Special Functions: Applications of Computer Algebra in Stochastics

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The Holonomic Framework

- Idea: Define functions by linear ordinary differential equations

$$p_0(x)f^{(0)}(x) + \cdots + p_d(x)f^{(d)}(x) = 0$$

with polynomial coefficients + initial values.

- Examples: rational functions, exp, sin, cos, Bessel, etc.
- Define sequences by linear recurrences

$$p_0(n)f_n + \cdots + p_d(n)f_{n+d} == 0$$

with polynomial coefficients + initial values.

- Examples: a^n, $n!$, $\sum_{k=1}^{n} 1/k$, etc.
- Several algorithms are available for the symbolic manipulation of holonomic functions and sequences.

Example 1: Proving (and Finding!) a Binomial Coefficients Identity

- $X \sim \mathbf{Hyp}(n, M, N)$
- Given $\mathrm{E}(X) = \frac{nM}{M+N}$,
- what is

$$v_n := \mathrm{Var}(X) = \sum_k \left(k - \frac{nM}{M+N}\right)^2 \frac{\binom{M}{k}\binom{N}{n-k}}{\binom{M+N}{n}} = ?$$

- Zeilberger's Algorithm (implemented in Mathematica) gives

$$-(n+1)(n-M-N+1)v_n + n(n-M-N)v_{n+1} = 0$$

- Hence

$$\mathrm{Var}(X) = \frac{nM(N + n(M-1))}{(M+N)(M+N-1)}.$$

Example 2: Solving a Recurrence from Ruin Theory

- Albrecher, Teugels, and Tichy (2001) consider the extended classical risk model with constant force of real interest i and risk parameter λ. Let $Z(t)$ denote the reserve at time t.
- They expand the non-ruin probability

$$U(x,t) = \mathrm{P}\{Z(s) \geq 0 \forall 0 \leq s \leq t \mid Z(0) = x\}$$

as a gamma series

$$U(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t) \frac{\gamma(n, \alpha x)}{\Gamma(n)}.$$

- Special case: For exponential claim size distribution $\mathbf{E}(\alpha)$ and $t \to \infty$ they derive the recurrence

$$a_{n+1}(\infty) = \frac{1}{\alpha c}((\lambda + \alpha c - in) a_n(\infty) + (in - i - \lambda) a_{n-1}(\infty)).$$

- Petkovsek's algorithm Hyper yields

$$\frac{a_{n+1}(\infty)}{a_n(\infty)} = \frac{\lambda - in}{\alpha c},$$

hence

$$a_n(\infty) = a_0(\infty) n! \binom{\lambda/i}{n} \left(\frac{i}{\alpha c}\right)^n.$$

Example 3: Factorial Transforms and Convolutions

- Fact: a_n holonomic $\iff \sum_{n\geq 0} a_n x^n$ holonomic
- Factorial transform

$$\mathsf{T}(\sum a_n x^n) = \sum n! a_n x^n$$

- ODE for $\sum a_n x^n \quad \mapsto \quad$ rec. for $a_n \quad \mapsto$
  rec. for $n!a_n \quad \mapsto \quad$ ODE for $\mathsf{T}(\sum a_n x^n)$
  (and vice versa)
- T and convolution:

$$\mathsf{T}(F * G) = \mathsf{T}(F) \cdot \mathsf{T}(G),$$

  where

$$(F * G)(x) = \int_{\mathbb{R}} F(x-y)\mathrm{d}G(y).$$

- Can be used to solve the renewal equation

$$U = p + qU * G$$

  (p, q parameters) for U:

$$\mathsf{T}(U) = p/(1 - q\mathsf{T}(G)).$$

- De Vylder, Marceau: Explicit analytic ruin probabilities for bounded claims (1995)

Example 4: Recurrences in the Collective Model

- Generating Functions of claim size X and claim number N

$$G_X(t) := \mathrm{E}(t^X), \qquad G_N(t) := \mathrm{E}(t^N)$$

- $$S := \sum_{i=1}^{N} X_i \quad \Longrightarrow \quad G_S = G_N \circ G_X$$

- If G_N is holonomic and G_X is algebraic, then G_S is holonomic.

  Example: $N \sim \mathbf{P}(\alpha)$, $X \sim \mathbf{Geo}(n, \vartheta)$ (with n fixed).

- If G_N is a polynomial and G_X is holonomic, then G_S is holonomic.

- In these cases, we can compute a recurrence of finite order for $\mathrm{P}(S = n)$.

Example 5: Recurrence for a Product of Holonomic Sequences

- Sichel distribution: Mix Poisson and generalized inverse Gaussian:

$$f_n := \frac{1}{n!}\left(\frac{\chi}{\psi}\right)^{n/2}\left(\frac{\psi+2}{\psi}\right)^{-(\lambda+n)/2}\frac{\mathrm{K}_{\lambda+n}(\sqrt{\chi(\psi+2)})}{\mathrm{K}_{\lambda}(\sqrt{\chi\psi})}$$

  where λ, χ, ψ are the parameters of the GIG, and K is a modified Bessel function of the third kind.

- Holonomic sequences are an algebra
- Using

$$\mathrm{K}_{\mu+1}(x) = \frac{2\mu}{x}\mathrm{K}_{\mu}(x) + \mathrm{K}_{\mu-1}(x)$$

  we can compute the recurrence

$$n(\psi+2)f_n = 2(\lambda+n-1)f_{n-1} + \frac{\chi}{n-1}f_{n-2}$$

  (Sichel 1971)

Conclusion

- Recent symbolic algorithms could be useful for working with probability densities and generating functions
- Implemented in the Mathematica packages `GeneratingFunctions` (Mallinger) and `Zb` (Paule, Schorn)

Available extensions:

- Identity provers for multivariate holonomic functions of discrete and continuous arguments (Chyzak, Zimmermann)
- Inequality provers (SG, Kauers)