An Implementation of the LIBOR Market Model

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August 30, 2006

The Libor Market Model (Brace et al. 1997)

- Based on observable market rates
- Models a discretization of the yield curve
- Easy calibration to caplet volatilities
- Transparent modelling of correlation of different points on the yield curve

Forward Rates in the LMM

- Fix a tenor structure $T_0 < \cdots < T_M$
- Let $F_k(t) = F(t; T_{k-1}, T_k)$ be the simply compounded interest rate prevailing at time t for borrowing from T_{k-1} to T_k
- For each modelled rate, reset T_{k-1} and maturity T_k are fixed, while time t increases
- Example: If $T_k - T_{k-1} = 3$ months, then $F_k(T_{k-1})$ is the 3-months-EURIBOR at T_{k-1}. At smaller time t, it is a forward 3-months-EURIBOR.

Forward Rates and Their Volatilities

- The LMM models a vector of spanning forward rates $(F_1(t), \ldots, F_M(t))$
- Deterministic volatility functions $\sigma_1(t), \ldots, \sigma_M(t)$
- Volatility $\sigma_k(t)$ is a function of time to reset $T_{k-1} - t$
- Parametric shape for the volatility functions
- Constant instantaneous correlation ρ_{ij}

The Forward Measure

- Fix some time horizon T
- Under the T-forward measure, for every traded asset V the process $V_t/B(t,T)$ is driftless
- $V_t/B(t,T)$ has constant expectation: $\frac{V_0}{B(0,T)} = \mathbf{E}^T[\frac{V_T}{B(T,T)}]$
- The price of V at time 0 is $B(0,T)\mathbf{E}^T[V_T]$

Forward Rates and Forward Measures

- $F_k(t) = \frac{1}{T_k - T_{k-1}} \left(\frac{B(t, T_{k-1})}{B(t, T_k)} - 1 \right)$
- Hence $F_k(t)$ is driftless under the T_k-forward measure
- $F_k(t)$ is *not* driftless under other T-forward measures
- Drift of $F_k(t)$ under T_j is a rational function of the other rates

Black's Formula for Caplets

- Assumes log-normal distribution of underlying
- Market standard for pricing caplets
- Volatilities for caps of various maturities and tenors are quoted
- LMM provides theoretical justification of Black's formula
- $F_k(T_{k-1})$ is log-normal under T_k-forward measure
- Libor Market Model $\approx$ multiple Black models

Volatility Calibration

- Volatility of $F_k(t)$ is

$$\sigma_k(t) = \phi_k(a + b(T_{k-1} - t))\mathrm{e}^{-c(T_{k-1}-t)} + d$$

- a, b, c, d independent of k
- Calibration to caplet prices:

$$\sqrt{\frac{1}{T_{k-1}}\int_0^{T_{k-1}} \sigma_k(t)^2 \mathrm{d}t} = T_{k-1} - \text{caplet volatility}$$

- Fit a, b, c, d by least squares, then determine ϕ_k for exact fit

Correlation Calibration

Correlation Matrix obtained from 1-month-log-returns, data from Jan 2001 to Jul 2007

$$
\begin{pmatrix}
1. & 0.88 & \ddots & 0.71 & 0.65 & 0.57 & 0.53 & 0.48 & 0.47 & 0.085 \\
0.88 & 1. & \ddots & 0.84 & 0.76 & 0.66 & 0.61 & 0.57 & 0.55 & 0.11 \\
0.82 & 0.97 & \ddots & 0.93 & 0.86 & 0.78 & 0.72 & 0.69 & 0.67 & 0.26 \\
0.77 & 0.92 & \ddots & 0.98 & 0.94 & 0.88 & 0.83 & 0.81 & 0.79 & 0.42 \\
0.71 & 0.84 & \ddots & 1. & 0.98 & 0.94 & 0.9 & 0.88 & 0.87 & 0.55 \\
0.65 & 0.76 & \ddots & 0.98 & 1. & 0.98 & 0.96 & 0.93 & 0.93 & 0.65 \\
0.57 & 0.66 & \ddots & 0.94 & 0.98 & 1. & 0.99 & 0.97 & 0.96 & 0.75 \\
0.53 & 0.61 & \ddots & 0.9 & 0.96 & 0.99 & 1. & 0.98 & 0.98 & 0.8 \\
0.48 & 0.57 & \ddots & 0.88 & 0.93 & 0.97 & 0.98 & 1. & 0.99 & 0.83 \\
0.47 & 0.55 & \ddots & 0.87 & 0.93 & 0.96 & 0.98 & 0.99 & 1. & 0.86 \\
0.085 & 0.11 & \ddots & 0.55 & 0.65 & 0.75 & 0.8 & 0.83 & 0.86 & 1.
\end{pmatrix}
$$

Correlation Calibration

- Parametric structure $\rho_{ij} = \ell + (1 - \ell)\mathrm{e}^{-\theta|i-j|}$
- Best fit for $\ell = 0.54$, $\theta = 0.25$
- Positive correlations
- Decrease when moving away from main diagonal
- Refined parametric structures allow increase along subdiagonals

An Example

- A structured note with 14 annual coupons
- Coupon V_k at T_k depends on $F_k, \ldots, F_{k+9}$ at T_{k-1}
- We propagate 22 rates $F_1, \ldots, F_{22}$
- Value at time 0 is

$$B(0, T_0)V_0 + B(0, T_M) \cdot \mathbf{E}^{T_M}[\sum_{k=1}^{13} \frac{V_k}{B(T_k, T_M)}]$$

Volatility Calibration

- A cap on $F_1(T_0)$ is a swaption with maturity T_0 and swap length one year
- ÖBFA provides swaption volatilities for maturities 1M, 3M, 6M, 1Y, . . . , 20Y
- Swaption volatilities for maturities $T_0, \ldots, T_M$ are interpolated from market data
- Parameters a, b, c, d are found by minimization with Mathematica
- Fairly good fit to swaption volatility surface

Monte Carlo Simulation

- Use dynamics of F under terminal measure

$$\frac{\mathrm{d}F_k(t)}{F_k(t)} = \mu(F_{k+1}(t), \dots, F_M(t))\,\mathrm{d}t + \sigma_k(t)\,\mathrm{d}W_k(t)$$

- W Brownian motion with instantaneous correlation matrix $\boldsymbol{\rho}$
- $W(t + \Delta t) - W(t) \sim \sqrt{\Delta t}\,\mathcal{N}(0, \boldsymbol{\rho})$
- Parameters: Number of time discretization points, number of Monte Carlo trials, number of factors

Conclusion

- Unlike short rate models, the LMM is based on observable market rates
- The LMM can price almost any interest rate product
- Mathematica + UnRisk provide comfortable implementation
- There are many variants and extensions