Crashcourse Interest Rate Models

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Interest Rate Models

- Model the evolution of the yield curve
- Can be used for forecasting the future yield curve or for pricing interest rate products
- Whole yield curve is more involved than the behaviour of an individual asset price
- Interest rates are used for discounting as well as for defining the payoff
- No generally accepted model (unlike Black-Scholes for stock options, e.g.)

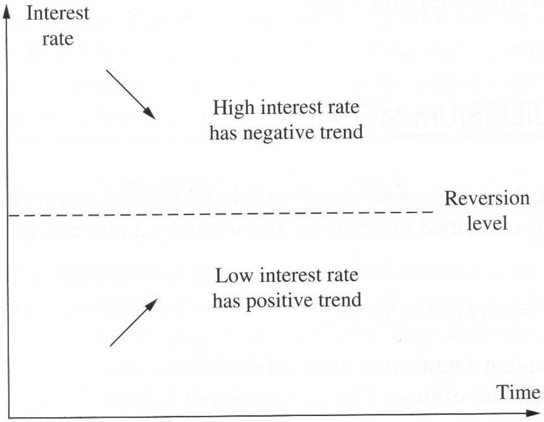
Desirable Properties of Interest Rate Models

- Realistic evolution of interest rates
- Can compute answers in reasonable time
- Required inputs can be observed or estimated
- Good fit of the model to market data

Desirable Properties of Interest Rate Models

- Positive interest rates
- Explicitly computable bond prices (hence spot rates, forward rates, swap rates)
- Explicitly computable bond option prices (hence caps, swaptions)
- Mean reversion

Mean Reversion



Short Rate Models

- Short rate (spot rate) r_t applies to an infinitesimally short period
- Artifical construct
- Approximation: Overnight money market rate
- Discount factor from time 0 to T is $\exp(-\int_0^T r_t \mathrm{d}t)$
- Special case: $\exp(-rT)$ if r_t is constant
- All rates (bond prices, EURIBOR, swap rates) are functions of the short rate

Risk Neutral Valuation

- Mathematical tool for pricing derivatives
- Events are assigned probabilities different from their real world probabilities
- In a risk neutral world, all assets grow at the risk free rate
- The price of a contract is the risk neutral expectation of its discounted payoff
- Example: The price of a zero coupon bond is $B(0, T) = \mathbf{E}[\exp(-\int_0^T r_t \mathrm{d}t)]$

The Risk Neutral World vs. the Real World

- Distribution of random variables differs
- We observe market data in the real world
- For pricing, the distribution in the risk neutral world matters
- Volatility is the same in both worlds

Vasicek Model (1977)

- Dynamics of the short rate under the risk-neutral measure
- Mean reversion level θ, reversion speed α
- $\mathrm{d}r_t = \alpha(\theta - r_t)\mathrm{d}t + \sigma\, \mathrm{d}W_t$
- $r_t - r_s \approx \alpha(\theta - r_s)(t - s) + \sigma(W_t - W_s), \qquad s < t$
- $W_t - W_s$ is normal with mean 0 and variance $t - s$

Vasicek Model: Distribution of the Short Rate

- Short rate r_t is normally distributed
- Mean $= r_0 \mathrm{e}^{-\alpha t} + \theta(1 - \mathrm{e}^{-\alpha t})$
- Mean decreases to θ at speed α
- Variance $= \frac{\sigma^2}{2\alpha}(1 - \mathrm{e}^{-2\alpha t})$
- Interest rates can become negative!

Vasicek Model: Bonds, Caps, and Floors

- Price of a zero coupon bond is

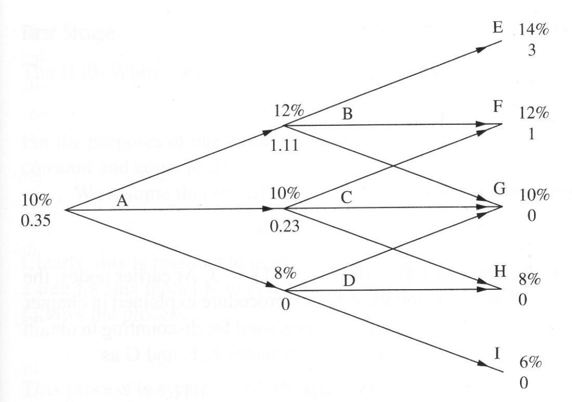
$$B(t, T) = A(t, T)\mathrm{e}^{-C(t,T)r_t}$$

- $A(t, T), C(t, T)$ deterministic functions
- There are explicit formulas for European call and put options on a zero coupon bond
- Give rise to explicit formulas for the prices of caplets and floorlets

Interest Rate Trees

- Discrete-time representation of the short rate
- R_t is the interest from t to $t + \Delta t$
- R_t is assumed to follow the same dynamics as r_t
- Transition probabilities are determined by the risk-neutral dynamics of the short rate
- Work backwards in time
- Discount factor varies from node to node
- Well suited for pricing American products

Example of a Trinomial Interest Rate Tree



- Payoff $\max\{100(R-0.11), 0\}$, where R is the Δt-period rate.
- Up, middle, and down probabilities are $0.25, 0.5, 0.25$, respectively.

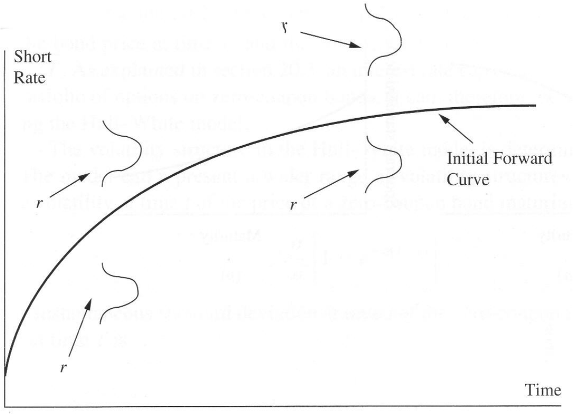
Vasicek Model: Summary

- Small number of parameters
- Does not reproduce initial yield curve
- Cannot reproduce some yield curve shapes (e.g., inverted)
- Normal distribution, hence rates can become negative
- Arbitrage-free (unless you can hide cash under the pillow)
- Only of theoretical and historical relevance

The Hull-White Model (1990)

- Extends Vasicek by a time-dependent drift
- $\mathrm{d}r_t = \alpha(\theta_t - r_t)\mathrm{d}t + \sigma\, \mathrm{d}W_t$
- θ_t is chosen so as to fit the initial term structure
- θ_t is a function of the instantaneous forward rate $f(0, T) = -\frac{\partial \log B(0,T)}{\partial T}$

Hull-White Model



- Short rate approximately follows initial forward rate curve

Hull-White Model

- Distribution of r_t is still normal
- Price of a zero coupon bond is $B(t,T) = A(t,T)\mathrm{e}^{-C(t,T)r_t}$
- $A(t,T), C(t,T)$ deterministic functions, involve initial term structure
- There are explicit formulas for European call and put options on a zero discount bond
- Give rise to explicit formulas for the prices of caplets and floorlets

Hull-White Model: Summary

- Fits initial term structure
- Calibration needs derivative of the yield curve
- Normal distribution, hence rates can become negative
- Arbitrage-free (unless you can hide cash under the pillow)
- Popular in practice

The Lognormal Models (Black-Derman-Toy 1990, Black-Karasinski 1991)

- $\mathrm{d} \log r_t = \alpha(\theta_t - \log r_t)\mathrm{d}t + \sigma \, \mathrm{d}W_t$
- Good fit to market volatility data
- The short rate cannot become negative
- Explosion of the bank account
- No analytic tractability, hence calibration is more difficult

One Factor Models

- The models considered so far are one factor models
- Only one source of randomness
- Bonds with different maturities are perfectly correlated
- No complete freedom in choosing the volatility term structure

Two Factor Models

- Two sources of randomness
- Richer pattern of term structure movements and volatility structures
- Interest rate trees become involved
- Require more computation time
- Rarely used in practice

Conclusion

- Hull-White and log-normal are favoured by practitioners
- Main difference: normal versus log-normal distribution
- Empirical studies do not favour any one of the two
- All short rate models are based on a theoretically constructed, not observable rate
- This shortcoming has led to the development of market models