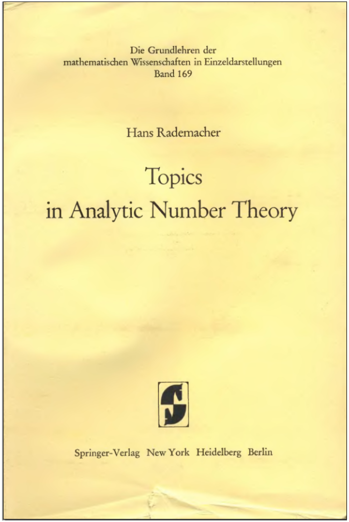
Disproof of a Conjecture by Rademacher on Partial Fractions

Stefan Gerhold

TU Wien

Joint work with Michael Drmota

April 2014



Die Grundlehren der
mathematischen Wissenschaften in Einzeldarstellungen
Band 169

Hans Rademacher

Topics
in Analytic Number Theory

Springer-Verlag New York Heidelberg Berlin

Hans Rademacher (1892-1969)

Integer partitions

- $p(n) =$ number of ways to write n as sum of positive integers (disregarding their order)
- Central object in additive number theory
- Euler: Recursion for $p(n)$
- Hardy, Ramanujan (1918):

$$p(n) \sim \frac{1}{4\sqrt{3}n} \exp\left(\pi\sqrt{\frac{2n}{3}}\right), \quad n \to \infty$$

- Rademacher (1937): Convergent series representation

Integer partitions: generating function

- Unrestricted partition generating function:

$$\sum_{n=1}^{\infty} p(n)x^n = \sum_{k_1 \geq 0} x^{k_1} \sum_{k_2 \geq 0} x^{2k_2} \cdots = \prod_{j=1}^{\infty} \frac{1}{1-x^j}$$

- Partitions into parts $\leq N$:

$$\sum_{n=1}^{\infty} p_N(n)x^n = \sum_{k_1 \geq 0} x^{k_1} \sum_{k_2 \geq 0} x^{2k_2} \cdots \sum_{k_N \geq 0} x^{Nk_N} = \prod_{j=1}^{N} \frac{1}{1-x^j}$$

(Same as partitions into at most N parts)

Partial fraction decomposition

- Rademacher (*Topics in Analytic Number Theory*, 1973):

$$\prod_{j=1}^{\infty} \frac{1}{1-x^j} = \sum_{\substack{0 \le h < k \\ \gcd(h,k)=1}} \sum_{\ell=1}^{\infty} \frac{C_{hk\ell}(\infty)}{(x - e^{2\pi i h/k})^{\ell}}$$

  He gave an explicit formula for $C_{hk\ell}(\infty)$.

- Restricted case:

$$\prod_{j=1}^{N} \frac{1}{1-x^j} = \sum_{\substack{0 \le h < k \le N \\ \gcd(h,k)=1}} \sum_{\ell=1}^{\lfloor N/k \rfloor} \frac{C_{hk\ell}(N)}{(x - e^{2\pi i h/k})^{\ell}}$$

- **Conjecture** (Rademacher):

$$\boxed{\lim_{N\to\infty} C_{hk\ell}(N) \stackrel{?}{=} C_{hk\ell}(\infty), \quad \text{for fixed } h, k, \ell.}$$

Topics in Analytic Number Theory (1973)

Let us write the unique algebraic partial fraction decomposition of (130.4) as

$$\frac{1}{\prod_{m=1}^{N}(1-x^m)} = \sum_{0\leq h<k\leq N}' \sum_{l=1}^{[N/k]} \frac{C_{hkl}(N)}{\left(x-e^{\frac{2\pi ih}{k}}\right)^l}. \tag{130.5}$$

The $C_{hkl}(N)$ can be obtained algebraically as expressions containing roots of unity, although the actual computation becomes soon very cumbersome with increasing N. No explicit formula for $C_{hkl}(N)$ is known, not even for the simplest case $h=0, k=1, l=1$, and variable N.

I conjecture now that the partial fraction decomposition (130.5) converges termwise to the expansion (130.1). More explicitly, I propose the

Topics in Analytic Number Theory (1973)

Conjecture.

$$\lim_{N\to\infty} C_{hkl}(N)$$

exists and is equal to

$$C_{hkl}(\infty) = -2\pi\left(\frac{\pi}{12}\right)^{3/2} \frac{\omega_{hk} e^{\frac{2\pi i h l}{k}}}{k^{5/2}} \Delta_\alpha^{l-1} L_{3/2}\left(-\frac{\pi^2}{6k^2}(\alpha+1)\right),$$

$$\alpha = \frac{1}{24}.$$

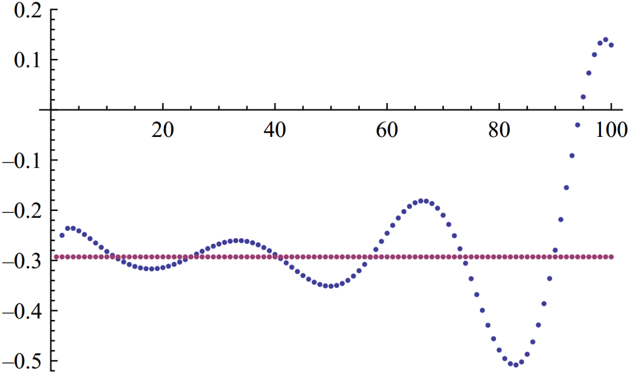
History

- Rademacher made some computations for $N = 1, \ldots, 5$; seemed ok.
- George Andrews: Rademacher discussed the conjecture in a course 1961/62.
- Ehrenpreis, Friedmann (1993): *"coefficients difficult to compute; inconclusive".*
- Davidson, Gagola (2002): $C_{011}(N)$ for $N \leq 45$. Oscillations.
- George Andrews (2003): *"Rademacher's conjecture lies at the interface of the theory of modular forms and the theory of q-series. Thus progress on this problem may require contributions from two areas that have had less contact in the past than might have been expected or hoped for."*

History

- Munagi (2008): Writes in favor of the conjecture.
- Sills, Zeilberger (2013): Recurrence for $C_{01\ell}(N)$.
  Computations up to $N = 1000$.
  No convergence, but oscillations and exponential growth, apparently
- Drmota, SG (2013): This is true, and Rademacher's conjecture is incorrect. Details to follow.
- O'Sullivan (2013): Disproof by another approach.

Plot from Sills and Zeilberger (2013)



$C_{011}(N)$ for $N = 1, \ldots, 100$ and Rademacher's conjectured limit

Disproof of the conjecture

Theorem (Drmota, SG 2013):

For any integer $\ell \geq 1$, we have the asymptotics

$$C_{0,1,\ell}(N) = b^N N^{-\ell-1} H_\ell(N) + O(b^N N^{-\ell-117/112}), \quad N \to \infty,$$

where $b \approx 1.07$, and H_ℓ is a bounded periodic function with period $p \approx 31.96$.

- Exponential growth + oscillations
- No convergence
- b and p defined by transcendental equation (involving dilogarithm)
- b and p independent of ℓ

Proof idea

- Contour integral representation of $C_{0,1,\ell}(N)$
- Split integration contour
- Left part dominates
- **Main Step 1 (left part):** Approximate the integrand (Mellin transform asymptotics)
- **Main Step 2 (left part):** Saddle point method
- **Main Step 3 (right part):** Direct estimates.

Integral representation by Cauchy's formula

$$\prod_{j=1}^{N} \frac{1}{1-x^j} = \cdots + \frac{C_{0,1,\ell}(N)}{(x-1)^\ell} + \ldots$$

For small $r > 0$:

$$C_{0,1,\ell}(N) = \frac{1}{2i\pi} \int_{|x-1|=r} (x-1)^{\ell-1} \prod_{j=1}^{N} \frac{1}{1-x^j}\, dx$$

$$= \frac{1}{2i\pi} \int_{|x|=r} x^{\ell-1} \prod_{j=1}^{N} \frac{1}{1-(x+1)^j}\, dx$$

From now on $\ell = 1$, for better readability.

Integral representation by Cauchy's formula

Substitute $x + 1 = e^{z/N}$

$$C_{0,1,\ell}(N) = \frac{1}{2i\pi} \int_{|x|=r} \prod_{j=1}^{N} \frac{1}{1-(x+1)^j} dx$$

$$= \frac{1}{2i\pi} \frac{1}{N} \int_{|z|=r} e^{z/N} \prod_{j=1}^{N} \frac{1}{1-e^{zj/N}} dz$$

Goal: Asymptotics for $N \to \infty$

Easy observation: Reflection formula

$$\prod_{j=1}^{N} \frac{1}{1-e^{zj/N}} = (-1)^N \prod_{j=1}^{N} \frac{e^{-zj/N}}{1-e^{-zj/N}}$$
$$= (-1)^N e^{-z(N+1)/2} \prod_{j=1}^{N} \frac{1}{1-e^{-zj/N}}$$

- $\Re z > 0 \implies$ exponential decay of $e^{-z(N+1)/2}$
- Left half-circle ($\Re z < 0$) dominates

Plan

- Define
$$g(z,N) := \log \prod_{j=1}^{N} \frac{1}{1-e^{zj/N}}$$
- Recall:
$$C_{0,1,\ell}(N) = \frac{1}{2i\pi}\frac{1}{N}\int_{|z|=r} e^{z/N+g(z,N)}dz$$
- Split the integration contour
- $\Re z < -N^{-7/8}$: Approximate g, saddle point method **(Steps 1, 2)**
- $\Re z \geq -N^{-7/8}$: Estimate g directly $\Rightarrow$ negligible **(Step 3)**
- We need two estimates for g with sufficient accuracy and sufficient validity region.

Approximate the integrand

- We want to approximate

$$g(z, N) = \log \prod_{j=1}^{N} \frac{1}{1 - e^{zj/N}}$$

- For Mellin transform, N has to be *real*
- Taylor series:

$$\begin{aligned} g(z, N) &= -\sum_{j=1}^{N} \log(1 - e^{zj/N}) \\ &= \sum_{j=1}^{N} \sum_{k=1}^{\infty} \frac{1}{k} e^{zjk/N} = \sum_{k=1}^{\infty} \frac{1}{k} \frac{1 - e^{kz}}{e^{-kz/N} - 1} \end{aligned}$$

Mellin transform, for fixed z with $\Re(z) < 0$, and $\Re(s) < -1$

$$
\begin{aligned}
\mathcal{M}g(z,\cdot)(s) &= \int_0^\infty g(z,x)x^{s-1}dx \\
&= \sum_{k=1}^{\infty} \frac{1-e^{kz}}{k} \int_0^\infty \frac{x^{s-1}}{e^{-kz/x}-1}dx \\
&= \sum_{k=1}^{\infty} \frac{1-e^{kz}}{k}(-kz)^s\Gamma(-s)\zeta(-s) \\
&= (-z)^s\Gamma(-s)\zeta(-s)\left(\sum_{k=1}^{\infty} k^{s-1} - \sum_{k=1}^{\infty} k^{s-1}e^{kz}\right) \\
&= (-z)^s\Gamma(-s)\zeta(-s)\Big(\zeta(1-s) - \mathrm{Li}_{1-s}(e^z)\Big)
\end{aligned}
$$

- Polylogarithm $\mathrm{Li}_\nu(w) = \sum_{k\geq 1} w^k/k^\nu$, for $|w| < 1$ and $\nu \in \mathbb{C}$

Mellin inversion: Poles map to asymptotic elements

- Mellin inversion formula:

$$g(z,N) = \frac{1}{2i\pi}\int_{-3/2-i\infty}^{-3/2+i\infty} \mathcal{M}g(z,\cdot)(s)N^{-s}ds$$

- Recall:

$$\mathcal{M}g(z,\cdot)(s) = (-z)^s\Gamma(-s)\zeta(-s)\Big(\zeta(1-s) - \mathrm{Li}_{1-s}(e^z)\Big)$$

- Shift integration path to the right ($\Re s = 8/7$)
- Collect residues
- $\Gamma(-s)$ has simple poles at $s = 0, 1, 2, \ldots$
- $\zeta(-s)$ has a simple pole at $s = -1$
- $\zeta(1-s)$ has a simple pole at $s = 0 \Longrightarrow$ double pole
- $\mathrm{Li}_{1-s}(e^z)$ is an entire function of s

Mellin inversion: Poles map to asymptotic elements

- Calculate residues:

$$res_{s=-1}\mathcal{M}g(z,\cdot)(s)N^{-s} = \frac{N}{z}(\zeta(2) - \mathrm{Li}_2(e^z))$$

$$res_{s=0}\mathcal{M}g(z,\cdot)(s)N^{-s} = \frac{1}{2}\log N + \frac{1}{2}\Big(\log 2\pi + \log(1-e^z) - \log(-z)\Big)$$

- Integrand expansion for fixed z with $\Re z < 0$, as $N \to \infty$:

$$g(z,N) = \frac{1}{z}\Big(\mathrm{Li}_2(e^z) - \frac{\pi^2}{6}\Big)N - \frac{1}{2}\log N - \frac{1}{2}\Big(\log 2\pi + \log(1-e^z) - \log(-z)\Big) + O(1/N)$$

Mellin inversion: Refined estimate

- We need uniformity w.r.t. z
- Refined estimate:

$$g(z,N)=\frac{1}{z}\Big(\mathrm{Li}_2(e^z)-\frac{\pi^2}{6}\Big)N-\frac{1}{2}\log N\\ -\frac{1}{2}\Big(\log 2\pi+\log(1-e^z)-\log(-z)\Big)+h(N)$$

The function h is

(i) uniformly $O(N^{-1/2})$ if $|\arg z|\geq\pi/2+\varepsilon$, z is bounded away from 0, and $z=O(N^{1/2})$,

(ii) uniformly $O(N^{33/112})$ if z is bounded, bounded away from 0 and $\pm 2i\pi$, $|\Im z|<8$, and $\Re z<-N^{-7/8}$.

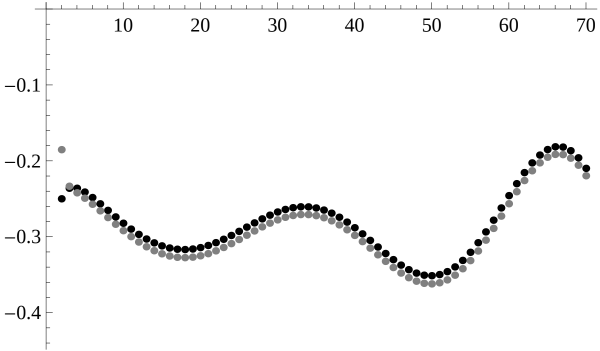
Mellin inversion: Refined estimate

Proof is based on the estimates ($\Re s = 8/7, \Im s \to +\infty$)

$$\begin{aligned}
|N^{-s}| &= N^{-\Re s}, \\
|(-z)^s| &= |z|^{\Re s} e^{-\Im(s)\arg(-z)}, \\
|\Gamma(-s)| &\sim \sqrt{2\pi} e^{-\frac{1}{2}\pi \Im s} (\Im s)^{-\Re s - 1/2}, \\
\zeta(-s) &= O((\Im s)^{\Re s + 1/2}), \\
\zeta(1-s) &= O((\Im s)^{\Re s - 1/2}), \\
\mathrm{Li}_{1-s}(e^z) &= O((\Im s)^{\Re s - 1/2})
\end{aligned}$$

Step 1 done (approximate integrand in left half-plane)

$$C_{0,1,1}(N) \approx \frac{1}{(2\pi N)^{3/2} i} \int_{\substack{|z|=5 \\ \Re z \leq 0}} \sqrt{\frac{-z}{1-e^z}} \exp\left(\frac{z}{N} + \frac{N}{z}\left(\mathrm{Li}_2(e^z) - \frac{\pi^2}{6}\right)\right) dz$$



Recall the proof idea

- Contour integral representation of $C_{0,1,\ell}(N)$
- Split integration contour
- Left part dominates
- ✓ **Main Step 1 (left part):** Approximate the integrand (Mellin transform asymptotics)
- **Main Step 2 (left part):** Saddle point method
- **Main Step 3 (right part):** Direct estimates.

Step 2: Saddle point asymptotics

- Method goes back to Riemann (1863) and Debye (1909)
- Move integration contour through saddle points
- Then: Laplace method
- Here: Two conjugate saddle points
- Location of saddle points yields exponential growth
- Local behavior of integrand yields subexponential factors

Saddle point asymptotics

- Dominating factor of integrand:

$$\exp\left(\frac{N}{z}\left(\mathrm{Li}_2(e^z)-\frac{\pi^2}{6}\right)\right)$$

- Saddle point equation:

$$\log(1-e^z)+\frac{1}{z}(\mathrm{Li}_2(e^z)-\pi^2/6)=0$$

- Saddle points: $z_0 \approx -1.61+7.42i$, and $\bar{z}_0$
- Independent of N

Axis of the saddle point

- Argument of the axis:

$$a = \frac{\pi}{2} - \frac{1}{2} \arg \frac{d^2}{dz^2} \left(\frac{1}{z} \left(\mathrm{Li}_2(e^z) - \frac{\pi^2}{6} \right) \right) \bigg|_{z=z_0}$$
$$= \frac{\pi}{2} - \frac{1}{2} \arg \left(\frac{e^{z_0}}{z_0(1 - e^{z_0})} \right)$$
$$\approx 1.79$$

- Direction of steepest descent:

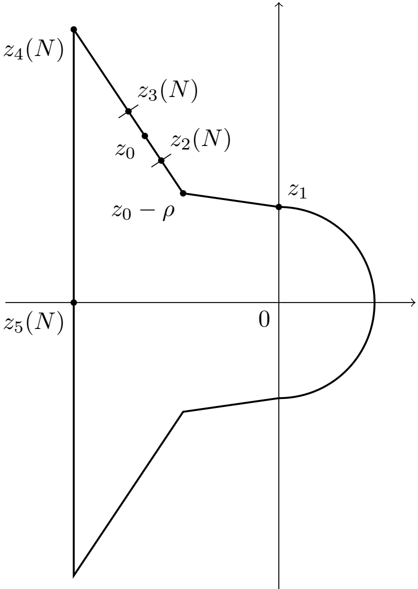
$$\rho = \exp(ia)$$

New integration contour

$$z_1 = 5i$$
$$z_5(N) = -\sqrt{N}$$

Width of central part is $O(N^{-39/112})$

$$z_2(N) = z_0 - \rho N^{-39/112}$$
$$z_3(N) = z_0 + \rho N^{-39/112}$$



Local expansion near the saddle point

- Saddle point segment:

$$z = z_0 + t\rho, \quad -N^{-39/112} \le t \le N^{-39/112}$$

- Expansion of integrand:

$$g(z, N) = -N \log(1 - e^{z_0}) - \tfrac{1}{2}\alpha N t^2 - \tfrac{1}{2} \log N + const + O(N^{-5/112}),$$

where

$$\alpha := -\frac{\rho^2 e^{z_0}}{z_0(1 - e^{z_0})} \approx 0.028$$

Gaussian integral

- From the second order term of the expansion:

$$\int_{-N^{-39/112}}^{N^{-39/112}} \exp(-\tfrac{1}{2}\alpha N t^2)dt \sim \frac{1}{\sqrt{\alpha N}} \int_{-\infty}^{\infty} e^{-x^2/2}dx = \sqrt{\frac{2\pi}{\alpha N}},$$

- Saddle point integral:

$$\int_{z_2(N)}^{z_3(N)} e^{f(z,N)}dz = \frac{\rho(-z_0)^{1/2}}{\sqrt{\alpha(1-e^{z_0})}}\frac{1}{N}(1-e^{z_0})^{-N}(1+O(N^{-5/112}))$$

There is also a lower saddle point

- Lower saddle point integral:

$$\int_{\bar{z}_3(N)}^{\bar{z}_2(N)} e^{g(z,N)} dz = -\overline{\int_{z_2(N)}^{z_3(N)} e^{g(z,N)} dz}$$

- Recall:

$$C_{0,1,1}(N) = \frac{1}{2i\pi} \frac{1}{N} \int_{|z|=r} e^{z/N} \prod_{j=1}^{N} \frac{1}{1 - e^{zj/N}} dz$$

- Contribution of both saddle points:

$$\frac{1}{\pi N} \Im \left(\int_{z_2(N)}^{z_3(N)} e^{g(z,N)} dz \right) \sim \frac{1}{\sqrt{\alpha}\pi N^2} \Im \left(\frac{\rho(-z_0)^{1/2}}{\sqrt{(1 - e^{z_0})}} (1 - e^{z_0})^{-N} \right)$$

Tail estimates

- Integrand $F(z, N) := e^{z/N} \prod_{j=1}^{N} \frac{1}{1-e^{zj/N}}$
- Estimates:

$$\int_{z_1}^{z_2} \mathbf{1}_{\{\Re z \le -N^{-7/8}\}} F(z, N) dz = O\Big(b^N \exp\Big(-\tfrac{1}{3}\alpha N^{17/56}\Big)\Big)$$

$$\int_{z_3}^{z_4} F(z, N) dz = O\Big(b^N \exp\Big(-\tfrac{1}{3}\alpha N^{17/56}\Big)\Big)$$

$$\int_{z_4}^{z_5} F(z, N) dz = \exp(O(N^{1/2}))$$

- All are $\ll b^N N^{-2}$
- Note: $\Re z \le -N^{-7/8}$ is validity region of Mellin estimate

Recall the proof idea

- Contour integral representation of $C_{0,1,\ell}(N)$
- Split integration contour
- Left part dominates
- ✓ **Main Step 1 (left part):** Approximate the integrand (Mellin transform asymptotics)
- ✓ **Main Step 2 (left part):** Saddle point method
- **Main Step 3 (right part):** Direct estimates.

Step 3: Estimate close to imaginary axis

Lemma:

$$\int_{z_1}^{z_2} \mathbf{1}_{\{\Re z \geq -N^{-7/8}\}} F(z,N) dz = O(0.85^N).$$

Proof:

- Directly from $F(z,N) = e^{z/N} \prod_{j=1}^{N} \frac{1}{1-e^{zj/N}}$
- Euler's summation formula
- Some (tedious) estimates, one of them proved by CAD (Cylindrical Algebraic Decomposition)

Step 3: Estimate in the right half-plane

Lemma:

$$\frac{1}{N}\frac{1}{2i\pi}\int_{|z|=5}\mathbf{1}_{\{\Re z>0\}}F(z,N)dz = O(0.95^N).$$

Proof:
Use reflection formula, recycle parts of proof from left half-plane

Conclusion

- Rademacher's conjecture: "partial fraction decompositon" and "$\lim_N \to \infty$" commute
- Disproved by us
- (Interesting question: Why didn't Rademacher do it?!)
- Is there *some* relation of the p.f. coefficients of $\prod_{j\geq 1}(1-x^j)^{-1}$ and $\prod_{j=1}^{N}(1-x^j)^{-1}$?
- Numerical observation by O'Sullivan: Maybe convergence of Cesàro means

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} C_{h,k,\ell}(n) \stackrel{?}{=} C_{h,k,\ell}(\infty)$$

References

- H. Rademacher, *Topics in analytic number theory*, 1973.
- L. Ehrenpreis, *Function theory for Rogers-Ramanujan-like partition identities*, 1993.
- G. E. Andrews, *Partitions: at the interface of q-series and modular forms*, 2003.
- C. O'Sullivan, *On the partial fraction decomposition of the restricted partition generating function*, 2012.
- A. V. Sills and D. Zeilberger, *Rademacher's infinite partial fraction conjecture is (almost certainly) false*, 2013.
- M. Drmota, S. Gerhold, *Disproof of a conjecture by Rademacher on partial fractions*, 2013.