The Sign Structure of Linear Recurrence Sequences

Stefan Gerhold, TU Vienna

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Linear Recurrence Sequences over $\mathbb{R}$

- Linear Recurrence of order h

$$f_{n+h} = c_1 f_{n+h-1} + \cdots + c_{h-1} f_{n+1} + c_h f_n, \qquad n \geq 0$$

- Rational generating function

$$\sum_{n=0}^{\infty} f_n z^n = \frac{\text{polynomial}(z)}{1 - c_1 z - \cdots - c_h z^h}$$

- Explicit formula

$$f_n = P_1(n)\alpha_1^n + \cdots + P_m(n)\alpha_m^n, \qquad n \geq 0,$$

where the *characteristic roots* $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ are the roots of the *characteristic polynomial* $z^h - c_1 z^{h-1} - \cdots - c_{h-1} z - c_h$, and $P_i(x) \in \mathbb{C}[x]$.

The Set of Zeros and the Positivity Set

- **Theorem** (Skolem-Mahler-Lech 1953). The set

$$\{n \in \mathbb{N} : f_n = 0\}$$

is the union of a finite set and finitely many arithmetic progressions.

- What about the positivity set

$$\{n \in \mathbb{N} : f_n > 0\}?$$

Sequences with no Positive Dominating Root

- What is the sign of $(2+\mathrm{i})^n + (2-\mathrm{i})^n$?
- **Question:** Are there infinitely many sign changes, if no dominating root is positive?
- (Side remark: Lower growth estimate

$$5^{n/2}/n^C < |(2+\mathrm{i})^n + (2-\mathrm{i})^n|$$

Schinzel 1967, based on Baker's theorem about linear forms in logarithms)

Are There Infinitely Many Sign Changes, if no Dominating Root is Positive?

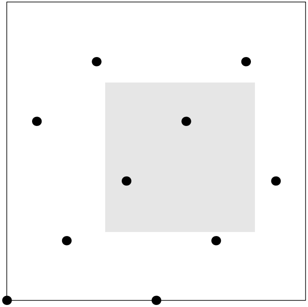
- 1 dominating root: trivial (root has to be real negative)
- 2 dominating roots: Burke, Webb 1981
  Example:
  $$(2+\mathrm{i})^n + (2-\mathrm{i})^n \lesseqgtr 0$$
- 3 or 4 dominating roots: SG 2005 (By Diophantine geometry)
  Example:
  $$(-\sqrt{5})^n + (2+\mathrm{i})^n + (2-\mathrm{i})^n \lesseqgtr 0$$

Normalization

- W.l.o.g. the dominating characteristic roots are simple and lie on the unit circle
- $$f_n = \sum_{i=1}^{d} a_i \cos(2\pi\theta_i n + \beta_i) + v - r_n, \qquad n \geq 0$$
- $\theta_i := \arg(\alpha_i)/2\pi$
- $a_i, \beta_i, v \in \mathbb{R}$
- $r_n = \mathrm{o}(1)$ smaller order part

Sequences with no Positive Dominating Root: Two Pairs of Conjugated Roots (SG 2005)

- $f_n = a_1 \cos(2\pi\theta_1 n + \beta_1) + a_2 \cos(2\pi\theta_2 n + \beta_2)$
- Example: $(\theta_1, \theta_2) = (\frac{7}{10}, \frac{1}{5})$
- The set $\{(7n/10, n/5) \bmod 1 : n \in N\}$



Sequences with no Positive Dominating Root: Two Pairs of Conjugated Roots (SG 2005)

- Does $(\cos(2\pi\theta_1 n + \beta_1), \cos(2\pi\theta_2 n + \beta_2))$ assume all four sign combinations

$$(+1, +1), (+1, -1), (-1, +1), (-1, -1)?$$

- Every square with side length $1/2$, parallel to the axes, contains a point (proof uses Minkowski's theorem from Diophantine geometry)
- Hence f_n oscillates for all a_1, a_2 (not both zero). $\square$

Pringsheim's Theorem

- **Theorem** (Pringsheim 1894). Let $f(z) = \sum_{n\geq 0} f_n z^n$ be analytic at zero and $f_n \geq 0$ for almost all $n \in \mathbb{N}$. Then $f(z)$ has a singularity at $z = R$, where R is the radius of convergence of $\sum_{n\geq 0} f_n z^n$.
- **Corollary.** Linear Recurrence sequences with no real positive dominating root oscillate.
- *Proof.* Singularities of generating function $f(z)$ are contained in $\{\alpha_1^{-1}, \ldots, \alpha_m^{-1}\}$. □
- But: Number theoretic methods yield a much stronger result!

Sequences with no Positive Dominating Root

- **Theorem** (Bell, SG 2007): The sets $\{n : f_n > 0\}$ and $\{n : f_n < 0\}$ have positive density.
- Idea: Study the function of $\mathbf{t} = (t_1, \ldots, t_d)$ resulting from f_n upon replacing each $n\theta_i$ by a real t_i in

$$f_n = \sum_{i=1}^{d} a_i \cos(2\pi\theta_i n + \beta_i) + v - r_n, \qquad n \geq 0.$$

- **Theorem** (Kronecker, Weyl): The sequence $(n\theta_1, \ldots, n\theta_d)$ is uniformly distributed modulo 1, if the numbers $1, \theta_1, \ldots, \theta_d$ are linearly independent over $\mathbb{Q}$.

Sequences with no Positive Dominating Root

- The other "extreme case": all θ_k are rational. Let q be a common denominator.

$$\begin{aligned}\sum_{j=0}^{q-1} f_j &= \sum_{j=0}^{q-1}\sum_{i=1}^{d} a_i \cos(2\pi j\theta_i + \beta_i)\\ &= \sum_{i=1}^{d} a_i \sum_{j=0}^{q-1} (\cos\beta_i \cdot \cos 2\pi j\theta_i - \sin\beta_i \cdot \sin 2\pi j\theta_i)\\ &= 0.\end{aligned}$$

Sequences with no Positive Dominating Root

- General case: some integer linear relations among $\theta_1, \ldots, \theta_d$
- Idea: Pass to a convenient basis

$$\{\tau_1, \ldots, \tau_{m+1}\}$$

  of the $\mathbb{Z}$-module

$$\mathbb{Z} + \mathbb{Z}\theta_1 + \cdots + \mathbb{Z}\theta_d.$$

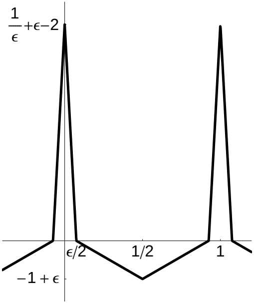
- $\tau_{m+1} = 1/g$, with g a positive integer, and $1, \tau_1, \ldots, \tau_m$ linearly independent over $\mathbb{Q}$
- Consider subsequences $(f_{gn+k})_{n \geq 0}$, $k = 0, \ldots, g-1$ $\square$

Which Numbers Occur as Density of the Positivity Set?

- Without restriction on the dominating roots: all numbers from $[0, 1]$ occur (Consider $f_n = \cos(2\pi\sqrt{2}n) - w$).
- No positive dominating root: Numerical experiments usually yield approximations $\approx 1/2$.
- **Theorem.** For every $\kappa \in]0, 1[$, there is a linear recurrence sequence f_n with no positive dominating root and

$$\text{density}(\{n \in N : f_n > 0\}) = \kappa.$$

Constructing Positivity Dets With Small Density



- Expand this function into a Fourier series

$$H(t) = \sum_{j=1}^{\infty} a_j \cos(2\pi j t),$$

truncate, replace t by $n\sqrt{2}$. □

Simultaneously Prescribing the Density of the Positivity Set and the Zero Set

- **Theorem.** r rational, $0 \leq \kappa, r \leq 1$, $\kappa + r \leq 1$. Then there is a linear recurrence sequence f_n with

$$\text{density}(\{n \in N : f_n > 0\}) = \kappa,$$

$$\text{density}(\{n \in N : f_n = 0\}) = r.$$

- r must be rational by the Skolem-Mahler-Lech theorem (zero set is periodic).

Investigating the Sign Pattern by Computer Algebra (Kauers, SG 2005)

- Example:

$$f_0 = 2 + \sqrt{2}, \quad f_1 = 2 + \sqrt{10}, \quad f_2 = -2 + 5\sqrt{2},$$
$$f_n = (4 + \sqrt{5}) f_{n-1} - (5 + 4\sqrt{5}) f_{n-2} + 5\sqrt{5} f_{n-3}, \qquad n \geq 3.$$

- Polynomial identity (found by computer algebra)

$$25 f_n^2 - \tfrac{10}{11}(14 + 13\sqrt{5}) f_n f_{n+1} - \tfrac{20}{11}(2 - 6\sqrt{5}) f_n f_{n+2} + (6 + 4\sqrt{5}) f_{n+1}^2 + f_{n+2}^2 - \tfrac{2}{11}(14 - 13\sqrt{5}) f_{n+1} f_{n+2} = 0, \qquad (*)$$

Computer Aided Induction Proofs (Kauers, SG 2005)

- replace f_n, f_{n+1}, f_{n+2} by real variables F_0, F_1, F_2
- Then the formula

$$\text{recurrence} \wedge (*) \wedge (F_0 \geq 0) \wedge (F_1 < 0) \wedge (F_2 \geq 0)$$

  is unsatisfiable over $\mathbb{R}$ (Cylindrical Algebraic Decomposition)
- Hence $(f_n \geq 0 \wedge f_{n+1} < 0) \Rightarrow f_{n+2} < 0$
- General sign pattern: $+^9[+ \mid 0] -^4 [- \mid 0]$

Computer Aided Induction Proofs (Kauers, SG 2005)

- The same procedure yields automatic proofs of many inequalities
- Scope: Objects need to be defined by linear or polynomial recursions
- Indeterminates are allowed
- Examples: Cauchy-Schwarz, Turán's inequality for Legendre polynomials, . . .

Conclusion

- It is unknown whether the problem "$f_n > 0$ for all n?" is decidable for linear recurrence sequences
- We have clarified the behaviour in the case of no positive dominating root
- **Problem:** Do we have

$$1 + \cos(2\pi\theta n) + (-1/2)^n > 0, \qquad \text{where } \theta = \sqrt[3]{2}?$$

(Remark: for almost all θ, this sequence is eventually positive.)