Crashcourse Interest Rate Models

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Interest Rate Models

- Model the evolution of the yield curve
- Can be used for forecasting the future yield curve or for pricing interest rate products
- Whole yield curve is more involved than the behaviour of an individual asset price
- Interest rates are used for discounting as well as for defining the payoff
- No generally accepted model (unlike Black-Scholes for stock options, e.g.)
Desirable Properties of Interest Rate Models

- Realistic evolution of interest rates
- Can compute answers in reasonable time
- Required inputs can be observed or estimated
- Good fit of the model to market data
Desirable Properties of Interest Rate Models

- Positive interest rates
- Explicitly computable bond prices (hence spot rates, forward rates, swap rates)
- Explicitly computable bond option prices (hence caps, swaptions)
- Mean reversion
Mean Reversion

![Diagram showing mean reversion with interest rate on the y-axis and time on the x-axis. High interest rate has a negative trend, and low interest rate has a positive trend. The dashed line indicates the reversion level.]
Short Rate Models

- Short rate (spot rate) \( r_t \) applies to an infinitesimally short period
- Artificial construct
- Approximation: Overnight money market rate
- Discount factor from time 0 to \( T \) is \( \exp(-\int_0^T r_t dt) \)
- Special case: \( \exp(-rT) \) if \( r_t \) is constant
- All rates (bond prices, EURIBOR, swap rates) are functions of the short rate
Risk Neutral Valuation

- Mathematical tool for pricing derivatives
- Events are assigned probabilities different from their real world probabilities
- In a risk neutral world, all assets grow at the risk free rate
- The price of a contract is the risk neutral expectation of its discounted payoff
- Example: The price of a zero coupon bond is $B(0, T) = \mathbb{E}[\exp(-\int_0^T r_t dt)]$
The Risk Neutral World vs. the Real World

- Distribution of random variables differs
- We observe market data in the real world
- For pricing, the distribution in the risk neutral world matters
- Volatility is the same in both worlds
Vasicek Model (1977)

- Dynamics of the short rate under the risk-neutral measure
- Mean reversion level $\theta$, reversion speed $\alpha$
- $dr_t = \alpha(\theta - r_t)dt + \sigma \, dW_t$
- $r_t - r_s \approx \alpha(\theta - r_s)(t - s) + \sigma(W_t - W_s), \quad s < t$
- $W_t - W_s$ is normal with mean 0 and variance $t - s$
Vasicek Model: Distribution of the Short Rate

- Short rate $r_t$ is normally distributed
- Mean $= r_0 e^{-\alpha t} + \theta (1 - e^{-\alpha t})$
- Mean decreases to $\theta$ at speed $\alpha$
- Variance $= \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha t})$
- Interest rates can become negative!
Price of a zero coupon bond is

\[ B(t, T) = A(t, T)e^{-C(t, T)r_t} \]

- \( A(t, T), C(t, T) \) deterministic functions
- There are explicit formulas for European call and put options on a zero coupon bond
- Give rise to explicit formulas for the prices of caplets and floorlets
Interest Rate Trees

- Discrete-time representation of the short rate
- $R_t$ is the interest from $t$ to $t + \Delta t$
- $R_t$ is assumed to follow the same dynamics as $r_t$
- Transition probabilities are determined by the risk-neutral dynamics of the short rate
- Work backwards in time
- Discount factor varies from node to node
- Well suited for pricing American products
Example of a Trinomial Interest Rate Tree

Payoff \( \max \{100(R - 0.11), 0\} \), where \( R \) is the \( \Delta t \)-period rate.

Up, middle, and down probabilities are 0.25, 0.5, 0.25, respectively.
Vasicek Model: Summary

- Small number of parameters
- Does not reproduce initial yield curve
- Cannot reproduce some yield curve shapes (e.g., inverted)
- Normal distribution, hence rates can become negative
- Arbitrage-free (unless you can hide cash under the pillow)
- Only of theoretical and historical relevance
The Hull-White Model (1990)

- Extends Vasicek by a time-dependent drift
- \( \text{dr}_t = \alpha (\theta_t - r_t) \text{dt} + \sigma \text{d}W_t \)
- \( \theta_t \) is chosen so as to fit the initial term structure
- \( \theta_t \) is a function of the instantaneous forward rate
  \( f(0, T) = -\frac{\partial \log B(0, T)}{\partial T} \)
Hull-White Model

- Short rate approximately follows initial forward rate curve
Hull-White Model

- Distribution of $r_t$ is still normal
- Price of a zero coupon bond is $B(t, T) = A(t, T)e^{-C(t, T)r_t}$
- $A(t, T), C(t, T)$ deterministic functions, involve initial term structure
- There are explicit formulas for European call and put options on a zero discount bond
- Give rise to explicit formulas for the prices of caplets and floorlets
Hull-White Model: Summary

- Fits initial term structure
- Calibration needs derivative of the yield curve
- Normal distribution, hence rates can become negative
- Arbitrage-free (unless you can hide cash under the pillow)
- Popular in practice
The Lognormal Models (Black-Derman-Toy 1990, Black-Karasinski 1991)

- \( \frac{d \log r_t}{r_t} = \alpha(\theta_t - \log r_t) d t + \sigma \ dW_t \)
- Good fit to market volatility data
- The short rate cannot become negative
- Explosion of the bank account
- No analytic tractability, hence calibration is more difficult
The models considered so far are one factor models
- Only one source of randomness
- Bonds with different maturities are perfectly correlated
- No complete freedom in choosing the volatility term structure
Two Factor Models

- Two sources of randomness
- Richer pattern of term structure movements and volatility structures
- Interest rate trees become involved
- Require more computation time
- Rarely used in practice
Conclusion

- Hull-White and log-normal are favoured by practitioners
- Main difference: normal versus log-normal distribution
- Empirical studies do not favour any one of the two
- All short rate models are based on a theoretically constructed, not observable rate
- This shortcoming has led to the development of market models