Recurrence Relations
and Inequalities

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Proving Inequalities
by Computer Algebra

(with M. Kauers, ISSAC 2005)

• We want to prove $a_n > 0$, $n \geq 0$

• $a_n$ polynomially recursive (not necessarily holonomic)

$$a_{n+s} = P(a_n, \ldots, n_{n+s-1}), \quad n \geq 0.$$  

Example: $2^{2^n}$

• induction step

$$a_n > 0, \ldots, a_{n+r-1} > 0 \implies a_{n+r} > 0$$

• Sufficient:

$$X_0 > 0, \ldots, X_{r-1} > 0 \implies X_r > 0 \quad (\ast)$$

for all real numbers $X_0, \ldots, X_r$ that satisfy polynomial equations arising from the recurrence of $a_n$.

• Increase $r$, if formula $(\ast)$ does not hold;
  or encode known inequalities/identities as additional inequalities/equations for the $X_k$.  

Proving Inequalities
by Computer Algebra

• \( a_n \) may involve undetermined parameters

• Works on many examples (Cauchy-Schwarz, Bernoulli, Turán, \ldots)

• But no a priori termination criterion known

• Example gallery (omitting some constraints):

\[(x + 1)^n \geq 1 + nx, \quad n \geq 0, x \geq -2 \ (!)
\]

\[P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0\]

\[
\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^2
\]

\[
\prod_{k=1}^{n} (1 - a_k) > 1 - \sum_{k=1}^{n} a_k
\]

\[(n - 1)\left(\sum_{k=1}^{n} a_k\right)^2 \geq 2n \sum_{k=1}^{n} a_k \sum_{i=1}^{k-1} a_i
\]

\[
\sqrt{n + \sqrt{(n - 1) + \sqrt{\cdots + \sqrt{2 + 1}}} < \sqrt{n + 1}}
\]
Linear Recurrences with Constant Coefficients

• Is there a subclass over which positivity is decidable?

• C-finite sequences are a natural candidate:

\[ c_0 a_n + c_1 a_{n+1} + \cdots + c_d a_{n+d} = 0, \quad n \geq 0. \]

• Question: \( a_n > 0, \) \( n \) large?

• power sum representation

\[ a_n = \sum_{k=1}^{d} b_k \alpha_k^n, \]

\( \alpha_k \) roots of characteristic polynomial (assumed to be simple here).

• dominating roots := roots of largest modulus

• Usually the sign is trivial:

\[ 3^n + (-2)^n > 0 \]
\[ (-3)^n + (-2)^n \leq 0 \]
Sequences with no Positive Dominating Root

• What is the sign of \((2 + i)^n + (2 - i)^n\)?

• (Side remark: Lower estimate

\[ 5^{n/2}/n^C < |(2 + i)^n + (2 - i)^n| \]

Schinzel 1967, based on Baker’s theorem about linear forms in logarithms)

• Conjecture: infinitely many sign changes, if no dominating root is positive.

• 1 dominating root: trivial (root has to be real negative)

• 2 dominating roots: Burke, Webb 1981

Example:

\[ (2 + i)^n + (2 - i)^n \leq 0 \]

• 3 or 4 dominating roots: SG 2005 (By Diophantine geometry)

Example:

\[ (-\sqrt{5})^n + (2 + i)^n + (2 - i)^n \leq 0 \]
Sequences with no Positive Dominating Root: Two Pairs of Conjugated Roots

- $a_n = w_1 \sin(2\pi n \theta_1 + \varphi_1) + w_2 \sin(2\pi n \theta_2 + \varphi_2)$
- Example: $(\theta_1, \theta_2) = (\frac{7}{10}, \frac{1}{5})$
- The set $\{(7n/10, n/5) \mod 1 : n \in \mathbb{N}\}$

- Every square with side length 1/2, parallel to the axes, contains a point (Minkowski’s theorem from Diophantine geometry)
- Hence $(\sin(2\pi n \theta_1 + \varphi_1), \sin(2\pi n \theta_2 + \varphi_2))$ assumes all four sign combinations $\:+1, +1), (+1, -1), (-1, +1), (-1, -1)\).$
- Hence $a_n$ oscillates for all $w_1, w_2$ (not both zero).
Sequences with no Positive Dominating Root: General Case

- **Theorem** (Bell, SG 2005): Let $a_n$ be a C-finite sequence, not identically zero, with no positive dominating root. Then the sets \( \{ n : a_n > 0 \} \) and \( \{ n : a_n < 0 \} \) have positive density.

- **Theorem** (Kronecker, Weyl): The sequence \( (n\theta_1, \ldots, n\theta_m) \) is uniformly distributed modulo 1, if the numbers \( 1, \theta_1, \ldots, \theta_m \) are linearly independent over \( \mathbb{Q} \).

- Idea: Study the function of \( t \) resulting from \( a_n \) upon replacing each \( n\theta_k \) by a real \( t_k \).

- The other “extreme case”: all \( \theta_k \) are rational. Let \( q \) be a common denominator.

\[
\sum_{j=0}^{q-1} a_j = \sum_{j=0}^{q-1} \sum_{i=1}^{m} w_i \sin(2\pi j \theta_i + \varphi_i)
= \sum_{i=1}^{m} w_i \sum_{j=0}^{q-1} (\cos \varphi_i \cdot \sin 2\pi j \theta_i + \sin \varphi_i \cdot \cos 2\pi j \theta_i)
= 0.
\]
Which Numbers Occur as Density of the Positivity Set?

- No positive dominating root: Numerical experiments usually yield approximations $\approx 1/2$.

- **Theorem:** For every $\kappa \in ]0, 1[$, there is a C-finite sequence $a_n$ with no positive dominating root and
  
  $$\text{density}(\{n \in \mathbb{N} : a_n > 0\}) = \kappa.$$ 

- With positive dominating root: all numbers from $[0, 1]$ occur.

- **Theorem:** $r$ rational, $0 \leq \kappa, r \leq 1, \kappa + r \leq 1$. Then there is a C-finite sequence $a_n$ with
  
  $$\text{density}(\{n \in \mathbb{N} : a_n > 0\}) = \kappa,$$
  
  $$\text{density}(\{n \in \mathbb{N} : a_n = 0\}) = r.$$ 

- $r$ must be rational by the Skolem-Mahler-Lech theorem (zero set is periodic).
Conclusion

- Inequalities are more difficult than identities
- Still, many examples can be done by the method we presented
- But even positivity of C-finite sequences is not known to be decidable
- Diophantine methods reveal oscillating behaviour
- Problem: Do we have

\[ 1 + \sin(2\pi \theta n) + (-1/2)^n > 0, \quad \text{where } \theta = \sqrt{2}? \]

(Remark: for almost all \( \theta \), this sequence is eventually positive.)