# Option Pricing in the Moderate Deviations Regime

Stefan Gerhold

TU Wien

Joint work with P. Friz and A. Pinter

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# Overview

- Small maturity asymptotics of call prices
- Strike depends on maturity, approaches spot
- Moderate deviations using heat kernel estimates
- Illustration in the Heston model
- Moderate deviations using the Gärtner-Ellis theorem

# Option pricing

- Underlying stochastic process $(S_t)_{t \geq 0}$ (stock price, FX rate,..)
- We fix a risk-neutral measure
- Normalization: $S_0 = 1$, $r = 0$
- Call price:
$$C(K, T) = \mathbb{E}[(S_T - K)^+]$$
- Other notation: $c(k, T) = C(e^k, T)$
- Strike $K$, log-strike $k$

# Applications of asymptotic approximations

- Fast approximate pricing (risk management)
- Fast approximate calibration (explicit formulas)
- Get good initial parameter values for "exact" calibration
- Understand qualitative influence of model parameters
- Parametrization design

# Small time asymptotics for diffusions

- **Large deviations (LD):** $k > 0$ fixed, $t \downarrow 0$. OTM.
- Well understood; many papers (Varadhan 1967,...)
- OTM call prices decay exponentially
- Rate function: Need to solve geodesic equation in general
- Other caveat: Quoted strikes approach spot as maturity shrinks
- **Central limit theorem (CLT):** $k = 0$, or more generally $k_t = o(t^{1/2})$. (almost) ATM.
- In general martingale model with diffusive component:

$$C(1, t) = c(0, t) \sim \frac{\sigma_0 \sqrt{t}}{\sqrt{2\pi}}, \quad t \downarrow 0.$$

with spot-vol $\sigma_0 > 0$ (Carr & Wu 2003, Muhle-Karbe & Nutz 2011, ...)

# Small time asymptotics for diffusions

- **Moderate deviations:** Intermediate regime. $k = \ell(t)t^{\beta}$, $\ell > 0$ slowly varying, $\beta \in (0, \frac{1}{2})$.
  (Recall: $\ell = const$ and $\beta = 0$ is LD; $\beta > \frac{1}{2}$ is CLT.)
- MOTM ("moderately OTM"): Not studied so far for diffusions.
- Strikes approach spot as $t \downarrow 0$
- Our results are easy to apply (need only local behavior of geodesic distance)

# What are moderate deviations?

- Law of large numbers (for zero-mean IID r.v.s )

$$\frac{1}{n}(X_1 + \cdots + X_n) \equiv \bar{X}_n \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{(LLN)}$$

- Central limit theorem (assume $\mathbb{V}X_1 = 1$)

$$\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n) \xrightarrow[n\to\infty]{d} \mathcal{N}(0,1) \qquad \text{(CLT)}$$

- Large deviation principle quantifies $\text{(LLN)}$. Cramér's theorem: with exp. moments, being at $x \neq 0\,(= \lim \bar{X}_n)$ is exponentially unlikely:

$$\mathbb{P}[\bar{X}_n \geq x] \underset{n\to\infty}{\approx} \exp\{-n\mathcal{I}(x)\} \qquad \text{(LDP)}$$

- **Moderate deviations**: scaling between $\text{LLN}$ and $\text{CLT}$,

$$\frac{1}{n^{1-\beta}}(X_1 + \cdots + X_n) \text{ with } \begin{cases} \beta = 0 & \text{(LLN)} \\ \beta \in (0, \frac{1}{2}) & \text{(moderate deviations)} \\ \beta = \frac{1}{2} & \text{(CLT)} \end{cases}$$

# Moderate deviations, continued

- Moderate deviation principle, with $\mathbb{V}X_1 = 1$:

$$\mathbb{P}[\bar{X}_n \geq n^{-\beta} x] \underset{n\to\infty}{\approx} \exp\left(-n^{1-2\beta} x^2/2\right). \quad \text{(MDP)}$$

- Heuristic derivation: $\mathcal{I}(y) = y^2/2 + ...$ for $y = n^{-\beta}x \approx 0$, as $n \to \infty$. Formal application of the LDP gives

$$\mathbb{P}[\bar{X}_n \geq y] \underset{n\to\infty}{\approx} \exp\left(-n\mathcal{I}(y)\right) \underset{n\to\infty}{\approx} \exp\left(-n^{1-2\beta} x^2/2\right).$$

- As a matter of terminology: $\mathrm{MDP}$ corresponds to $\mathrm{LDP}$ for $\left(n^{\beta}\bar{X}_n\right)$ with modified speed function ($n^{1-2\beta}$ instead of $n$) and quadratic rate function $x^2/2$.

# The MOTM regime

The **moderately OTM regime** ( $\leftrightarrow$ moderate deviation regime) has several advantages when compared to OTM ($\leftrightarrow$ large deviation) regime:

- rate function computable (in fact, quadratic)
- refined expansions possible (and also computable)
- AATM/MOTM regimes reflect the reality of option data!

  (Range of) strikes decrease with remaining option life time; that is,

  $$|k_t| \downarrow 0, \text{ with } t \downarrow 0.$$

- This is not at all captured by large deviations which deal with

  $$\text{fixed log-strike } k > 0, \text{ with } t \downarrow 0.$$

# Main result: Assumptions

- Underlying $S_t$ given by stochastic volatility model
- Locally uniform estimate for pdf of underlying:

$$q(k,t) \sim c_0 t^{-1/2} \exp\left(-\frac{\Lambda(k)}{t}\right), \quad t \downarrow 0$$

  $2\Lambda(k)$ is the squared Riemannian distance from $(S_0, \sigma_0)$ to $\{(K,\sigma) : \sigma > 0\}$ with $K = S_0 e^k$ (Deuschel, Friz, Jacquier, Violante 2014)

- Local volatility, defined by Dupire formula, converges locally uniformly to spot-vol:

$$\sigma_{loc}(K,t) \to \sigma_0, \quad t \downarrow 0$$

  (Berestycki, Busca, Florent 2004)

# Main result

## Theorem (Friz-G.-Pinter)

*Assume a generic Markovian stochastic vol model with spot vol $\sigma_0$. Consider MOTM log-strikes,*

$$k_t = t^\beta \ell(t) \quad \text{with } \beta \in (0, \tfrac{1}{2}).$$

*Then*

$$c(k_t, t) = \exp\left( -\frac{\Lambda''(0)}{2} \frac{k_t^2}{t} \Big( 1 + o(1) \Big) \right) \qquad \text{as } t \downarrow 0.$$

*Useful fact:* $1/\Lambda''(0) = \sigma_0^2 = v_0$.

- Similar to (A)ATM: robust asymptotics, only depending on spot vol.
- Different from AATM: not a trivial perturbation of ATM, certainly not true in presence of jumps! (In this case $c(k_t, t) = O(t)$.)

# Main result (for $0 < \beta < \frac{1}{3}$)

## Theorem (Friz-G.-Pinter)

*Under the previous assumptions, consider a moderately out-of-the-money call,*

$$k_t = t^\beta \ell(t) \quad \textit{with } \beta \in (0, \tfrac{1}{3}).$$

*Moderate second order expansion:*

$$-\log c(k_t, t) = \frac{1}{2}\Lambda''(0)\frac{k_t^2}{t} + \frac{1}{6}\Lambda'''(0)\frac{k_t^3}{t}(1 + o(1)) \quad \textit{as } t \downarrow 0$$

*Novel skew formula:*

$$\frac{\partial}{\partial k}\Big|_{k=0} \left\{\sigma_{impl}^2(0, k)\right\} = 2v_0^2 \lim_{t\downarrow 0}\left(\frac{t}{k_t^3}\log\frac{c\,(k_t, t)}{c_{BS}\,(k_t, t; \sigma_0)}\right)$$

- $\Lambda''(0), \Lambda'''(0)$ are easily computable from the diffusion coefficients of the SV model).

# Main result: refined expansion

Assumptions as before (here no $\ell$ for simplicity)

$$k_t = t^{\beta} \quad \text{with } \beta \in (0, \tfrac{1}{2}).$$

Refined call asymptotics (and not just log-asymptotics):

$$c(k_t, t) \sim c_0 v_0^2 \, t^{3/2-2\beta} \exp\left( - \sum_{m=2}^{\lfloor 1/\beta \rfloor} \frac{\Lambda^{(m)}(0)}{m!} t^{m\beta-1} \right), \quad t \downarrow 0.$$

# Main result: proof idea

Recall Dupire's formula:

$$\sigma_{loc}^2(K,t) = \frac{\partial_t C}{\frac{1}{2}K^2\partial_{KK}C}$$

Hence

$$\begin{aligned} C(K,t) &= \int_0^t \partial_t C(K,u)du \\ &= \int_0^t \frac{1}{2}\partial_{KK}C(K,u)K^2\sigma_{loc}^2(K,u)du. \end{aligned}$$

Now use:

- density asymptotics
- convergence of $\sigma_{loc}$
- Laplace method

# Main result: proof idea

Laplace method yields

$$c(k_t, t) \sim \frac{1}{2} c_0 v_0 \frac{t^{3/2}}{\Lambda(k_t)} \exp\left(-\frac{\Lambda(k_t)}{t}\right)$$

Now use Taylor expansion

$$\begin{aligned} \Lambda(k_t) &= \tfrac{1}{2}\Lambda''(0) k_t^2 + \dots \\ &= \frac{1}{2v_0} k_t^2 + \dots \end{aligned}$$

# How does all this look like in Heston?

Consider Heston with correlation $\rho$, spot variance $v_0$, vvol $\eta$. Then (Forde-Jacquier-Lee '12)

$$t \log c(k,t) \sim -\Lambda(k)$$

where $\Lambda(k) = \sup_x\{kx - \Gamma(x)\}$ is Legendre-transform of

$$\begin{aligned}\Gamma(p) &= \frac{v_0 p}{\eta[\bar{\rho}\cot(\eta\bar{\rho}p/2) - \rho]} \\ &= \cdots \\ &= \frac{v_0}{2}p^2 + \frac{v_0\eta\rho}{4}p^3 + O(p^4)\end{aligned}$$

# Heston continued

Write $g(k)$ for the maximizer in def. of $\Lambda(k)$. Then $g(0) = 0$,

$$\begin{aligned} \Lambda''(k) &= \frac{1}{\Gamma''(g(k))}, \\ \Lambda'''(k) &= -(\Lambda''(k))^3 \Gamma'''(g(k)), \end{aligned}$$

and hence in Heston

$$\begin{aligned} \Lambda''(0) &= \frac{1}{v_0}, \\ \Lambda'''(0) &= -\frac{3}{v_0^2} \frac{\eta\rho}{2}. \end{aligned}$$

# Heston concluded

As a consequence,

$$\begin{aligned}\log c(k_t, t) &= -\frac{1}{2}\Lambda''(0)\frac{k_t^2}{t} - \frac{1}{6}\Lambda'''(0)\frac{k_t^3}{t}(1+o(1)) \\ &= -\frac{1}{2v_0}\frac{k_t^2}{t} + \frac{1}{2v_0^2}\frac{\eta\rho}{2}\frac{k_t^3}{t}(1+o(1))\end{aligned}$$

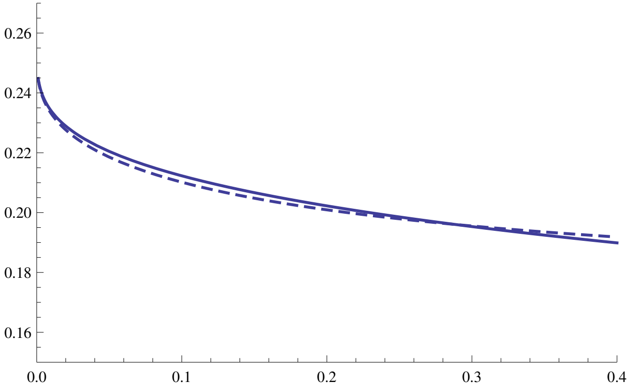
from which, with spot-vol $\sigma_0 = \sqrt{v_0}$, and with $t \downarrow 0$,

$$2v_0^2\left(\frac{t}{k_t^3}\log\frac{c\left(k_t, t\right)}{c_{BS}\left(k_t, t; \sigma_0\right)}\right) \to \frac{\eta\rho}{2},$$

which agrees exactly with the (well-known) Heston skew

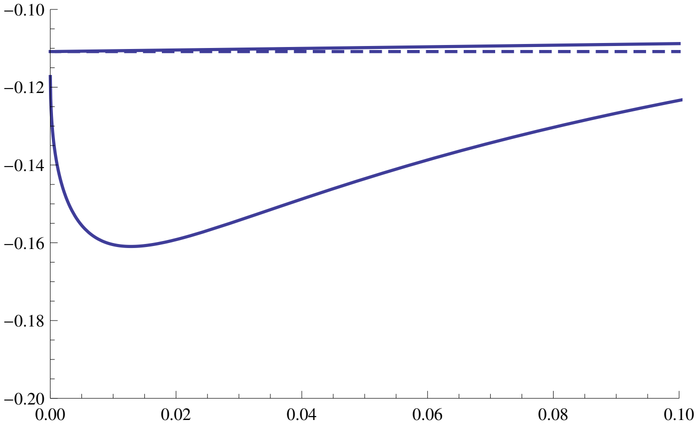
$$\left.\frac{\partial}{\partial k}\right|_{k=0}\left\{\sigma_{impl}^2\left(0, k\right)\right\} = \frac{\eta\rho}{2}.$$

# Implied volatility asymptotics in the Heston model, $\beta = 0.3$




log-strike $k = 0.4\, t^{0.3}$; $\sigma_0 = 0.2557$.
Dashed: exact MOTM implied volatility $\sigma_{\mathrm{imp}}(k_t, t)$
Solid: $\sigma_0 + \frac{\eta\rho}{4\sigma_0} k_t$

# Heston variance skew asymptotics




Approximation of the Heston variance skew.
Dashed line: Limit of the Heston variance skew $\eta\rho/2$
Upper line: Heston variance skew
Lower line: Approximation from theorem

# Alternative approach: Gärtner-Ellis theorem

- mgf (moment generating function) of log-spot $X_t = \log S_t$:

$$M(p, t) := \mathbb{E}[e^{pX_t}]. \tag{1}$$

- **Assumption GE:** For all $\beta \in (0, \frac{1}{2})$, the rescaled mgf satisfies

$$\lim_{t \downarrow 0} t^{1-2\beta} \log M(t^{\beta-1} p, t) = \frac{1}{2}\sigma_0^2 p^2, \quad p \in \mathbb{R}. \tag{2}$$

- Now: no assumptions on density or local vol.

# Alternative approach: Discussion of the assumption

- Heuristically: density asymptotics (s. above) $\Longrightarrow$ assumption GE.
- Necessary: Critical moment

  $$p_+(t) := \sup\{p \geq 0 : M(p,t) < \infty\}.$$

  satisfies

  $$\lim_{t \downarrow 0} t^{1-\beta} p_+(t) = \infty$$

- Heston: $p_+(t) \approx 1/t \gg t^{\beta-1}$ (Keller-Ressel 2011). Indeed, Heston satisfies GE (affine calculation).

# Recall the Gärtner-Ellis Theorem

- $(Z_t)_{t \geq 0}$ stochastic process
- $t \mapsto a_t$ real function with $0 < a_t = o(1)$ as $t \downarrow 0$
- Assume existence and smoothness of

$$\Gamma_Z(p) := \lim_{t \downarrow 0} a_t \log M_Z(p/a_t, t), \quad p \in \mathbb{R}.$$

- **Gärtner-Ellis Theorem** The laws of $Z_t$ satisfy an LDP for $t \downarrow 0$, with rate $a_t^{-1}$ and rate function $\Gamma_Z^*$ (Legendre transform).
- Thus:

$$\mathbb{P}[Z_t \in B] \approx \exp(-a_t^{-1} \inf_B \Gamma_Z^*)$$

# Alternative approach: Result

- **Theorem** Under assumption GE, for $k_t = \theta t^\beta$ with $\beta \in (0, \frac{1}{2})$ and $\theta > 0$,

$$\mathbb{P}[X_t \geq k_t] = \exp\left(-\frac{1}{2\sigma_0^2}\frac{k_t^2}{t}(1 + o(1))\right), \quad t \downarrow 0.$$

- Proof idea: Define $Z_t := t^{-\beta} X_t$, with mgf $M_Z(s,t) = \mathbb{E}[e^{sZ_t}]$, and

$$a_t := t^{1-2\beta} = o(1), \quad t \downarrow 0.$$

  Then GE is equivalent to

$$\Gamma_Z(p) := \lim_{t \downarrow 0} a_t \log M_Z(p/a_t, t) = \tfrac{1}{2}\sigma_0^2 p^2, \quad p \in \mathbb{R}.$$

  $\Gamma_Z$ is finite on $\mathbb{R}$; by Gärtner-Ellis, $(Z_t)_{t \geq 0}$ satisfies an LDP as $t \downarrow 0$, with rate $a_t^{-1}$ and rate function $\Gamma_Z^*$.

# Summary

- Small time asymptotics for stochastic volatility models
- New regime ("moderately out-of-the-money")
- Mild assumptions on density and local vol of the model
- Alternative assumption: limit of scaled characteristic function
- Numerically tractable: asymptotic expansions can be easily evaluated