Automatisches Beweisen von Identitäten

Stefan Gerhold

30. Oktober 2006

The General Scheme

- Parametric sums

$$\sum_k summand(n, k) = answer(n)$$

- Parametric integrals

$$\int_{-\infty}^{\infty} integrand(x, y)dy = answer(x)$$

Zeilberger's Summation Algorithm ($\sim$ 1990)

- Goal: derive a recurrence for the sum $s(n) = \sum_k f(n, k)$.
- Summand must be *hypergeometric* in n and k: The quotients $\frac{f(n+1,k)}{f(n,k)}$ and $\frac{f(n,k+1)}{f(n,k)}$ must be rational functions of n and k.
- E.g., $f(n, k)$ a product of binomal coefficients
- Provides algorithmic proofs of many combinatorial identities

A Simple Application of Zeilberger's Algorithm

- Input: $f(n,k) := \binom{n}{k}x^k$
- Output (Δ_k is the forward difference operator):

$$-(x+1)f(n,k) + f(n+1,k) = \Delta_k \left(-\frac{k}{n-k+1}f(n,k)\right).$$

- Sum both sides for $k = 0, \ldots, n+1$:

$$-(x+1)\sum_{k=0}^{n} f(n,k) + \sum_{k=0}^{n+1} f(n+1,k) = 0,$$

hence

$$\sum_{k=0}^{n} f(n,k) = (x+1)^n.$$

Example Gallery

$$\sum_{k=0}^{n}(-1)^k\binom{x-k+1}{k}\binom{x-2k}{n-k}=\begin{cases}1 & n \text{ even}\\ 0 & n \text{ odd}\end{cases}$$

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3=(-1)^n\frac{(3n)!}{n!^3}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x}{k+x}=\binom{x+n}{n}^{-1}$$

What Zeilberger's Algorithm Does

- We want to do the sum $\sum_k f(n,k)$.
- Define the forward shift $S_n f(n) := f(n+1)$.
- Zb finds an identity

$$P(S_n, n) f(n,k) = (S_k - 1) Q(S_n, n, S_k) f(n,k)$$

  with polynomials P and Q.
- Upon summation, the right-hand side vanishes:

$$P(S_n, n) \sum_k f(n,k) = 0.$$

- P and Q are constructed by solving a system of linear equations with rational function coefficients.

Rogers-Ramanujan Identities

Rogers (1894), Ramanujan, Schur (1917)

$$1 + \sum_{k=1}^{\infty} \frac{q^{k^2}}{(1-q)(1-q^2)\dots(1-q^k)} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})}$$

$$1 + \sum_{k=1}^{\infty} \frac{q^{k(k+1)}}{(1-q)(1-q^2)\dots(1-q^k)} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+2})(1-q^{5j+3})}$$

- The formulas are the limits $n \to \infty$ of finite versions (Andrews 1974).
- Zeilberger (1990) gave a computer proof, greatly simplified by Paule (1994).

The Bieberbach Conjecture

- An injective holomorphic function $f : \{|z| < 1\} \to \mathbb{C}$ is called *schlicht*, if $f(z) = z + \sum_{n\geq 2} a_n z^n$.
- Conjecture (Bieberbach 1916): $|a_n| \leq n$ for $n \geq 2$.
- Bieberbach (1916): $|a_2| \leq 2$.
- Nevanlinna (1920): Conjecture holds if image is star-shaped.
- Loewner (1923): $|a_3| \leq 3$.
- Littlewood (1925): $|a_n| \leq e \cdot n$.
- ...
- De Branges (1985) finally settled the conjecture.

The Bieberbach Conjecture

- De Branges' proof uses that a certain ${}_3F_2$ is non-negative.
- Askey, Gasper (1976)

$$\frac{(\alpha+2)_n}{n!} \cdot {}_3F_2\left(\begin{matrix} -n, n+2+\alpha, \frac{\alpha+1}{2} \\ \alpha+1, \frac{\alpha+3}{2} \end{matrix} \middle| x\right) = \sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{(\frac{1}{2})_j(\frac{\alpha}{2}+1)_{n-j}(\frac{\alpha+3}{2})_{n-2j}(\alpha+1)_{n-2j}}{j!(\frac{\alpha+3}{2})_{n+j}(\frac{\alpha+1}{2})_{n-2j}(n-2j)!} \cdot {}_3F_2\left(\begin{matrix} 2j-n, n-2j+\alpha-1, \frac{\alpha+1}{2} \\ \alpha+1, \frac{\alpha+2}{2} \end{matrix} \middle| x\right).$$

- The identity can be proven with Zeilberger's algorithm and implies the inequality.

The Irrationality of $\zeta(3)$

- $\zeta(2) = \sum_{n\geq 1} n^{-2} = \frac{1}{6}\pi^2$ is irrational, and similarly for $\zeta(2n)$.
- Apéry (1978): $\zeta(3) = \sum_{n\geq 1} n^{-3}$ is irrational.
- The proof uses that the sequence

$$b_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfies the recurrence

$$n^3 b_n + (n-1)^3 b_{n-2} = (34n^3 - 51n^2 + 27n - 5) b_{n-1}, \qquad n \geq 2.$$

Generalizations of Zeilberger's Algorithm (Chyzak 1998, Schneider 2001)

- General idea: summand satisfies recurrences, not necessarily of first order
- Compute recurrence for the sum
- Works also for integrands that depend on a continuous parameter
- Compute differential equation for the integral

Example Gallery

$$\sum_{i+j+k=n} \binom{i+j}{i} \binom{j+k}{j} \binom{k+i}{k} = \sum_{k=0}^{n} \binom{2k}{k}$$

$$\sum_{n=0}^{\infty} P_n(x) y^n = \frac{1}{\sqrt{1-2xy+y^2}}$$

$$\int_{-1}^{1} \frac{e^{-px} T_n(x)}{\sqrt{1-x^2}} dx = \pi(-1)^n I_n(p)$$

$$\int_{0}^{\infty} xe^{-px^2} J_\nu(ax) J_\nu(bx) dx = \frac{1}{2p} \exp(-\frac{a^2+b^2}{4p}) \cdot I_\nu(\frac{ab}{2p})$$

When is 0.999... equal to 1?

- Balogh and Pemantle (2004) came across the sum

$$S := \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}$$

  in a run time analysis of the simplex algorithm on a certain polytope. ($H_n := \sum_{k=1}^{n} 1/k$.)

- Easy bound: $0.999197 \leq S \leq 1.00093$.
- **Theorem** (Schneider 2006)

$$\begin{aligned} S &= -4\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5) \\ &= 0.999222\ldots \end{aligned}$$

  Proof: truncate the sums, find a closed form, pass to the limit

Further Uses of Recurrences

- Recurrences are not only useful for proving identities
- Rapid computation of sums and integrals
- Example: Recurrences for basis functions in higher order finite element schemes (Paule, Schöberl et al. 2006)
- Asymptotics via recurrences and differential equations for generating functions

From Recurrences to Asymptotics

- Weiss, Glebsky (2005): The number of limit states in a certain Schelling population model is

$$s(n) = 4\sum_{k=1}^{2n} \binom{n-1}{k-1}\binom{n-k-1}{k-1} + 2\sum_{k=1}^{2n} \binom{n-1}{k}\binom{n-k-1}{k-1}$$
$$+ \text{ three similar sums.}$$

- What is the behaviour as $n \to \infty$?
- Let

$$a(n) := 4\sum_{k=1}^{2n} \binom{n-1}{k-1}\binom{n-k-1}{k-1}$$

denote the first sum.

From Recurrences to Asymptotics

- Generating function

$$A(z) := \sum_{n \geq 0} a(n) z^n.$$

- Zeilberger's algorithm $\Rightarrow$ recurrence for $a(n)$ $\Rightarrow$ ODE for $A(z)$.
- Fuchs-Frobenius theory yields

$$A(z) \sim \text{const} \cdot (1 - 3z)^{-1/2}, \qquad z \to \tfrac{1}{3}^{-}.$$

- Singularity analysis (Flajolet, Odlyzko 1990) then gives

$$a(n) \sim \text{const} \cdot 3^n / \sqrt{n}, \qquad n \to \infty.$$

Automatic Proofs of Inequalities (M. Kauers, SG 2005)

- Inequality must depend on some discrete parameter n
- Quantities must satisfy polynomial recurrences
- Induction step is reduced to CAD (Cylindrical Algebraic Decomposition)
- Check finitely many initial values

Turán's Inequality for Legendre Polynomials

- Turán (1946):

$$\Delta_n(x) := P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0, \qquad x \in [-1,1],\ n \geq 1.$$

- Introduce real variables N, Y_{-1}, Y_0, Y_1, Y_2 representing n, $P_{n-1}(x)$, $P_n(x)$, $P_{n+1}(x)$, $P_{n+2}(x)$
- Sufficient condition for induction step:

$$\begin{aligned} \forall\ N, X, Y_{-1}, Y_0, Y_1, Y_2 \in \mathbb{R} : \big(N \geq 1 \wedge -1 \leq X \leq 1 \wedge \\ (N+2)Y_2 = -(N+1)Y_0 + (3X + 2NX)Y_1 \wedge \\ (N+1)Y_1 = -NY_{-1} + (X + 2NX)Y_0\big) \\ \implies \big(Y_0^2 - Y_{-1}Y_1 \geq 0 \implies Y_1^2 - Y_0Y_2 \geq 0\big). \end{aligned}$$

Turán's Inequality: Refinements

- SG, Kauers (2005):

$$|x|P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0, \qquad x \in [-1,1],\ n \geq 1,$$

- Alzer, SG, Kauers (2006):

$$\alpha_n\,(1-x^2) \leq P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \leq \beta_n\,(1-x^2)$$

with the best possible factors

$$\alpha_n = \mu_{\lfloor n/2 \rfloor}\,\mu_{\lfloor (n+1)/2 \rfloor} \quad \textit{and} \quad \beta_n = \frac{1}{2}, \qquad \mu_n := 2^{-2n}\binom{2n}{n}.$$

Other Inequalities

The following inequalities can be proven automatically:

$$\Big(\sum_{k=1}^{n} x_k y_k\Big)^2 \le \sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^2 \qquad \text{(Cauchy-Schwarz)}$$

$$(x+1)^n \ge 1 + n \cdot x, \qquad n \ge 0, x \ge -1 \qquad \text{(Bernoulli)}$$

$$\prod_{k=1}^{n} (1 - a_k) > 1 - \sum_{k=1}^{n} a_k, \qquad 0 < a_k < 1, \sum a_k < 1 \text{ (Weierstraß)}$$

$$\sqrt{n - \tfrac{3}{4}} \le a_n - \tfrac{1}{2} \le \sqrt{n + \tfrac{1}{4}}, \qquad a_1 = 1, a_{n+1} = 1 + \frac{n}{a_n}$$

Conclusion

- Symbolic methods are useful for working with special functions
- Classical Tables like Abramowitz-Stegun can be partially reproduced by computer algebra
- NIST's Digital Library of Mathematical Functions will have a chapter on these methods (Chyzak, Paule)