The Small Maturity Implied Volatility Slope for Lévy Models

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Introduction and motivation

- There are many results on asymptotics of implied volatility
- Our focus:
  - Lévy models
  - Slope of implied vol (strike derivative)
  - Short maturity
- Lee's moment formula + refinements: steepness of wings
- We focus on the ATM slope
- Diffusion models: limit smile as $T \to 0$
- Lévy models: Implied vol explodes off-the-money, no limit smile

Introduction and Motivation

- Good slope approximation for very short maturities
- Identify the sign of the slope for short maturities
- $\rightarrow$ simple parameter constraint for market calibration
- Qualitative influence of parameters on model behavior
- Our asymptotic method should be of general applicability
- Method extends to other time-asymptotics problems for Lévy processes (densities, option prices ...)

Implied volatility slope and digitals

- For simplicity: $S_0 = 1, r = 0$
- Implied volatility $\sigma_{\rm imp} = \sigma_{\rm imp}(K, T)$:

$$C_{\rm BS}(K, \sigma_{\rm imp}, T) = C(K, T) := \mathbb{E}[(S_T - K)^+]$$

- Implied volatility slope:

$$\partial_K \sigma_{\rm imp} = -\frac{\partial_K C_{\rm BS} - \partial_K C}{\partial_\sigma C_{\rm BS}}$$

- If law of S_T is continuous:

$$\partial_K \sigma_{\rm imp} = -\frac{\partial_K C_{\rm BS} + \mathbb{P}[S_T \geq K]}{\partial_\sigma C_{\rm BS}}$$

Implied volatility slope and digitals

- More explicitly:

$$\partial_K \sigma_{\mathrm{imp}} = \frac{\Phi(-\sigma_{\mathrm{imp}}\sqrt{T}/2) - \mathbb{P}[S_T \geq K]}{K\sqrt{T}\, n(\sigma_{\mathrm{imp}}\sqrt{T}/2)}$$

- Under mild assumptions we have $\sigma_{\mathrm{imp}}\sqrt{T} = o(1),\ T \to 0$, and so

$$\partial_K \sigma_{\mathrm{imp}} \sim \frac{\sqrt{2\pi}}{K\sqrt{T}} \left(\frac{1}{2} - \mathbb{P}[S_T \geq K] - \frac{\sigma_{\mathrm{imp}}\sqrt{T}}{2\sqrt{2\pi}} + O((\sigma_{\mathrm{imp}}\sqrt{T})^3) \right)$$

- Two cases for ATM slope asymptotics:
  - $\mathbb{P}[S_T \geq S_0] \not\to \frac{1}{2}$: Need first order term of $\mathbb{P}[S_T \geq S_0]$
  - $\mathbb{P}[S_T \geq S_0] \to \frac{1}{2}$: Need second order term of $\mathbb{P}[S_T \geq S_0]$ and first term of $\sigma_{\mathrm{imp}}\sqrt{T}$

ATM digital calls: Small-time expansions for Lévy models

- $S_t = \exp(X_t)$, X Lévy process with characteristic triplet (σ, ν, b)

$$\log E[e^{zX_1}] = \frac{\sigma^2 z^2}{2} + bz + \int_{-\infty}^{\infty} (e^{zx} - 1 - zx\mathbf{1}_{\{|x| \leq 1\}})\nu(dx).$$

- If $\sigma > 0$, then

$$\lim_{T \to 0} \mathbb{P}[S_T > S_0] = \lim_{T \to 0} \mathbb{P}[X_T > 0] = \tfrac{1}{2}$$

- If $\sigma = 0$, then the limit can attain all values from $[0, 1]$.

Warm-up: Jump diffusions

- Log-underlying

$$X_T = \sigma W_T + bT + compound\ \ Poisson$$

- Let τ be the first jump time. Then

$$\begin{aligned}
\mathbb{P}[X_T > 0] &= \mathbb{P}[X_T > 0 | \tau \le T] \cdot \mathbb{P}[\tau \le T] \\
&\qquad + \mathbb{P}[X_T > 0 | \tau > T] \cdot \mathbb{P}[\tau > T] \\
&= O(T) + \mathbb{P}[\sigma W_T + bT > 0](1 + O(T)) \\
&= \mathbb{P}[\sigma W_T + bT > 0] + O(T)
\end{aligned}$$

- Hence

$$\mathbb{P}[X_T > 0] = \frac{1}{2} + \frac{b}{\sigma\sqrt{2\pi}}\sqrt{T} + O(T), \quad T \to 0.$$

Warm-up: Jump diffusions

- Recall

$$\partial_K \sigma_{\mathrm{imp}} \sim \frac{\sqrt{2\pi}}{K\sqrt{T}} \left(\frac{1}{2} - \mathbb{P}[S_T \geq K] - \frac{\sigma_{\mathrm{imp}}\sqrt{T}}{2\sqrt{2\pi}} + O((\sigma_{\mathrm{imp}}\sqrt{T})^3) \right)$$

- From $\sigma_{\mathrm{imp}} \to \sigma$ and

$$\mathbb{P}[X_T > 0] = \frac{1}{2} + \frac{b}{\sigma\sqrt{2\pi}}\sqrt{T} + O(T)$$

we get the ATM slope

$$\lim_{T\to 0} \partial_K \sigma_{\mathrm{imp}}|_{K=1} = -\frac{b}{\sigma} - \frac{\sigma}{2}$$

- What about infinite activity?

Case Study: The Normal Inverse Gaussian Model

- Moment generating function

$$\mathbb{E}[e^{zX_T}] = \exp(T(\tfrac{1}{2}\sigma^2 z^2 + bz + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + z)^2})))$$

- First suppose $\sigma = 0$. Thus we need the first term of $\mathbb{P}[X_T > 0]$.
- Special case of thm by Rosenbaum, Tankov (2011): $\varepsilon^{-1} X_{\varepsilon t}$ converges in law to a 1-stable Lévy process X^*.
- Hence

$$\begin{aligned}\lim_{T \to 0} \mathbb{P}[X_T > 0] &= \lim_{\varepsilon \to 0} \mathbb{P}[\varepsilon^{-1} X_\varepsilon > 0] \\ &= \mathbb{P}[X_1^* > 0] = \tfrac{1}{2} + \tfrac{1}{\pi} \arctan(b/\delta).\end{aligned}$$

Case Study: The Normal Inverse Gaussian Model

- Recall

$$\partial_K \sigma_{\mathrm{imp}} \sim \frac{\sqrt{2\pi}}{K\sqrt{T}} \left(\frac{1}{2} - \mathbb{P}[S_T \geq K] - \frac{\sigma_{\mathrm{imp}}\sqrt{T}}{2\sqrt{2\pi}} + O((\sigma_{\mathrm{imp}}\sqrt{T})^3) \right)$$

- From

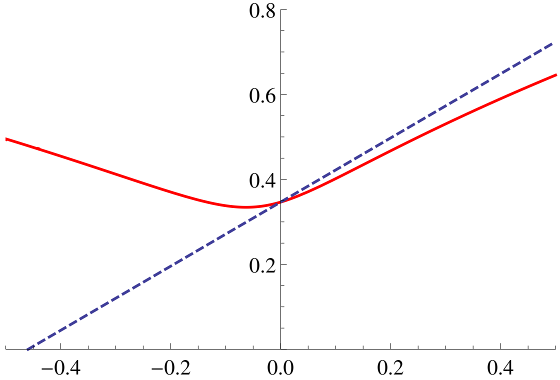
$$\lim_{T\to 0} \mathbb{P}[X_T > 0] = \frac{1}{2} + \frac{1}{\pi}\arctan(b/\delta)$$

we thus get

$$\partial_K \sigma_{\mathrm{imp}}|_{K=1} \sim -\sqrt{2/\pi}\,\arctan(b/\delta)\frac{1}{\sqrt{T}}$$

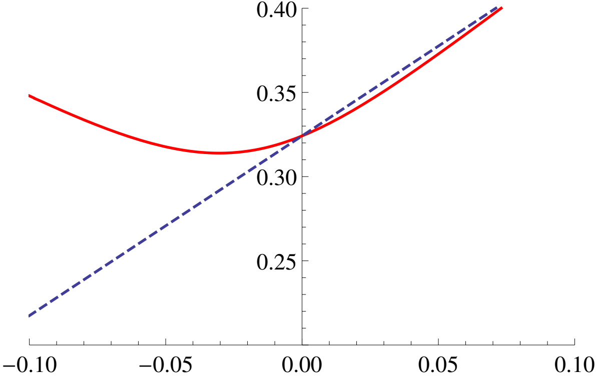
- ATM slope explodes

NIG implied vol, with $\sigma = 0$



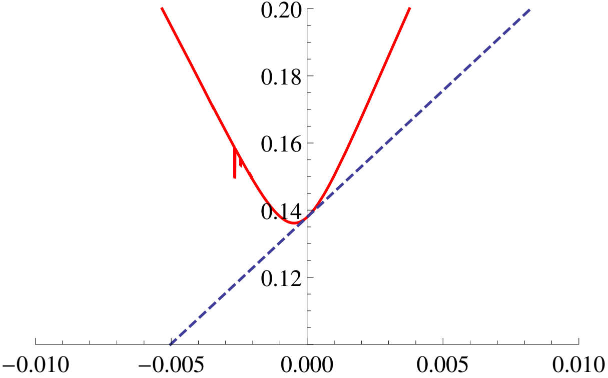
Maturity $T = 0.1$

NIG implied vol, with $\sigma = 0$



Maturity $T = 0.05$

NIG implied vol, with $\sigma = 0$



Maturity $T = 0.001$

Robustness of Lee's moment formula for $\sigma = 0$

- ATM slope $\sim \mathit{const} \cdot T^{-1/2}$ extends to other models (Variance gamma, CGMY, Meixner)
- Lee's moment formula (Lee 2004; $K = \log k$):

$$\sigma_{\mathrm{imp}}(K, T) \sim \mathit{const} \cdot \sqrt{k/T}, \qquad k \to \infty$$
$$\sigma_{\mathrm{imp}}(K, T) \sim \mathit{const} \cdot \sqrt{-k/T}, \qquad k \to -\infty$$

- Fact: For these models, the ATM slope is positive for small T if and only if the right smile wing is steeper

Case Study: The Normal Inverse Gaussian Model. Now $\sigma > 0$.

- What about $\sigma > 0$?
- Econometric studies suggest the necessity of a diffusion component (Ait-Sahalia, Jacod 2009, 2010)
- We need the second order term in

$$\mathbb{P}[X_T > 0] = \frac{1}{2} + ?, \quad T \to 0$$

- The methods we've mentioned so far do not apply

Mellin transform asymptotics

- Goal: Asymptotics of a function $H(T)$, as $T \to 0$. (We: $H(T) = \mathbb{P}[X_T > 0]$.)
- Mellin transform

$$\mathcal{M}(s) = \int_0^\infty T^{s-1} H(T) dT$$

- Inversion formula

$$H(T) = \frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} T^{-s} \mathcal{M}(s) ds$$

- Move contour to the left. Suppose $\mathcal{M}$ has poles at $0 \geq s_0 > s_1 > \dots$
- Wrap contour around poles; residue theorem $\Longrightarrow$

$$H(T) = res_{s=s_0} T^{-s} \mathcal{M}(s) + res_{s=s_1} T^{-s} \mathcal{M}(s) + \dots$$

Mellin transform asymptotics: Lévy processes

- ATM digital:

$$H(T) = \mathbb{P}[X_T > 0] = \frac{1}{2i\pi}\int_{a-i\infty}^{a+i\infty} \frac{1}{z} e^{-T\psi(z)} dz$$

- Mellin transform (using Fubini)

$$\begin{aligned}\mathcal{M}(s) &= \frac{1}{2i\pi}\int_{a-i\infty}^{a+i\infty} \frac{1}{z}\Big(\int_0^\infty T^{s-1} e^{-T\psi(z)} dT\Big) dz \\ &= \frac{\Gamma(s)}{2i\pi}\int_{a-i\infty}^{a+i\infty} \frac{1}{z}\psi^{-s}(z) dz =: \Gamma(s)F(s)\end{aligned}$$

- $\Gamma(s)$ has poles at $s = 0, -1, -2, \dots$
- If $F(s)$ has meromorphic continuation, we get asymptotics of $H(T)$

Mellin transform asymptotics

- Poles of $\mathcal{M}$ correspond to terms in the asymptotic expansion of H
- Simple poles yield powers of T:

$$res_{s=s_0} T^{-s} \left(\frac{c}{s - s_0} + \dots \right) = cT^{-s_0}$$

- Double poles yield logarithmic terms:

$$res_{s=s_0} T^{-s} \left(\frac{c_2}{(s-s_0)^2} + \frac{c_1}{s - s_0} + \dots \right)$$
$$= T^{-s_0} res_{s=s_0} (1 - (\log T)(s - s_0) + \dots) \left(\frac{c_2}{(s-s_0)^2} + \frac{c_1}{s - s_0} + \dots \right.$$

Case study: NIG with $\sigma > 0$

- Lévy exponent

$$\psi(z) = -(\tfrac{1}{2}\sigma^2 z^2 + bz + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + z)^2}))$$

- Fourier representation of ATM digital:

$$\mathbb{P}[X_T > 0] = \frac{1}{\pi}\mathrm{Re}\int_0^\infty \frac{e^{-T\psi(a+iy)}}{a+iy}dy$$

- Mellin transform:

$$\int_0^\infty \frac{e^{-T\psi(a+iy)}}{a+iy}dy \quad \longrightarrow \quad \Gamma(s)\int_0^\infty \frac{\psi^{-s}(a+iy)}{a+iy}dy$$

Case study: NIG with $\sigma > 0$

- Mellin transform:

$$\Gamma(s) \int_0^\infty \frac{\psi^{-s}(a+iy)}{a+iy} dy$$

- Analytic for $\mathrm{Re}(s) > 0$
- We need meromorphic continuation to $\Re(s) \leq 0$
- Goal: "analytic function" + "explicit meromorphic function"
- y large:

$$\psi^{-s}(a+iy) = (\sigma^2 y^2/2)^{-s} + O(y^{-2s-1})$$

Case study: NIG with $\sigma > 0$

- y large:

$$\psi^{-s}(a+iy) = (\sigma^2 y^2/2)^{-s} + O(y^{-2s-1})$$

- Add and subtract convergence-inducing summand

$$\int_0^\infty \frac{(\sigma^2 y^2/2)^{-s}}{a+iy} dy = \frac{analytic}{\sin 2\pi s}$$

$\implies$ meromorphic continuation to $-\frac{1}{2} < \Re s$

- Second convergence-inducing summand $\implies$ continuation to $-1 < \Re s$
- $res_{s=0} T^{-s} \mathcal{M}(s) = const = \frac{\pi}{2} + imaginary$
- $res_{s=-1/2} T^{-s} \mathcal{M}(s) = const \cdot \sqrt{T}$

Case study: NIG with $\sigma > 0$

- Result of Mellin transform asymptotics:

$$\mathbb{P}[X_T > 0] = \frac{1}{2} + \frac{b}{\sigma\sqrt{2\pi}}\sqrt{T} + O(T \log T)$$

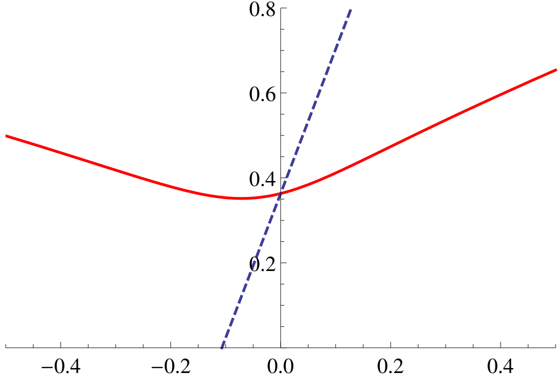
- ATM slope

$$\lim_{T\to 0} \partial_K \sigma_{\mathrm{imp}}|_{K=1} = -\frac{b}{\sigma} - \frac{\sigma}{2}$$

with

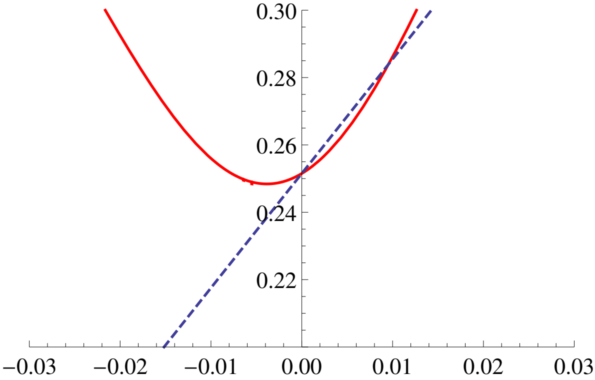
$$b = -\tfrac{1}{2}\sigma^2 - \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+1)^2})$$

NIG implied vol, with $\sigma > 0$



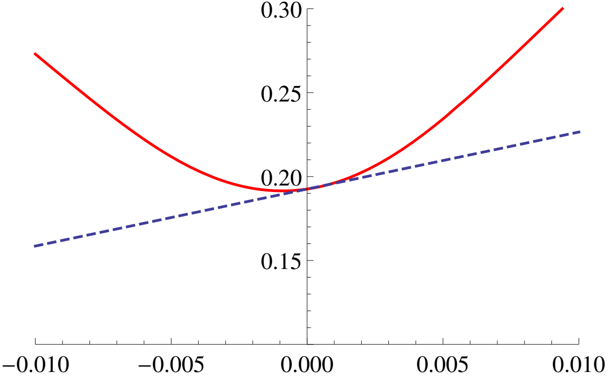
$\sigma = .1$, maturity $T = 0.1$

NIG implied vol, with $\sigma > 0$



$\sigma = .1$, maturity $T = 0.05$

NIG implied vol, with $\sigma > 0$



$\sigma = .1$, maturity $T = 0.001$

Further results, outlook

- Method works for CGMY, variance gamma, Meixner,... with $\sigma > 0$:
$$\lim_{T \to 0} \partial_K \sigma_{\mathrm{imp}}|_{K=1} = -\frac{b}{\sigma} - \frac{\sigma}{2}$$
- b such that $S = e^X$ martingale
- b depends on all model parameters
- Work in progress: general theorem (with conditions on Lévy triplet)

- Refined expansion for NIG, $\sigma = 0$:

$$\partial_K \sigma_{\mathrm{imp}}|_{K=1} = -\sqrt{2/\pi}\, \arctan(b/\delta) \frac{1}{\sqrt{T}} + \frac{\beta\delta}{\pi} T \log T + O(T)$$

Summary

- Implied vol slope of Lévy models for small maturity
- Related to transition probabilities (ATM digital calls)
- Slope sign gives useful constraint for calibration
- Mellin transform technique extends to other problems (time asymptotics of Lévy processes)

References

Y. Ait-Sahalia, J. Jacod, *Is Brownian motion necessary to model high frequency data?* Annals of Statistics 2010.

L. Andersen, A. Lipton, *Asymptotics for exponential Lévy processes and their volatility smile: survey and new results*, IJTAF 2013.

S. Gerhold, I.C. Gülüm, *The small maturity implied volatility slope for Lévy models,* working paper.

P. Tankov, *Pricing and hedging in exponential Lévy models: review of recent results,* Paris-Princeton Lecture Notes in Mathematical Finance, 2010.