Option Pricing in the Moderate Deviations Regime

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Overview

- Small maturity asymptotics of call prices
- Strike depends on maturity, approaches spot
- Moderate deviations using heat kernel estimates
- Illustration in the Heston model
- Moderate deviations using the Gärtner-Ellis theorem
Option pricing

- Underlying stochastic process \((S_t)_{t\geq 0}\) (stock price, FX rate,..)
- We fix a risk-neutral measure
- Normalization: \(S_0 = 1, \ r = 0\)
- Call price:
  \[
  C(K, T) = \mathbb{E}[(S_T - K)^+] 
  \]
- Other notation: \(c(k, T) = C(e^k, T)\)
- Strike \(K\), log-strike \(k\)
Applications of asymptotic approximations

- Fast approximate pricing (risk management)
- Fast approximate calibration (explicit formulas)
- Get good initial parameter values for “exact” calibration
- Understand qualitative influence of model parameters
- Parametrization design
Small time asymptotics for diffusions

- **Large deviations (LD):** \( k > 0 \) fixed, \( t \downarrow 0 \). OTM.
- Well understood; many papers (Varadhan 1967,...)
- OTM call prices decay exponentially
- Rate function: Need to solve geodesic equation in general
- Other caveat: Quoted strikes approach spot as maturity shrinks
- **Central limit theorem (CLT):** \( k = 0 \), or more generally \( k_t = o(t^{1/2}) \). (almost) ATM.
- In general martingale model with diffusive component:

\[
C(1, t) = c(0, t) \sim \frac{\sigma_0 \sqrt{t}}{\sqrt{2\pi}}, \quad t \downarrow 0.
\]

with spot-vol \( \sigma_0 > 0 \) (Carr & Wu 2003, Muhle-Karbe & Nutz 2011, ...
Small time asymptotics for diffusions

- **Moderate deviations**: Intermediate regime. $k = \ell(t)t^\beta$, $\ell > 0$ slowly varying, $\beta \in (0, \frac{1}{2})$.
  (Recall: $\ell = \text{const}$ and $\beta = 0$ is LD; $\beta > \frac{1}{2}$ is CLT.)

- MOTM (“moderately OTM”): Not studied so far for diffusions.

- Strikes approach spot as $t \downarrow 0$

- Our results are easy to apply (need only local behavior of geodesic distance)
What are moderate deviations?

- **Law of large numbers** (for zero-mean IID r.v.s)
  \[
  \frac{1}{n} \left( X_1 + \cdots + X_n \right) \equiv \bar{X}_n \xrightarrow{n \to \infty} 0 \quad \text{(LLN)}
  \]

- **Central limit theorem** (assume \( \forall X_1 = 1 \))
  \[
  \frac{1}{\sqrt{n}} \left( X_1 + \cdots + X_n \right) \xrightarrow{d} \mathcal{N}(0, 1) \quad \text{(CLT)}
  \]

- **Large deviation principle** quantifies (LLN). Cramér’s theorem: with exp. moments, being at \( x \neq 0 \) (\( \equiv \lim \bar{X}_n \)) is exponentially unlikely:
  \[
  \mathbb{P}[\bar{X}_n \geq x] \xrightarrow{n \to \infty} \exp \{-n \mathcal{I}(x)\} \quad \text{(LDP)}
  \]

- **Moderate deviations**: scaling between LLN and CLT,
  \[
  \frac{1}{n^{1-\beta}} \left( X_1 + \cdots + X_n \right) \text{ with } \begin{cases} 
  \beta = 0 & \text{(LLN)} \\
  \beta \in (0, \frac{1}{2}) & \text{(moderate deviations)} \\
  \beta = \frac{1}{2} & \text{(CLT)}
  \end{cases}
  \]
Moderate deviations, continued

- Moderate deviation principle, with $\mathbb{V}X_1 = 1$:

\[
P[\bar{X}_n \geq n^{-\beta} x] \xrightarrow{n \to \infty} \exp \left(-n^{1-2\beta} x^2 / 2\right). \quad (\text{MDP})
\]

- Heuristic derivation: $I(y) = y^2 / 2 + \ldots$ for $y = n^{-\beta} x \approx 0$, as $n \to \infty$. Formal application of the LDP gives

\[
P[\bar{X}_n \geq y] \xrightarrow{n \to \infty} \exp \left(-nI(y)\right) \xrightarrow{n \to \infty} \exp \left(-n^{1-2\beta} x^2 / 2\right).
\]

- As a matter of terminology: MDP corresponds to LDP for $\left(n^\beta \bar{X}_n\right)$ with modified speed function ($n^{1-2\beta}$ instead of $n$) and quadratic rate function $x^2 / 2$. 

The MOTM regime

The moderately OTM regime (↔ moderate deviation regime) has several advantages when compared to OTM (↔ large deviation) regime:

▶ rate function computable (in fact, quadratic)
▶ refined expansions possible (and also computable)
▶ AATM/MOTM regimes reflect the reality of option data!

(Range of) strikes decrease with remaining option life time; that is,

\[ |k_t| \downarrow 0, \text{ with } t \downarrow 0. \]

▶ This is not at all captured by large deviations which deal with fixed log-strike \( k > 0 \), with \( t \downarrow 0 \).
Main result: Assumptions

- Underlying $S_t$ given by stochastic volatility model

- Locally uniform estimate for pdf of underlying:

$$q(k, t) \sim c_0 t^{-1/2} \exp \left( -\frac{\Lambda(k)}{t} \right), \quad t \downarrow 0$$

$2\Lambda(k)$ is the squared Riemannian distance from $(S_0, \sigma_0)$ to $\{(K, \sigma) : \sigma > 0\}$ with $K = S_0 e^k$ (Deuschel, Friz, Jacquier, Violante 2014)

- Local volatility, defined by Dupire formula, converges locally uniformly to spot-vol:

$$\sigma_{loc}(K, t) \to \sigma_0, \quad t \downarrow 0$$

(Berestycki, Busca, Florent 2004)
Main result

Theorem (Friz-G.-Pinter)

Assume a generic Markovian stochastic vol model with spot vol $\sigma_0$. Consider MOTM log-strikes,

$$k_t = t^\beta \ell(t) \quad \text{with} \quad \beta \in (0, \frac{1}{2}).$$

Then

$$c(k_t, t) = \exp \left( -\frac{\Lambda''(0) k_t^2}{2t} \left(1 + o(1)\right) \right) \quad \text{as} \quad t \downarrow 0.$$ 

Useful fact: $1/\Lambda''(0) = \sigma_0^2 = \nu_0$.

- Similar to (A)ATM: robust asymptotics, only depending on spot vol.
- Different from AATM: not a trivial perturbation of ATM, certainly not true in presence of jumps! (In this case $c(k_t, t) = O(t)$.)
Main result (for $0 < \beta < \frac{1}{3}$)

Theorem (Friz-G.-Pinter)

Under the previous assumptions, consider a moderately out-of-the-money call,

$$k_t = t^\beta \ell(t) \text{ with } \beta \in (0, \frac{1}{3}).$$

Moderate second order expansion:

$$- \log c(k_t, t) = \frac{1}{2} \Lambda''(0) \frac{k_t^2}{t} + \frac{1}{6} \Lambda'''(0) \frac{k_t^3}{t} (1 + o(1)) \quad \text{as } t \downarrow 0$$

Novel skew formula:

$$\left. \frac{\partial}{\partial k} \right|_{k=0} \left\{ \sigma_{impl}^2(0, k) \right\} = 2v_0^2 \lim_{t \downarrow 0} \left( \frac{t}{k_t^3} \log \frac{c(k_t, t)}{c_{BS}(k_t, t; \sigma_0)} \right)$$

- $\Lambda''(0), \Lambda'''(0)$ are easily computable from the diffusion coefficients of the SV model).
Main result: refined expansion

Assumptions as before (here no \( \ell \) for simplicity)

\[ k_t = t^\beta \text{ with } \beta \in (0, \frac{1}{2}). \]

Refined call asymptotics (and not just log-asymptotics):

\[
c(k_t, t) \sim c_0 \nu_0^2 \ t^{3/2 - 2\beta} \exp \left( - \sum_{m=2}^{\left\lfloor 1/\beta \right\rfloor} \frac{\Lambda^{(m)}(0)}{m!} t^{m\beta-1} \right), \quad t \downarrow 0.
\]
Main result: proof idea

Recall Dupire’s formula:

\[ \sigma_{loc}^2(K, t) = \frac{\partial_t C}{\frac{1}{2} K^2 \partial_{KK} C} \]

Hence

\[
C(K, t) = \int_0^t \partial_t C(K, u) du = \int_0^t \frac{1}{2} \partial_{KK} C(K, u) K^2 \sigma_{loc}^2(K, u) du.
\]

Now use:
- density asymptotics
- convergence of \( \sigma_{loc} \)
- Laplace method
Main result: proof idea

Laplace method yields

\[ c(k_t, t) \sim \frac{1}{2} c_0 v_0 \frac{t^{3/2}}{\Lambda(k_t)} \exp \left( -\frac{\Lambda(k_t)}{t} \right) \]

Now use Taylor expansion

\[ \Lambda(k_t) = \frac{1}{2} \Lambda''(0) k_t^2 + \ldots \]

\[ = \frac{1}{2 v_0} k_t^2 + \ldots \]
How does all this look like in Heston?

Consider Heston with correlation $\rho$, spot variance $v_0$, vvol $\eta$. Then (Forde-Jacquier-Lee ’12)

$$
\log c(k, t) \sim -\Lambda(k)
$$

where $\Lambda(k) = \sup_x \{kx - \Gamma(x)\}$ is Legendre-transform of

$$
\Gamma(p) = \frac{v_0 \rho}{\eta [\bar{\rho} \cot (\eta \bar{\rho} p/2) - \rho]}
$$

$$
= \cdots
$$

$$
= \frac{v_0}{2} p^2 + \frac{v_0 \eta \rho}{4} p^3 + O(p^4)
$$
Write $g(k)$ for the maximizer in def. of $\Lambda(k)$. Then $g(0) = 0$,

\[ \Lambda''(k) = \frac{1}{\Gamma''(g(k))}, \]

\[ \Lambda'''(k) = - (\Lambda''(k))^3 \Gamma'''(g(k)), \]

and hence in Heston

\[ \Lambda''(0) = \frac{1}{v_0}, \]

\[ \Lambda'''(0) = - \frac{3}{v_0^2} \frac{\eta \rho}{2}. \]
Heston concluded

As a consequence,

\[
\log c(k_t, t) = -\frac{1}{2} \Lambda''(0) \frac{k_t^2}{t} - \frac{1}{6} \Lambda'''(0) \frac{k_t^3}{t} (1 + o(1))
\]

\[
= - \frac{1}{2v_0} \frac{k_t^2}{t} + \frac{1}{2v_0^2} \frac{\eta \rho k_t^3}{2} (1 + o(1))
\]

from which, with spot-vol \( \sigma_0 = \sqrt{v_0} \), and with \( t \downarrow 0 \),

\[
2v_0^2 \left( \frac{t}{k_t^3} \log \frac{c(k_t, t)}{c_{BS}(k_t, t; \sigma_0)} \right) \to \frac{\eta \rho}{2},
\]

which agrees exactly with the (well-known) Heston skew

\[
\frac{\partial}{\partial k} \bigg|_{k=0} \left\{ \sigma_{impl}^2(0, k) \right\} = \frac{\eta \rho}{2}.
\]
Implied volatility asymptotics in the Heston model, $\beta = 0.3$

Dashed: exact MOTM implied volatility $\sigma_{\text{imp}}(k_t, t)$
Solid: $\sigma_0 + \frac{\eta \rho}{4\sigma_0} k_t$

log-strike $k = 0.4 \ t^{0.3}$; $\sigma_0 = 0.2557$. 
Approximation of the Heston variance skew.
Dashed line: Limit of the Heston variance skew $\eta \rho / 2$
Upper line: Heston variance skew
Lower line: Approximation from theorem
Alternative approach: Gärtner-Ellis theorem

- mgf (moment generating function) of log-spot $X_t = \log S_t$:
  \[
  M(p, t) := \mathbb{E}[e^{pX_t}]. \tag{1}
  \]

- **Assumption GE:** For all $\beta \in (0, \frac{1}{2})$, the rescaled mgf satisfies
  \[
  \lim_{t \downarrow 0} t^{1-2\beta} \log M(t^{\beta-1}p, t) = \frac{1}{2} \sigma_0^2 p^2, \quad p \in \mathbb{R}. \tag{2}
  \]

- Now: no assumptions on density or local vol.
Alternative approach: Discussion of the assumption

- Heuristically: density asymptotics (s. above) \(\Rightarrow\) assumption GE.

- Necessary: Critical moment

\[
p_+(t) := \sup\{p \geq 0 : M(p, t) < \infty\}.
\]

satisfies

\[
\lim_{t \downarrow 0} t^{1-\beta} p_+(t) = \infty
\]

- Heston: \(p_+(t) \approx 1/t \gg t^{\beta-1}\) (Keller-Ressel 2011). Indeed, Heston satisfies GE (affine calculation).
Recall the Gärtner-Ellis Theorem

- \((Z_t)_{t \geq 0}\) stochastic process
- \(t \mapsto a_t\) real function with \(0 < a_t = o(1)\) as \(t \downarrow 0\)
- Assume existence and smoothness of

\[
\Gamma_Z(p) := \lim_{t \downarrow 0} a_t \log M_Z(p/a_t, t), \quad p \in \mathbb{R}.
\]

- **Gärtner-Ellis Theorem** The laws of \(Z_t\) satisfy an LDP for \(t \downarrow 0\), with rate \(a_t^{-1}\) and rate function \(\Gamma_Z^*\) (Legendre transform).
- Thus:

\[
P[Z_t \in B] \approx \exp(-a_t^{-1} \inf_B \Gamma_Z^*)
\]
Alternative approach: Result

**Theorem** Under assumption GE, for \( k_t = \theta t^\beta \) with \( \beta \in (0, \frac{1}{2}) \) and \( \theta > 0 \),

\[
P[X_t \geq k_t] = \exp \left( - \frac{1}{2 \sigma_0^2} \frac{k_t^2}{t} (1 + o(1)) \right), \quad t \downarrow 0.
\]

**Proof idea:** Define \( Z_t := t^{-\beta} X_t \), with mgf \( M_Z(s, t) = \mathbb{E}[e^{s Z_t}] \), and

\[a_t := t^{1-2\beta} = o(1), \quad t \downarrow 0.\]

Then GE is equivalent to

\[
\Gamma_Z(p) := \lim_{t \downarrow 0} a_t \log M_Z(p/a_t, t) = \frac{1}{2} \sigma_0^2 p^2, \quad p \in \mathbb{R}.
\]

\( \Gamma_Z \) is finite on \( \mathbb{R} \); by Gärtner-Ellis, \( (Z_t)_{t \geq 0} \) satisfies an LDP as \( t \downarrow 0 \), with rate \( a_t^{-1} \) and rate function \( \Gamma_Z^* \).
Summary

➤ Small time asymptotics for stochastic volatility models
➤ New regime ("moderately out-of-the-money")
➤ Mild assumptions on density and local vol of the model
➤ Alternative assumption: limit of scaled characteristic function
➤ Numerically tractable: asymptotic expansions can be easily evaluated