Recurrence Relations and Inequalities

Stefan Gerhold

RISC
(Research Institute for Symbolic Computation)

Linz/Hagenberg, Austria

Proving Inequalities by Computer Algebra

(with M. Kauers, ISSAC 2005)

- We want to prove $a_n > 0$, $n \geq 0$
- a_n polynomially recursive (not necessarily holonomic)

$$a_{n+s} = P(a_n, \ldots, n_{n+s-1}), \qquad n \geq 0.$$

  Example: 2^{2^n}

- induction step

$$a_n > 0, \ldots, a_{n+r-1} > 0 \Longrightarrow a_{n+r} > 0$$

- Sufficient:

$$X_0 > 0, \ldots, X_{r-1} > 0 \Longrightarrow X_r > 0 \qquad (*)$$

  for all real numbers $X_0, \ldots, X_r$ that satisfy polynomial equations arising from the recurrence of a_n.

- Increase r, if formula $(*)$ does not hold;
  or encode known inequalities/identities as additional inequalities/equations for the X_k.

Proving Inequalities by Computer Algebra

- a_n may involve undetermined parameters
- Works on many examples (Cauchy-Schwarz, Bernoulli, Turán, ...)
- But no a priori termination criterion known
- Example gallery (omitting some constraints):

$$(x+1)^n \geq 1 + nx, \qquad n \geq 0, x \geq -2 \ (!)$$

$$P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0$$

$$\Big(\sum_{k=1}^n x_k y_k\Big)^2 \leq \sum_{k=1}^n x_k^2 \sum_{k=1}^n y_k^2$$

$$\prod_{k=1}^n (1-a_k) > 1 - \sum_{k=1}^n a_k$$

$$(n-1)\Big(\sum_{k=1}^n a_k\Big)^2 \geq 2n \sum_{k=1}^n a_k \sum_{i=1}^{k-1} a_i$$

$$\sqrt{n + \sqrt{(n-1) + \sqrt{\cdots + \sqrt{2 + \sqrt{1}}}}} < \sqrt{n} + 1$$

Linear Recurrences with Constant Coefficients

- Is there a subclass over which positivity is decidable?

- C-finite sequences are a natural candidate:

$$c_0 a_n + c_1 a_{n+1} + \cdots + c_d a_{n+d} = 0, \qquad n \geq 0.$$

- Question: $a_n > 0$, n large?
- power sum representation

$$a_n = \sum_{k=1}^{d} b_k \alpha_k^n,$$

  α_k roots of characteristic polynomial (assumed to be simple here).

- dominating roots := roots of largest modulus
- Usually the sign is trivial:

$$3^n + (-2)^n > 0$$
$$(-3)^n + (-2)^n \lesseqgtr 0$$

Sequences with no Positive Dominating Root

- What is the sign of $(2+\mathrm{i})^n + (2-\mathrm{i})^n$?
- (Side remark: Lower estimate

$$5^{n/2}/n^C < |(2+\mathrm{i})^n + (2-\mathrm{i})^n|$$

  Schinzel 1967, based on Baker's theorem about linear forms in logarithms)
- Conjecture: infinitely many sign changes, if no dominating root is positive.
- 1 dominating root: trivial (root has to be real negative)
- 2 dominating roots: Burke, Webb 1981

  Example:

$$(2+\mathrm{i})^n + (2-\mathrm{i})^n \lesseqgtr 0$$

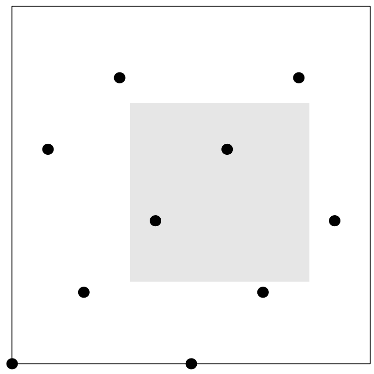
- 3 or 4 dominating roots: SG 2005 (By Diophantine geometry)

  Example:

$$(-\sqrt{5})^n + (2+\mathrm{i})^n + (2-\mathrm{i})^n \lesseqgtr 0$$

Sequences with no Positive Dominating Root: Two Pairs of Conjugated Roots

- $a_n = w_1 \sin(2\pi n\theta_1 + \varphi_1) + w_2 \sin(2\pi n\theta_2 + \varphi_2)$
- Example: $(\theta_1, \theta_2) = (\frac{7}{10}, \frac{1}{5})$
- The set $\{(7n/10, n/5) \bmod 1 : n \in \mathbb{N}\}$



- Every square with side length 1/2, parallel to the axes, contains a point (Minkowsi's theorem from Diophantine geometry)
- Hence $(\sin(2\pi n\theta_1 + \varphi_1), \sin(2\pi n\theta_2 + \varphi_2))$ assumes all four sign combinations

$$(+1, +1), (+1, -1), (-1, +1), (-1, -1).$$

- Hence a_n oscillates for all w_1, w_2 (not both zero).

Sequences with no Positive Dominating Root: General Case

- **Theorem** (Bell, SG 2005): Let a_n be a C-finite sequence, not identically zero, with no positive dominating root. Then the sets $\{n : a_n > 0\}$ and $\{n : a_n < 0\}$ have positive density.

- **Theorem** (Kronecker, Weyl): The sequence $(n\theta_1, \ldots, n\theta_m)$ is uniformly distributed modulo 1, if the numbers $1, \theta_1, \ldots, \theta_m$ are linearly independent over $\mathbb{Q}$.

- Idea: Study the function of $\mathbf{t}$ resulting from a_n upon replacing each $n\theta_k$ by a real t_k.

- The other "extreme case": all θ_k are rational. Let q be a common denominator.

$$\begin{aligned}
\sum_{j=0}^{q-1} a_j &= \sum_{j=0}^{q-1} \sum_{i=1}^{m} w_i \sin(2\pi j\theta_i + \varphi_i) \\
&= \sum_{i=1}^{m} w_i \sum_{j=0}^{q-1} (\cos\varphi_i \cdot \sin 2\pi j\theta_i + \sin\varphi_i \cdot \cos 2\pi j\theta_i) \\
&= 0.
\end{aligned}$$

Which Numbers Occur as Density of the Positivity Set?

- No positive dominating root: Numerical experiments usually yield approximations $\approx 1/2$.
- **Theorem**: For every $\kappa \in]0,1[$, there is a C-finite sequence a_n with no positive dominating root and

$$\text{density}(\{n \in \mathbb{N} : a_n > 0\}) = \kappa.$$

- With positive dominating root: all numbers from $[0,1]$ occur.
- **Theorem**: r rational, $0 \le \kappa, r \le 1$, $\kappa + r \le 1$. Then there is a C-finite sequence a_n with

$$\text{density}(\{n \in \mathbb{N} : a_n > 0\}) = \kappa,$$

$$\text{density}(\{n \in \mathbb{N} : a_n = 0\}) = r.$$

- r must be rational by the Skolem-Mahler-Lech theorem (zero set is periodic).

Conclusion

- Inequalities are more difficult than identities
- Still, many examples can be done by the method we presented
- But even positivity of C-finite sequences is not known to be decidable
- Diophantine methods reveal oscillating behaviour
- Problem: Do we have

  $$1 + \sin(2\pi\theta n) + (-1/2)^n > 0, \qquad \text{where } \theta = \sqrt{2}?$$

  (Remark: for almost all θ, this sequence is eventually positive.)