Non-Holonomic Sequences and Functions

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- Rather general way to specify concrete functions/sequences in finite terms
- Several algorithms are available for the symbolic manipulation of holonomic functions and sequences.

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- Examples of holonomic sequences: α^n, $n!$, $\sum_{k=1}^{n} 1/k$, $\sharp$(involutions), etc.
- Sum and Product of two holonomic sequences (functions) are holonomic
- a_n is holonomic iff $\sum_{n\geq 0} a_n z^n$ is holonomic
- Definition can be extended to several variables $(n_1, \ldots, n_r, z_1, \ldots, z_s)$

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- Holonomic sequences $\approx$ algebraic numbers
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- If a sequence (function) is not obviously holonomic, it is usually not holonomic
- But how to come up with a rigorous proof?

Why?

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- Relations to many areas of mathematics:
- Analytic combinatorics, complex analysis, algebraic geometry, number theory
- Many ways to define sequences (functions) $\Longrightarrow$ ample opportunities for research

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- Hence 2^{2^n} and 2^{n^2} are not holonomic (What about n^n ?).
- Other argument: Singularities of a holonomic function $f(z)$ are roots of $p_d(z)$.
- Hence $\tan z$, $z/(\mathrm{e}^z - 1)$, and $\prod(1 - z^n)^{-1}$ are not holonomic.

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- Euler:
$$f_{\text{even}}(z) = \frac{1}{2} - \frac{\pi}{2} z \cot \pi z$$
- Molteni (2001):
$$f_{\text{even}}(z)/z - f_{\text{odd}}(z) = \gamma - \psi(1+z),$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function.

Known Non-Trivial Results

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- Van der Put, Singer (1997): The reciprocal $1/a_n$ of a holonomic sequence is holonomic iff a_n is an interlacement of hypergeometric sequences.
- Pólya-Carlson (1921): If $f(z)$ has integer coefficients and radius of convergence 1, then $f(z)$ is rational or has the unit circle for its natural boundary.

Proofs by Number Theory (SG, 2004)

- W.l.o.g. the polynomials $p_k(n)$ have coefficients in $\mathbb{Q}(a_j : j \geq 0)$.

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- $a_n = \sqrt{n}$ does not satisfy a recurrence

$$p_0(n)a_n + \cdots + p_d(n)a_{n+d} = 0$$

with coefficients in $\mathbb{Q}(\sqrt{j} : j \geq 0)[n]$, since

$$[\mathbb{Q}(\sqrt{\rho_1}, \ldots, \sqrt{\rho_s}) : \mathbb{Q}] = 2^s$$

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for distinct primes $\rho_1, \ldots, \rho_s$.
- Other argument: transcendence of e implies non-holonomicity of n^n.

Proofs by Asymptotics (P. Flajolet, SG, B. Salvy, 2005)

- Fuchs-Frobenius theory: Asymptotic expansion of holonomic functions as $|z| \to \infty$ must be linear combination of series of the form
$$\mathrm{e}^{P(z^{1/r})} z^{\alpha} \sum_{j \geq 0} Q_j(\log z) z^{-js},$$
where P and Q_j are polynomials, the Q_j have bounded degree, $r \in \mathbb{N}$, $\alpha \in \mathbb{C}$, $0 < s \in \mathbb{Q}$.

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where P and Q_j are polynomials, the Q_j have bounded degree, $r \in \mathbb{N}$, $\alpha \in \mathbb{C}$, $0 < s \in \mathbb{Q}$.
- Hence $\log \log z$, $\mathrm{e}^{\mathrm{e}^{z}-1}$, and Lambert W are not holonomic.

Proofs by Asymptotics

- **Basic Abelian theorem.** Let $\phi(x)$ be any of the functions

$$x^{\alpha}(\log x)^{\beta}(\log\log x)^{\gamma}, \qquad \alpha \geq 0, \quad \beta,\gamma \in \mathbb{C}. \qquad (1)$$

Let (a_n) be a sequence that satisfies the asymptotic estimate

$$a_n \underset{n\to\infty}{\sim} \phi(n).$$

Then the generating function $f(z) := \sum_{n\geq 0} a_n z^n$ satisfies the asymptotic estimate

$$f(z) \underset{z\to 1-}{\sim} \Gamma(\alpha+1)\frac{1}{(1-z)}\phi\left(\frac{1}{1-z}\right). \qquad (2)$$

Proofs by Asymptotics

- The sequence of prime numbers:

$$n\text{-th prime} = n \log n + n \log \log n + \mathrm{O}(n),$$

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- The sequences of powers ($\alpha \in \mathbb{C} \setminus \mathbb{Z}$):

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k k^\alpha \sim \frac{(\log n)^{-\alpha}}{\Gamma(1-\alpha)}.$$

$e^{1/n}$ by Asymptotics (P. Flajolet, SG, B. Salvy, 2005)

- Lindelöf integral representation

$$\sum_{n\geq 1} e^{1/n}(-z)^n = -\frac{1}{2i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{z^s e^{1/s}}{\sin \pi s} ds$$

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- Asymptotics (saddle point method)

$$\sum_{n\geq 1} \mathrm{e}^{1/n}(-z)^n \sim -\frac{\mathrm{e}^{2\sqrt{\log z}}}{2\sqrt{\pi}(\log z)^{1/4}} \qquad \text{as} \quad |z| \to \infty.$$

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- Hence $\mathrm{e}^{1/n}$ is not holonomic.
- Work in progress: generalize asymptotics to α^{n^β}.

Closed-Form Sequences (J.P. Bell, SG, M. Klazar, F. Luca, 2006)

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- The recurrence means that

$$F(z; p_0, \ldots, p_d) := \sum_{k=0}^{d} p_k(z) f(z+k)$$

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  vanishes at $z = 0, 1, 2, \ldots$.
- Example: $f(z) = z^\alpha$. If F vanishes identically, the left hand side of

$$f(z) = -\frac{1}{p_0(z)} \sum_{k=1}^{d} p_k(z) f(z+k)$$

  is meromorphic at $z = 0$, hence $\alpha \in \mathbb{Z}$.

Zeroes of Analytic and of Elementary Functions

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- Khovanskiĭ investigates the geometry of the zero set of elementary functions in his book "Fewnomials".
- **Definition.** *Elementary functions* are built by composing rational functions, $\exp(x)$, $\log(x)$, $\sin(x)$, $\cos(x)$, $\tan x$, $\arcsin(x)$, $\arccos(x)$, and $\arctan(x)$. The domain of definition must be such that arguments of sin and cos are bounded.

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- **Theorem** (Khovanskiĭ). An elementary function has only finitely many simple zeros in its domain of definition.

Results proved using Carlson or Khovanskiĭ

- For distinct complex $u_1, \ldots, u_s$, the sequence $\Gamma(n-u_1)^{\alpha_1} \ldots \Gamma(n-u_s)^{\alpha_s}$ is holonomic if and only if $\alpha_1, \ldots, \alpha_s$ are integers.

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- If a sequence from $\mathbb{R}(n, \mathrm{e}^n)$ is holonomic, then the denominator has just one summand.
- If $(f(n))_{n \geq 1}$ is holonomic for an algebraic function $f :]1, \infty] \to \mathbb{R}$, then f is a rational function.

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- Opportunity to apply methods and results from various areas
- There is a good chance that holonomicity of a sequence can be decided if it has (i) a closed form representation or (ii) a known asymptotic expansion.