equality (4) is just the definition of ${}_2F_2$.

Now we are going to calculate the integral I_2. First of all by the Mellin–Barnes integral representation of U, see equation 13.4.18 in [DLMF], the integrand can be written as

$$z^{b_1-a_1+\theta-1}\mathrm{e}^{-z}\mathrm{U}\left(a_1-1,b_1;z\right)=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma\left(b_1-1+t\right)\Gamma\left(t\right)}{\Gamma\left(a_1-1+t\right)}z^{\theta-a_1-t}dt,$$

where the contour of the integral passes all the poles of $t\mapsto\Gamma\left(b_1-1+t\right)\Gamma\left(t\right)$ on the right hand side. This formula holds in fact for all complex z with $\arg\left(z\right)\leq\frac{\pi}{2}$ therefore especially in our case. Furthermore note that $b_1-1>0$ and we are therefore able to choose the contour arbitrary close but on the right of the imaginary line. When calculating I_2 we are going to interchange the integrals, which is valid by Fubini's theorem as long as $\theta-a_1-\mathrm{Re}\left(t\right)+1<0$. Since we have already seen that we can choose $\mathrm{Re}\left(t\right)$ to be arbitrarily small, it suffices to ensure that $\theta-a_1+1<0$, which in turn can be done by choosing λ large enough. In fact recalling that

$$a_1-1=\eta-\frac{a}{\sigma^2}=\sqrt{\frac{a^2}{\sigma^4}+\frac{2\lambda}{\sigma^2}}-\frac{a}{\sigma^2}$$

simple calculations show that λ has to be chosen such that

$$\lambda>\frac{\theta^2}{2\sigma^2}+a\theta.$$

Therefore after applying Fubini's theorem and solving the inner integral we arrive at

$$I_2=-\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma\left(b_1-1+t\right)\Gamma\left(t\right)}{\Gamma\left(a_1-1+t\right)\left(\theta-a_1+1-t\right)}z_0^{\theta-a_1+1-t}dt$$

The next step is to apply the residue theorem. Therefore we are going to look at the poles of the integrand. The corresponding residues can be calculated by elementary properties of the residue and the gamma function. As a reminder we recall that

$$\mathrm{Res}\left(\Gamma,-n\right)=\frac{\left(-1\right)^n}{n!},\qquad\text{for } n\in\mathbb{N}_0.$$

Now the residues of the integrand of I_2 are located and given by

- $t=\theta-a_1+1$, which is - after the appropriate choice of λ - in the left half-plane and therefore accounts in the calculations of the residues. The residue is given by

$$-\frac{\Gamma\left(b_1-a_1+\theta\right)\Gamma\left(\theta-a_1+1\right)}{\Gamma\left(\theta\right)}.$$

- $t=-n$ with $n\in\mathbb{N}_0$. The residue is given by

$$\frac{\Gamma\left(b_1-1-n\right)}{\Gamma\left(a_1-1-n\right)\left(\theta-a_1+1+n\right)n!}\left(-1\right)^n z_0^{\theta-a_1+1+n}.$$

- $t = 1 - b_1 - n$ with $n \in \mathbb{N}_0$. The residue is given by

$$\frac{\Gamma(1-b_1-n)}{\Gamma(a_1-b_1-n)(\theta-a_1+b_1+n)\,n!}(-1)^n z_0^{\theta-a_1+b_1+n}.$$

The above holds true if the cases do not coincide. But since we are trying to calculate the Laplace moment transform, which is in fact a C^∞ function in λ and θ, it suffices to know it except for those special cases which are a zero set anyway. Summing up all the residues we get

$$\begin{aligned} I_2 = {} & \frac{\Gamma(b_1-a_1+\theta)\,\Gamma(\theta-a_1+1)}{\Gamma(\theta)} \\ & - z_0^{\theta-a_1+1}\underbrace{\sum_{n=0}^{\infty}\frac{\Gamma(b_1-1-n)}{\Gamma(a_1-1-n)(\theta-a_1+1+n)\,n!}(-1)^n z_0^n}_{=:H_1} \\ & - z_0^{\theta-a_1+b_1}\underbrace{\sum_{n=0}^{\infty}\frac{\Gamma(1-b_1-n)}{\Gamma(a_1-b_1-n)(\theta-a_1+b_1+n)\,n!}(-1)^n z_0^n}_{=:H_2}. \end{aligned}$$

In fact H_1 can be further simplified. There holds

$$H_1 = \frac{\Gamma(b_1-1)\,{}_2F_2([2-a_1, 1+\theta-a_1]\,[2-b_1, 2+\theta-a_1], -z_0)}{\Gamma(a_1-1)(1+\theta-a_1)}.$$

In order to derive this equation we will show that the coefficients of the powerseries expansion of the right hand side coincides with the coefficients of H_1. From the definition of ${}_2F_2$ and by multiplying with $n!$ the coefficients are given by

$$\frac{\Gamma(b_1-1)}{\Gamma(a_1-1)(1+\theta-a_1)}\frac{(2-a_1)_n(1+\theta-a_1)_n}{(2-b_1)_n(2+\theta-a_1)_n}$$

writing out the pochhammer symbols and using elementary properties of the gamma function this equals

$$\begin{aligned} & \frac{\Gamma(b_1-1)}{\Gamma(a_1-1)(1+\theta-a_1)}\frac{\frac{\Gamma(2-a_1+n)}{\Gamma(2-a_1)}\frac{\Gamma(1+\theta-a_1+n)}{\Gamma(1+\theta-a_1)}}{\frac{\Gamma(2-b_1+n)}{\Gamma(2-b_1)}\frac{\Gamma(2+\theta-a_1+n)}{\Gamma(2+\theta-a_1)}} \\ & = \frac{\Gamma(b_1-1)}{\Gamma(a_1-1)(1+\theta-a_1)}\frac{\Gamma(2-a_1+n)}{\Gamma(2-a_1)}\frac{\Gamma(1+\theta-a_1+n)}{\Gamma(1+\theta-a_1)}\frac{\Gamma(2-b_1)}{\Gamma(2-b_1+n)}\frac{\Gamma(2+\theta-a_1)}{\Gamma(2+\theta-a_1+n)} \\ & = \frac{\Gamma(b_1-1)}{\Gamma(a_1-1)}\frac{\Gamma(2-a_1+n)}{\Gamma(2-a_1)}\frac{\Gamma(2-b_1)}{\Gamma(2-b_1+n)}\frac{1}{(1+\theta-a_1+n)} \end{aligned}$$

Since for $z \in \mathbb{C} \setminus \mathbb{Z}$ there holds

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{18}$$

one can rewrite the above as

$$\begin{aligned}
&\frac{\Gamma(b_1-1)}{\Gamma(a_1-1)}\frac{\Gamma(2-a_1+n)}{\Gamma(2-a_1)}\frac{\Gamma(2-b_1)}{\Gamma(2-b_1+n)}\frac{1}{(1+\theta-a_1+n)} \\
&= \frac{\sin(\pi(a_1-1))}{\sin(\pi(b_1-1))}\frac{\Gamma(2-a_1+n)}{\Gamma(2-b_1+n)}\frac{1}{(1+\theta-a_1+n)} \\
&= \frac{\sin(\pi(a_1-1-n))}{\sin(\pi(b_1-1-n))}\frac{\Gamma(2-a_1+n)}{\Gamma(2-b_1+n)}\frac{1}{(1+\theta-a_1+n)}.
\end{aligned}$$

Note that the last equation is valid even tough the sine function is 2π periodic. Now using again equation (18) yields the assertion. Analogously one can verify that

$$H_2 = \frac{\Gamma(1-b_1)\,{}_2F_2([1-a_1+b_1,\theta-a_1+b_1][b_1,1+\theta-a_1+b_1],-z_0)}{\Gamma(a_1-b_1)(\theta-a_1+b_1)}.$$

$\square$

Together with remark 2.2 one immediately gets

Theorem 5.3. *In the α-hypergeometric model the Laplace moment transform of v is given by*

$$\int_0^\infty \mathrm{e}^{-\lambda t}\mathbb{E}\left(\mathrm{e}^{\theta v_t}\right)dt = \int_0^\infty \mathrm{e}^{-\lambda t}\mathbb{E}\left(\mathrm{e}^{\frac{\theta}{\alpha}\tilde{v}_t}\right)dt,$$

where the process $\tilde{v}$ with starting value $\tilde{v}_0 = \alpha v_0$ follows the SDE

$$d\tilde{v}_t = \left(\alpha a - \alpha b \mathrm{e}^{\tilde{v}_t}\right)dt + \alpha\sigma dB_t,$$

which can be calculated using proposition 5.2.

5.4 The variance swap

The expected annualized variance until t is given by

$$\mathrm{VS}(t) = \frac{1}{t}\mathbb{E}\left(\int_0^t \mathrm{e}^{2v_s}ds\right) = \frac{1}{t}\int_0^t \mathbb{E}\left(\mathrm{e}^{2v_s}\right)ds.$$

Its Laplace transform is therefore given by

$$\int_0^\infty \mathrm{e}^{-\lambda t} t\mathrm{VS}(t)\,dt = \frac{1}{\lambda}\int_0^\infty \mathrm{e}^{-\lambda t}\mathbb{E}\left(\mathrm{e}^{2v_t}\right)dt.$$

Note that we are already able to calculate the above right hand side.

5.5 Transforms of S

In the following we want to compute the Laplace transform in time of the Mellin transform of the forward S. As seen above for the process v we were able to calculate the Laplace moment transform for general $\alpha > 0$, see remark 2.2. The same strategy can however not be performed when dealing with the forward itself. In the proof of the next theorem we therefore set $\alpha = 1$. The interested reader can follow the first part of the proof with general $\alpha > 0$ but will soon run into a problem. According to chapter 9 in [Henry-Labordère, 2008] the 1-hypergeometric model lies in the class of solvable stochastic volatility model which are related to Natanzon superpotentials. Hereafter we will therefore focus on the 1-hypergeometric model. The dynamics are therefore given by

$$\begin{aligned} dS_t &= S_t \mathrm{e}^{v_t} dW_t, \\ dv_t &= \left(a - b\mathrm{e}^{v_t}\right) dt + \sigma dB_t, \\ dW_t dB_t &= \rho dt. \end{aligned}$$

Theorem 5.6. *Let S and v be given by the 1-hypergeometric model with $\rho\sigma < b$. Furthermore let $\theta \in (\theta^\star, \theta_+)$ where*

$$\begin{aligned} \theta^\star &= \frac{9\sigma - 16b\rho + 3\sqrt{32b^2 + 9\sigma^2 - 32b\rho\sigma}}{2\sigma\left(9 - 8\rho^2\right)}, \\ \theta_+ &= \frac{\sigma - 2b\rho + \sqrt{\left(\sigma - 2b\rho\right)^2 + 4b^2\left(1 - \rho^2\right)}}{2\sigma\left(1 - \rho^2\right)} \end{aligned}$$

and $\lambda > 0$ such that

$$\left(\frac{a^2}{\sigma^4} + \frac{2\lambda}{\sigma^2}\right)^{\frac{1}{2}} - \frac{\left(b - \theta\rho\sigma\right)\left(\frac{a}{\sigma^2} + \frac{1}{2}\right)}{\sqrt{\left(b - \theta\rho\sigma\right)^2 + \sigma^2\theta\left(1 - \theta\right)}} + \frac{1}{2} > 0.$$

Then the Laplace transform in time of the Mellin transform of S is given by

$$\int_0^\infty \mathrm{e}^{-\lambda t}\mathbb{E}\left(S_t^\theta\right) dt = \frac{1}{\sigma^2}\mathrm{e}^{-\frac{a}{\sigma^2}v_0 + \left(\frac{b}{\sigma^2} - \frac{\theta\rho}{\sigma}\right)\mathrm{e}^{v_0}}\left(J_1 + J_2\right).$$

With

$$\begin{aligned} J_1 &= 2\frac{\Gamma\left(a_2\right)}{\Gamma\left(b_2\right)}\mathrm{e}^{-\frac{z_0}{2}} z_0^\eta \mathrm{U}\left(a_2, b_2; z_0\right)\left(2\nu_2\right)^{-\theta - \frac{a}{\sigma^2}} I_1, \\ J_2 &= 2\frac{\Gamma\left(a_2\right)}{\Gamma\left(b_2\right)}\mathrm{e}^{-\frac{z_0}{2}} z_0^\eta \mathrm{M}\left(a_2, b_2; z_0\right)\left(2\nu_2\right)^{-\theta - \frac{a}{\sigma^2}} I_2, \end{aligned}$$

where

$$I_1 = \sum_{n=0}^\infty \frac{\left(a_2\right)_n}{\left(b_2\right)_n n!} i_n,$$

with i_n is given by

$$i_n = (-\delta(\theta))^{-\eta-\frac{a}{\sigma^2}-n}\,\gamma\left(\eta+\frac{a}{\sigma^2}+n, -\delta(\theta)\, z_0\right),$$

where γ denotes the lower incomplete gamma function. Alternatively i_n satisfies the following recurrence relation

$$\delta(\theta)\, i_{n+1} = z_0^{\eta+\frac{a}{\sigma^2}+n}\mathrm{e}^{\delta(\theta) z_0} - \left(\eta+\frac{a}{\sigma^2}+n\right) i_n.$$

Furthermore

$$I_2 = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{\Gamma(b_2-1-n)}{\Gamma(a_2-n)}j(-n) + \frac{\Gamma(1-b_2-n)}{\Gamma(a_2+1-b_2-n)}j(1-b_2-n)\right).$$

The function j is given by

$$j : t \mapsto \zeta^{\eta-\frac{a}{\sigma^2}+t}\Gamma\left(-\eta+\frac{a}{\sigma^2}-t, z_0\zeta\right),$$

where $\Gamma(\cdot,\cdot)$ denotes the upper incomplete gamma function and $\zeta = -\frac{1}{2} - \frac{\theta\rho\sigma - b}{2\nu_2\sigma^2}$ with $a_2 = \eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}$, $b_2 = 1+2\eta$, $\nu_1 = \frac{(b-\theta\rho\sigma)}{\sigma^2}\left(\frac{a}{\sigma^2}+\frac{1}{2}\right)$, $\nu_2 = \frac{1}{\sigma^2}\sqrt{(\theta\rho\sigma - b)^2 + \sigma^2\theta(1-\theta)}$, $z_0 = 2\nu_2\mathrm{e}^{v_0}$ and $\eta^2 = \frac{a^2}{\sigma^4} + \frac{2\lambda}{\sigma^2}$.

Proof. Since S is given by a stochastic exponential and by introducing the standard Brownian motion $\left(B_t, B_t^{\perp}\right)_{t\geq 0}$ we have

$$\begin{aligned}\mathbb{E}\left(S_t^{\theta}\right) &= \mathbb{E}\left(\exp\left(\theta\int_0^t \mathrm{e}^{v_s} dW_s - \frac{\theta}{2}\int_0^t \mathrm{e}^{2v_s} ds\right)\right)\\ &= \mathbb{E}\left(\exp\left(\theta\rho\int_0^t \mathrm{e}^{v_s} dB_s + \theta\sqrt{1-\rho^2}\int_0^t \mathrm{e}^{v_s} dB_s^{\perp} - \frac{\theta}{2}\int_0^t \mathrm{e}^{2v_s} ds\right)\right).\end{aligned}$$

With the sigma algebra $\mathcal{F}_s := \sigma(B_u : 0 \leq u \leq s)$ and using elementary properties of the conditional expectation and Ito integrals we arrive at

$$\begin{aligned}\mathbb{E}\left(S_t^{\theta}\right) &= \mathbb{E}\left(\exp\left(\theta\rho\int_0^t \mathrm{e}^{v_s} dB_s - \frac{\theta}{2}\int_0^t \mathrm{e}^{2v_s} ds\right)\mathbb{E}\left(\exp\left(\theta\sqrt{1-\rho^2}\int_0^t \mathrm{e}^{v_s} dB_s^{\perp}\right)\middle|\mathcal{F}_t\right)\right)\\ &= \mathbb{E}\left(\exp\left(\theta\rho\int_0^t \mathrm{e}^{v_s} dB_s + \frac{\theta^2(1-\rho^2)-\theta}{2}\int_0^t \mathrm{e}^{2v_s} ds\right)\right).\end{aligned}$$

Now note that by the Ito formula applied to e^{v_t} we have

$$\sigma\int_0^t \mathrm{e}^{v_s} dB_s = \mathrm{e}^{v_t} - \mathrm{e}^{v_0} - \int_0^t \mathrm{e}^{v_s}(a - b\mathrm{e}^{v_s})\, ds - \frac{\sigma^2}{2}\int_0^t \mathrm{e}^{v_s} ds.$$

Therefore we arrive at[3]

$$\begin{aligned}
\mathbb{E}\left(S_t^\theta\right) &= \mathbb{E}\left(\exp\left(\theta\rho\int_0^t \mathrm{e}^{v_s} dB_s + \frac{\theta^2\left(1-\rho^2\right)-\theta}{2}\int_0^t \mathrm{e}^{2v_s} ds\right)\right) \\
&= \mathbb{E}\left(\exp\left(\frac{\theta\rho}{\sigma}\left(\mathrm{e}^{v_t} - \mathrm{e}^{v_0} - \int_0^t \mathrm{e}^{v_s}\left(a - b\mathrm{e}^{v_s}\right) ds - \frac{\sigma^2}{2}\int_0^t \mathrm{e}^{v_s} ds\right)\right.\right. \\
&\qquad\qquad \left.\left. + \frac{\theta^2\left(1-\rho^2\right)-\theta}{2}\int_0^t \mathrm{e}^{2v_s} ds\right)\right) \\
&= \exp\left(-\frac{\theta\rho}{\sigma}\mathrm{e}^{v_0}\right)\mathbb{E}\left(\exp\left(\frac{\theta\rho}{\sigma}\mathrm{e}^{v_t} - \frac{\theta\rho}{\sigma}\left(a + \frac{\sigma^2}{2}\right)\int_0^t \mathrm{e}^{v_s} ds\right.\right. \\
&\qquad\qquad \left.\left. - \frac{\theta - \theta^2\left(1-\rho^2\right) - \frac{2\theta\rho b}{\sigma}}{2}\int_0^t \mathrm{e}^{2v_s} ds\right)\right) \\
&= \exp\left(-\alpha_0 \mathrm{e}^{v_0}\right)\mathbb{E}\left(\exp\left(\alpha_0 \mathrm{e}^{v_t} + \alpha_1\int_0^t \mathrm{e}^{v_s} ds - \frac{\alpha_2}{2}\int_0^t \mathrm{e}^{2v_s} ds\right)\right),
\end{aligned}$$

with

$$\alpha_0 = \frac{\theta\rho}{\sigma}, \qquad \alpha_1 = -\frac{\theta\rho}{\sigma}\left(a + \frac{\sigma^2}{2}\right), \qquad \alpha_2 = \theta - \theta^2\left(1-\rho^2\right) - \frac{2\theta\rho b}{\sigma}.$$

Now as in the proof of proposition 5.2 let

$$\tilde{B}_s := B_s + \int_0^s \frac{a - b\mathrm{e}^{v_u}}{\sigma} du \tag{19}$$

and let $\tilde{\mathbb{P}}^t$, given by the Girsanov theorem, denote the probability measure under which $\left(\tilde{B}_s\right)_{0\le s\le t}$ is a Brownian motion. The density is therefore given by

$$\begin{aligned}
\frac{d\tilde{\mathbb{P}}^t}{d\mathbb{P}} &= \mathcal{E}\left(-\int_0^\cdot \frac{a - b\mathrm{e}^{v_u}}{\sigma} dB_u\right)_t \\
&= \exp\left(-\int_0^t \frac{a - b\mathrm{e}^{v_u}}{\sigma} dB_u - \frac{1}{2}\int_0^t \left(\frac{a - b\mathrm{e}^{v_u}}{\sigma}\right)^2 du\right).
\end{aligned}$$

The new dynamics under $\tilde{\mathbb{P}}^t$ are therefore given by

$$\begin{aligned}
dv_t &= \sigma d\tilde{B}_t, \\
d\mathrm{e}^{v_t} &= \frac{\sigma^2}{2}\mathrm{e}^{v_t} dt + \sigma \mathrm{e}^{v_t} d\tilde{B}_t
\end{aligned}$$

[3]Note that this step would lead to another integral term with integrand $\mathrm{e}^{(1+\alpha)v_s}$ for general $\alpha > 0$.

and in integrated version

$$v_t - v_0 = \sigma \tilde{B}_t, \tag{20}$$

$$\mathrm{e}^{v_t} - \mathrm{e}^{v_0} = \frac{\sigma^2}{2}\int_0^t \mathrm{e}^{v_s} ds + \sigma \int_0^t \mathrm{e}^{v_s} d\tilde{B}_s. \tag{21}$$

Performing this measure change and using (19), (20) and (21) we arrive at[4]

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left(S_t^\theta\right) &= \mathrm{e}^{-\alpha_0 \mathrm{e}^{v_0}} \mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\exp\left(\alpha_0 \mathrm{e}^{v_t} + \alpha_1 \int_0^t \mathrm{e}^{v_s} ds - \frac{\alpha_2}{2}\int_0^t \mathrm{e}^{2v_s} ds \right.\right. \\
&\qquad \left.\left. + \int_0^t \frac{a - b\mathrm{e}^{v_s}}{\sigma} dB_s + \frac{1}{2}\int_0^t \left(\frac{a - b\mathrm{e}^{v_s}}{\sigma}\right)^2 ds\right)\right) \\
&= \mathrm{e}^{-\alpha_0 \mathrm{e}^{v_0}} \mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\exp\left(\alpha_0 \mathrm{e}^{v_t} + \alpha_1 \int_0^t \mathrm{e}^{v_s} ds - \frac{\alpha_2}{2}\int_0^t \mathrm{e}^{2v_s} ds \right.\right. \\
&\qquad \left.\left. + \int_0^t \frac{a - b\mathrm{e}^{v_s}}{\sigma} d\tilde{B}_s - \frac{1}{2}\int_0^t \left(\frac{a - b\mathrm{e}^{v_s}}{\sigma}\right)^2 ds\right)\right) \\
&= \mathrm{e}^{-\alpha_0 \mathrm{e}^{v_0}} \mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\exp\left(\alpha_0 \mathrm{e}^{v_t} + \alpha_1 \int_0^t \mathrm{e}^{v_s} ds - \frac{\alpha_2}{2}\int_0^t \mathrm{e}^{2v_s} ds \right.\right. \\
&\qquad + \frac{a}{\sigma^2}\left(v_t - v_0\right) - \frac{b}{\sigma^2}\left(\mathrm{e}^{v_t} - \mathrm{e}^{v_0} - \frac{\sigma^2}{2}\int_0^t \mathrm{e}^{v_s} ds\right) \\
&\qquad \left.\left. - \frac{a^2 t}{2\sigma^2} + \frac{ab}{\sigma^2}\int_0^t \mathrm{e}^{v_s} ds - \frac{b^2}{2\sigma^2}\int_0^t \mathrm{e}^{2v_s} ds\right)\right) \\
&= \mathrm{e}^{-\frac{a}{\sigma^2}v_0 + \left(\frac{b}{\sigma^2} - \alpha_0\right)\mathrm{e}^{v_0} - \frac{a^2 t}{2\sigma^2}} \mathbb{E}^{\tilde{\mathbb{P}}^t}\left(\exp\left(\frac{a}{\sigma^2}v_t + \beta_0 \mathrm{e}^{v_t} + \beta_1 \int_0^t \mathrm{e}^{v_s} ds - \frac{\beta_2^2}{2}\int_0^t \mathrm{e}^{2v_s} ds\right)\right)
\end{aligned}$$

with

$$\begin{aligned}
\beta_0 &= \alpha_0 - \frac{b}{\sigma^2} = \frac{\theta\rho\sigma - b}{\sigma^2}, \\
\beta_1 &= \alpha_1 + \frac{b}{2} + \frac{ab}{2\sigma^2} = (b - \theta\rho\sigma)\left(\frac{a}{\sigma^2} + \frac{1}{2}\right), \\
\beta_2^2 &= \alpha_2 + \frac{b^2}{\sigma^2} = -\theta^2\left(1 - \rho^2\right) + \theta\left(1 - \frac{2\rho b}{\sigma}\right) + \frac{b^2}{\sigma^2} \\
&= \frac{1}{\sigma^2}\left(\left(\theta\rho\sigma - b\right)^2 + \sigma^2\theta\left(1 - \theta\right)\right).
\end{aligned}$$

Note that in the definition of β_2^2 we need to ensure positivity of the right hand side. One of course could simply calculate the zeros of the right hand side as a function of θ, but since

[4]Note that these calculations are also valid for general $\alpha > 0$ but lead to extra integral terms with integrands of the forms $\mathrm{e}^{\alpha v_s}, \mathrm{e}^{(\alpha+1)v_s}$ and $\mathrm{e}^{2\alpha v_s}$, which collapse into each other in the case $\alpha = 1$.

$\beta_2^2|_{\theta=0} = \frac{b^2}{\sigma^2} > 0$ and $\beta_2^2|_{\theta=1} = \frac{(\rho\sigma-b)^2}{\sigma^2} \geq 0$ we have strict positivity at least for $\theta \in [0,1)$ and in the case $\rho\sigma \neq b$ even in an open neighbourhood of the interval $[0,1]$. Now let

$$\nu_1 = \frac{\beta_1}{\sigma^2}, \qquad \nu_2^2 = \frac{\beta_2^2}{\sigma^2}$$

and let $F(t,v)$ denote the the above expectation with initial condition

$$F(0,v) = \exp\left(\frac{a}{\sigma^2} v + \beta_0 \mathrm{e}^v\right).$$

From now on we follow exactly the proof and notation of theorem 5.2. Again by performing a Laplace transform in time and with $\eta^2 := \frac{a^2}{\sigma^4} + \frac{2\lambda}{\sigma^2}$ we have

$$\int_0^\infty \mathrm{e}^{-\lambda t} \mathbb{E}^{\mathbb{P}}\left(S_t^\theta\right) dt = \frac{1}{\sigma^2} \mathrm{e}^{-\frac{a}{\sigma^2}v_0 + \left(\frac{b}{\sigma^2} - \alpha_0\right)\mathrm{e}^{v_0}} \int_{-\infty}^\infty G\left(v_0, y, \frac{\eta^2}{2}\right) F(0,y)\, dy.$$

Again we need to ensure that $\eta > \frac{\nu_1}{\nu_2} - \frac{1}{2}$, which can be done by choosing λ sufficiently large. Explicitly in terms of model parameters one needs

$$\left(\frac{a^2}{\sigma^4} + \frac{2\lambda}{\sigma^2}\right)^{\frac{1}{2}} - \frac{(b - \theta\rho\sigma)\left(\frac{a}{\sigma^2} + \frac{1}{2}\right)}{\sqrt{(b-\theta\rho\sigma)^2 + \sigma^2\theta(1-\theta)}} + \frac{1}{2} > 0.$$

In order to calculate the integral on the right hand side we plug in the Green function and split up the integral such that

$$\int_{-\infty}^\infty G\left(v_0, y, \frac{\eta^2}{2}\right) F(0,y)\, dy = J_1 + J_2,$$

again simplifying J_1 and J_2 leads to

$$\begin{aligned} J_1 &= \frac{\Gamma\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}\right)}{\nu_2 \Gamma(1+2\eta)} \mathrm{e}^{-\frac{v_0}{2}} W_{\frac{\nu_1}{\nu_2},\eta}\left(2\nu_2 \mathrm{e}^{v_0}\right)\left(2\nu_2\right)^{\frac{1}{2} - \frac{a}{\sigma^2}} \\ &\quad \cdot \underbrace{\int_0^{z_0} z^{\eta - 1 + \frac{a}{\sigma^2}} \mathrm{e}^{\left(-\frac{1}{2} + \frac{\beta_0}{2\nu_2}\right)z} \mathrm{M}\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}, 1 + 2\eta; z\right) dz}_{=: I_1} \\ J_2 &= \frac{\Gamma\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}\right)}{\nu_2 \Gamma(1+2\eta)} \mathrm{e}^{-\frac{v_0}{2}} M_{\frac{\nu_1}{\nu_2},\eta}\left(2\nu_2 \mathrm{e}^{v_0}\right)\left(2\nu_2\right)^{\frac{1}{2} - \frac{a}{\sigma^2}} \\ &\quad \cdot \underbrace{\int_{z_0}^{\infty} z^{\eta - 1 + \frac{a}{\sigma^2}} \mathrm{e}^{\left(-\frac{1}{2} + \frac{\beta_0}{2\nu_2}\right)z} \mathrm{U}\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}, 1 + 2\eta; z\right) dz}_{=: I_2}. \end{aligned}$$

Note the main difference to the proof of theorem 5.2 is the exponential factor given by $\delta(\theta) := -\frac{1}{2} + \frac{\beta_0}{2\nu_2}$ in the above integrals. We will now investigate the function δ a little more.

Let θ_-, θ_+ denote the zeros of β_2^2 as a function of θ. Simple calculations show that - see theorem 2.8 -

$$\theta_+ = \frac{\sigma - 2b\rho + \sqrt{(\sigma - 2b\rho)^2 + 4b^2(1-\rho^2)}}{2\sigma(1-\rho^2)}.$$

We have already seen that $\theta_- < 0 < 1 < \theta_+$. Since

$$\delta(\theta) = -\frac{1}{2} + \frac{1}{2}\frac{\theta\rho\sigma - b}{\sqrt{(\theta\rho\sigma - b)^2 + \sigma^2\theta(1-\theta)}}$$

we immediately have that $\delta(1) = -1$, since $b > \rho\sigma$. Simple calculations yield that

$$\operatorname{sgn}(\delta'(\theta)) = (\rho\sigma - 2b)\theta + b,$$

which is negative for $\theta \in (1, \theta_+)$ and therefore δ is monotonically decreasing for $\theta \in (1, \theta_+)$. Furthermore since δ has a pole at θ_+ we have that $\lim_{\theta \to \theta_+} \delta(\theta) = -\infty$. Later on we will need to choose θ in such a way that $\delta(\theta) < -2$. Solving $\delta(\theta) = -2$ leads to finding the largest zero of the polynomial

$$8(\theta\rho\sigma - b)^2 + 9\sigma^2\theta(1-\theta),$$

which we denote by $\theta^\star$ and is given by

$$\theta^\star = \frac{9\sigma - 16b\rho + 3\sqrt{32b^2 + 9\sigma^2 - 32b\rho\sigma}}{2\sigma(9 - 8\rho^2)}.$$

First we consider I_1. Note that since $\eta + \frac{a}{\sigma^2} > 0$ for all $\lambda > 0$ the integral I_1 stays finite. Again, as in the proof of theorem 5.2, by interchanging the integral and the powerseries expansion of the Kummer function we arrive at

$$\begin{aligned} I_1 &= \int_0^{z_0} z^{\eta - 1 + \frac{a}{\sigma^2}} \mathrm{e}^{\delta(\theta)z} \mathrm{M}\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}, 1 + 2\eta; z\right) dz \\ &= \sum_{n=0}^{\infty} \frac{\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}\right)_n}{(1+2\eta)_n\, n!} \underbrace{\int_0^{z_0} z^{\eta - 1 + \frac{a}{\sigma^2} + n} \mathrm{e}^{\delta(\theta)z} dz}_{:= i_n} \end{aligned}$$

If $\delta(\theta) < 0$ then a simple change of variables leads to

$$i_n = (-\delta(\theta))^{-\eta - \frac{a}{\sigma^2} - n} \gamma\left(\eta + \frac{a}{\sigma^2} + n, -\delta(\theta) z_0\right),$$

where γ denotes the lower incomplete gamma function. Integration by parts immediately leads to the recurrence relation

$$\delta(\theta)\, i_{n+1} = z_0^{\eta + \frac{a}{\sigma^2} + n} \mathrm{e}^{\delta(\theta) z_0} - \left(\eta + \frac{a}{\sigma^2} + n\right) i_n.$$

We now consider I_2. By equation 13.2.6 in [DLMF] we know that

$$\mathrm{U}(a,b,z) \sim z^{-a}$$

as $z \to \infty$. Therefore I_2 is finite if and only if

$$\int_{z_0}^{\infty} z^{\frac{\nu_1}{\nu_2}+\frac{a}{\sigma^2}-\frac{3}{2}} \mathrm{e}^{\delta(\theta)z} dz < \infty,$$

which holds if and only if $\delta(\theta) < 0$ or $\delta(\theta) = 0$ and $\frac{\nu_1}{\nu_2} + \frac{a}{\sigma^2} < \frac{1}{2}$. Assuming now $\delta(\theta) < -1$ and using the integral representation of U again we have

$$z^{\eta-1+\frac{a}{\sigma^2}} \mathrm{e}^{\delta(\theta)z} \mathrm{U}\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}, 1+2\eta; z\right) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(b_2-1+t)\Gamma(t)}{\Gamma(a_2+t)} z^{-\eta-1+\frac{a}{\sigma^2}-t} \mathrm{e}^{(\delta(\theta)+1)z} dt,$$

with $a_2 = \eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}$ and $b_2 = 1+2\eta$ and where the contour of the integral again passes all the poles of $t \mapsto \Gamma(b_2-1+t)\Gamma(t)$ on the right hand side. As before note that $b_2 - 1 > 0$ and we are therefore able to choose the contour arbitrary close but on the right of the imaginary line. By our assumption $\delta(\theta) < -1$ we are able to interchange the integrals in I_2 because of Fubini's theorem. We obtain

$$\begin{aligned} I_2 &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(b_2-1+t)\Gamma(t)}{\Gamma(a_2+t)} \int_{z_0}^{\infty} z^{-\eta-1+\frac{a}{\sigma^2}-t} \mathrm{e}^{(\delta(\theta)+1)z} dz dt \\ &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(b_2-1+t)\Gamma(t)}{\Gamma(a_2+t)} \underbrace{\zeta^{\eta-\frac{a}{\sigma^2}+t} \Gamma\left(-\eta + \frac{a}{\sigma^2} - t, z_0\zeta\right)}_{=:j(t)} dt, \end{aligned}$$

with $\zeta = -\delta(\theta) - 1$ and $\Gamma(\cdot,\cdot)$ denoting the upper incomplete gamma function. Since $z_0\zeta \neq 0$ the map $a \mapsto \Gamma(a, z_0\zeta)$ is an entire function. We conclude that j dose not contribute to any poles of the integrand. First let us compute the poles and corresponding residues of the integrand. They are located at:

- $t = -n$ with $n \in \mathbb{N}_0$. The residue is given by

$$\frac{\Gamma(b_2-1-n)}{\Gamma(a_2-n)\,n!} (-1)^n j(-n).$$

- $t = 1 - b_2 - n$ with $n \in \mathbb{N}_0$. The residue is given by

$$\frac{\Gamma(1-b_2-n)}{\Gamma(a_2+1-b_2-n)\,n!} (-1)^n j(1-b_2-n).$$

Therefore I_2 can be written as

$$I_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\Gamma(b_2-1-n)}{\Gamma(a_2-n)} j(-n) + \frac{\Gamma(1-b_2-n)}{\Gamma(a_2+1-b_2-n)} j(1-b_2-n) \right).$$

To ensure that the residue theorem is applicable we need to check if the integral along the semi-circle $t = re^{i\varphi}$ with $\varphi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ converges to zero as $r \to \infty$. Therefore we need certain asymptotics of the gamma function as well as the incomplete gamma functions. As in [DLMF] equations 8.2.4 and 8.2.5 the normalized incomplete gamma functions are given by

$$P(a, z) = \frac{\gamma(a, z)}{\Gamma(a)},$$
$$Q(a, z) = \frac{\Gamma(a, z)}{\Gamma(a)}$$

and trivially satisfy

$$P(a, z) + Q(a, z) = 1.$$

Therefore the integrand can be written as

$$\begin{aligned}
&\frac{\Gamma(b_2 - 1 + t)\Gamma(t)}{\Gamma(a_2 + t)} j(t) \\
&= \frac{\Gamma(b_2 - 1 + t)\Gamma(t)}{\Gamma(a_2 + t)} \zeta^{\eta - \frac{a}{\sigma^2} + t} \Gamma\left(-\eta + \frac{a}{\sigma^2} - t, z_0\zeta\right) \\
&= \frac{\Gamma(b_2 - 1 + t)\Gamma(t)\Gamma\left(-\eta + \frac{a}{\sigma^2} - t\right)}{\Gamma(a_2 + t)} Q\left(-\eta + \frac{a}{\sigma^2} - t, z_0\zeta\right) \zeta^{\eta - \frac{a}{\sigma^2} + t} \\
&= \frac{\Gamma(b_2 - 1 + t)\Gamma(t)\Gamma\left(-\eta + \frac{a}{\sigma^2} - t\right)}{\Gamma(a_2 + t)} \left(1 - P\left(-\eta + \frac{a}{\sigma^2} - t, z_0\zeta\right)\right) \zeta^{\eta - \frac{a}{\sigma^2} + t}.
\end{aligned}$$

Furthermore since for $z \in \mathbb{C} \setminus \mathbb{Z}$ there holds

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}$$

and because of 8.11.5 in [DLMF] we have

$$P(a, z) \sim \frac{z^a e^{-z}}{\Gamma(1 + a)}$$

as $a \to \infty$ with $|\arg(a)| < \pi$ the behaviour of the integrand of I_2 - ignoring multiplicative constants - along the semi-circle as $r \to \infty$ is given by the quantities

$$k_1 = \frac{\Gamma(1 - a_2 - t)\Gamma\left(-\eta + \frac{a}{\sigma^2} - t\right)}{\Gamma(2 - b_2 - t)\Gamma(1 - t)} \frac{\sin(\pi(a_2 + t))}{\sin(\pi(b_2 - 1 + t))\sin(\pi t)} \zeta^t,$$
$$k_2 = \frac{\Gamma(1 - a_2 - t)\Gamma\left(-\eta + \frac{a}{\sigma^2} - t\right)}{\Gamma(2 - b_2 - t)\Gamma(1 - t)\Gamma\left(1 - \eta + \frac{a}{\sigma^2} - t\right)} \frac{\sin(\pi(a_2 + t))}{\sin(\pi(b_2 - 1 + t))\sin(\pi t)} z_0^{-t}.$$

By 5.11.12 in [DLMF] we have for $a, b \in \mathbb{R}$

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b}$$

as $z \to \infty$ with $|\arg(z)| < \pi$. Furthermore, since for $\varphi \neq \pi$

$$\lim_{r\to\infty} \frac{|\sin(\pi(a + r\mathrm{e}^{i\varphi}))|^2}{\mathrm{e}^{2r\pi|\sin(\varphi)|}} = \lim_{r\to\infty} \frac{\sin^2(\pi a + \pi r\cos(\varphi)) + \sinh^2(\pi r \sin(\varphi))}{\mathrm{e}^{2r\pi|\sin(\varphi)|}} = 1,$$

there holds

$$\left|\sin\left(\pi\left(a + r\mathrm{e}^{i\varphi}\right)\right)\right| \sim \mathrm{e}^{r\pi|\sin(\varphi)|}$$

as $r \to \infty$. Therefore we have

$$|k_1| \sim r^{b_2 - a_2 - \eta + \frac{a}{\sigma^2} - 2} \mathrm{e}^{-r\pi|\sin(\varphi)|} \mathrm{e}^{r\log(\zeta)\cos(\varphi)}.$$

To ensure that the contour integral along the semi-circle in fact converges to zero we have to ensure that $\zeta \geq 1$. Also note if $\zeta < 1$ the contour integral along the semi-circle will not converge to zero. This is can be see by looking at the contour part where φ is sufficiently close to π, which in turn results in $|\sin(\varphi)|$ being close to zero. Then the part $\mathrm{e}^{r\log(\zeta)\cos(\varphi)}$ will dominate and the integral will not converge to zero. The fact that $\zeta \geq 1$ is equivalent to $\delta(\theta) \leq -2$, which we have already seen to be possible and satisfied for $\theta \in (\theta^\star, \theta_+)$. In order to quantify the behaviour of k_2 note that by 5.11.7 in [DLMF] we have

$$\Gamma(z+b) \sim \sqrt{2\pi}\mathrm{e}^{-z} z^{z+b-\frac{1}{2}}$$

for $b \in \mathbb{C}$ as $z \to \infty$ with $|\arg(z)| < \pi$. Therefore we have

$$\left|\Gamma\left(b - r\mathrm{e}^{i\varphi}\right)\right| \sim \sqrt{2\pi}\mathrm{e}^{r\cos(\varphi)} \mathrm{e}^{-r\log(r)\cos(\varphi) + r(\varphi+\pi)\sin(\varphi) + \left(b - \frac{1}{2}\right)\log(r)}$$

as $r \to \infty$. Putting all together we have

$$|k_2| \sim r^{b_2 - a_2 - \eta + \frac{a}{\sigma^2} - 2} \mathrm{e}^{-r\pi|\sin(\varphi)|} \mathrm{e}^{r\log(z_0)\cos(\varphi)} \sqrt{2\pi}\mathrm{e}^{-r\cos(\varphi)} \mathrm{e}^{r\log(r)\cos(\varphi) - r(\varphi+\pi)\sin(\varphi) - \left(\frac{1}{2} - \eta + \frac{a}{\sigma^2}\right)\log(r)}.$$

Note that the term $\mathrm{e}^{r\log(r)\cos(\varphi)}$ always guaranties that the contour integral along the semi-circle converges to zero, because of the choice of φ. This shows that the residue theorem is applicable and we therefore conclude the proof. $\square$

6 Pricing vanilla options

Following section 6.7.8 in [Jeanblanc et al., 2009] and section 3.3 in [Da Fonseca and Martini, 2014] we are going to use the method of Mellin transformation to perform option pricing. Note that the Mellin transform of a call option in the strike can be expressed in terms of the moments of the forward:

$$\int_0^\infty \mathbb{E}\left(S_t - K\right)^+ K^{\theta-2} dK = \frac{1}{\theta\left(\theta-1\right)} \mathbb{E}\left(S_t^\theta\right)$$

for $\theta > 1$. Applying this to the 1-hypergeometric model and choosing λ and θ as in theorem 5.6 and Laplace transforming the above in time leads to

$$\int_0^\infty \mathrm{e}^{-\lambda t} \int_0^\infty \mathbb{E}\left(S_t - K\right)^+ K^{\theta-2} dK dt = \frac{1}{\theta\left(\theta-1\right)} \underbrace{\int_0^\infty \mathrm{e}^{-\lambda t} \mathbb{E}\left(S_t^\theta\right) dt}_{=:g(\theta,\lambda)},$$

which can already be calculated. Let $L\left(K,\lambda\right)$ denote the Laplace transform in time of a call option with strike K, i.e.

$$L\left(K,\lambda\right) = \int_0^\infty \mathrm{e}^{-\lambda t} \mathbb{E}\left(S_t - K\right)^+ dt.$$

By Fubini's theorem there holds

$$\int_0^\infty L\left(K,\lambda\right) K^{\theta-2} dK = \frac{g\left(\theta,\lambda\right)}{\theta\left(\theta-1\right)}.$$

In order to calculate the value of a call option we are going to numerically invert the above transform. In order to invert the Mellin transform we need the following

Lemma 6.1. *For $\theta \in \left(\theta^\star, \theta_+\right)$ and λ as in theorem 5.6 the function*

$$c \mapsto \frac{g\left(\theta + ic, \lambda\right)}{\left(\theta + ic\right)\left(\theta + ic - 1\right)}$$

is $L^1\left(\mathbb{R}\right)$.

Proof. The result follows immediately from

$$\begin{aligned}\left|\frac{g\left(\theta + ic, \lambda\right)}{\left(\theta + ic\right)\left(\theta + ic - 1\right)}\right| &\leq \frac{\int_0^\infty \mathrm{e}^{-\lambda t} \mathbb{E}\left(\left|S_t^{\theta+ic}\right|\right) dt}{\left|\left(\theta + ic\right)\left(\theta + ic - 1\right)\right|} \\ &\leq \frac{\int_0^\infty \mathrm{e}^{-\lambda t} \mathbb{E}\left(S_t^\theta\right) dt}{\left(\theta-1\right)^2 + c^2}\end{aligned}$$

and the fact that $\theta^\star > 1$, see the proof of theorem 5.6. $\square$

Therefore we can obtain L by using Mellin's inversion formula:

$$L(K,\lambda) = \int_{\theta+i\mathbb{R}} \frac{g(\tau,\lambda)}{\tau(\tau-1)} K^{-\tau+1} d\tau. \tag{22}$$

The last step to do is use for example the modified Talbot method to invert the Laplace transform, see [Dingfelder and Weideman, 2013]. The method to calculate the value of a call option is as follows: We discretize the integral in (22) with a simple quadrature of size $N \in \mathbb{N}$. Then we calculated g at the necessary points using the results of theorem 5.6. To calculated the value of the call option we then used the modified Talbot method to invert L. The valuation of a variance swap was done analogously.

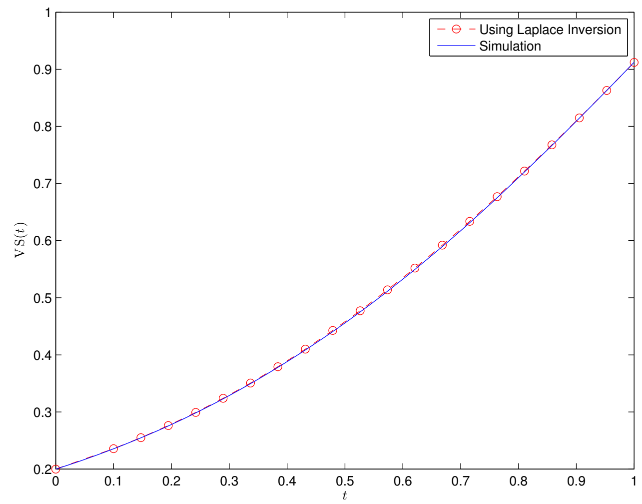


Figure 10: Plot of $t \mapsto \mathrm{VS}(t)$ with initial instantaneous variance V_0 of 20% and model parameters given by $a = 0.8$, $b = 0.4$, $\alpha = 1$ and $\sigma = 1$. Numerical inversion was performed using the modified Talbot method.

7 Appendix

7.1 Black Scholes Formula

Theorem 7.2. *The value at time t of a European call with maturity T and strike K of the asset S with dynamic $dS_t = S_t \sigma dW_t$ is given by $\mathcal{BS}(S_t, \sigma, t)$, where*

$$\mathcal{BS}(x, \sigma, t) = x\Phi\left(d_+\left(\frac{x}{K}, T - t\right)\right) - K\Phi\left(d_-\left(\frac{x}{K}, T - t\right)\right),$$

with

$$d_\pm(y, u) = \frac{1}{\sqrt{\sigma^2 u}} \ln(y) \pm \frac{\sqrt{\sigma^2 u}}{2}.$$

Proof. See theorem 2.3.2.1 in [Jeanblanc et al., 2009]. □

Theorem 7.3. *The value at time t of a European call with maturity T and strike K of the asset S with dynamic $dS_t = S_t \sigma_t dW_t$ with a deterministic function σ is given by $\mathcal{BS}(S_t, \Sigma_t, t)$, where*

$$\Sigma_t^2 = \frac{1}{T - t} \int_t^T \sigma_s^2 ds.$$

Proof. See Exercise 2.3.2.5 in [Jeanblanc et al., 2009]. □

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