Local volatility asymptotics and small-time central limit theorems

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Talk consists of two parts:

- PART I: Local volatility approximations
- PART II: Small-time central limit theorems for semimartingales
PART I: Overview

- Local vol model: recreates marginals of a given diffusion (via call price surface)
- Question: Given market dynamics, what does the associated local vol surface look like?
- Asymptotics for small time, large strike
- Applications: model risk; parametrization design
- Short end of the local vol surface: regularization
Local volatility

Given call price surface

\[ C = C(K, T) \]

Reproduced by local volatility model

\[ \frac{dS_t}{S_t} = \sigma_{\text{loc}}(S_t, t) dW_t \]

Dupire’s formula (1994)

\[ \sigma^2_{\text{loc}}(K, T) = \frac{2 \partial_T C}{K^2 \partial_{KK} C} \]
Heston Model

- Consider Call price surface $C_{\text{Hes}}(K, T)$ generated by Heston:

\[
\begin{align*}
    dS_t &= S_t \sqrt{V_t} dW_t, \quad S_0 = 1, \\
    dV_t &= (a + bV_t) dt + c \sqrt{V_t} dZ_t, \quad V_0 = v_0 > 0,
\end{align*}
\]

- Correlated Brownian motions

\[
    d\langle W, Z \rangle_t = \rho dt, \quad \rho \in [-1, 1]
\]

- Parameters

\[
a \geq 0, \ b \leq 0, \ c > 0
\]
Local vol in the Heston model (De Marco, Friz, SG 2013)

- Heston dynamics $\implies$ Call prices $\implies$ local vol surface
- Dupire’s formula

$$
\sigma^2_{\text{loc}}(K, T) = \frac{2\partial_T C_{\text{Hes}}}{K^2 \partial_{KK} C_{\text{Hes}}}
$$

- New wing asymptotics ($k = \log K$)

$$
\sigma^2_{\text{loc}}(K, T) \sim \text{const} \times k, \quad K \to \infty \\
\sigma^2_{\text{loc}}(K, T) \sim \text{const} \times |k|, \quad K \to 0
$$

- Looks like Lee’s moment formula for implied vol
- Similarly for the Stein-Stein model (Friz, De Marco 2012; large deviations)
Local variance for Heston model computed with Dupire’s formula. Call price derivatives computed via 1D integration of Heston characteristic function on a fixed integration contour.
Local variance for Heston model computed with Dupire’s formula. Adaptive contour with shift into saddle point. Note the linear increase.
Application 1: Design local vol parametrizations

- Example: Gatheral’s SVI parametrization
- Popular parametrization of the **implied vol** surface

\[ \sigma_{\text{imp}}(K, T)^2 T \approx \text{SVI}(k; a, b, c, m, s) \]

\[ k \mapsto a + b \left( (-m + k)c + \sqrt{(-m + k)^2 + s} \right) \]

- Gatheral, Jacquier 2011: Heston, \( T \to \infty \Rightarrow \text{SVI} \)
- Wings \( (k \to \pm \infty) \) compatible with Lee’s formula
- Our asymptotic result motivates SVI parametrization also for **local vol** \( \sigma_{\text{loc}}(K, T) \)
Application 2: Model risk

- Consider a path-dependent exotic
- $SV =$ price under **stochastic vol model**
- $LV =$ price under associated **local vol model**
- Note: local vol model recreates marginals of stoch vol model, but not the full law $\implies$ in general $SV \neq LV$
- Similar price: low model risk (e.g., variance swap)
- Different price: high model risk (e.g., volatility swap)
- **Toxicity index** (Reghai 2011)

\[
I = \frac{|SV - LV|}{|SV + LV|}
\]
Application 2: Model risk

- How to calculate local vol of a stochastic vol model?
- We need $\sigma_{\text{loc}}(K, T)$ in particular for large/small $K$ (Monte Carlo requires it)
- Dupire’s formula + Fourier inversion: unstable for large/small $K$
- Conditioning:

$$\sigma_{\text{loc}}^2(K, T) = E[\sigma_{\text{stoch}}^2(T) | S_T = K]$$

Difficult for $K \gg S_0$ (condition on unlikely events)

$\Rightarrow$ Wing approximation useful for computation
Towards a general wing approximation of local vol

- Moment generating function ($X_T = \log S_T$):

$$M(s, T) := E[\exp(sX_T)], \quad m(s, T) := \log M(s, T)$$

- Dupire’s formula + Fourier inversion

$$\sigma_{loc}^2(K, T) = \frac{2\partial_T C}{K^2 \partial_{KK} C} = \frac{2 \int_{-i\infty}^{i\infty} \frac{\partial_T m(s, T)}{s(s-1)} e^{-ks} M(s, T) ds}{\int_{-i\infty}^{i\infty} e^{-ks} M(s, T) ds}$$

- Saddle point method: Leading terms are integrands evaluated at saddle point $\longrightarrow$ cancellation
General wing formula for local vol (De Marco, Friz, SG 2013)

- log moment generating function ($X_T = \log S_T$)

$$m(s, T) = \log E[\exp(sX_T)]$$

- saddle point $\hat{s}(k, T)$

$$\frac{\partial}{\partial s} m(s, T) \bigg|_{s=\hat{s}} = k$$

- “Lee type” wing formula for $k \to \infty$: 

$$\sigma_{\text{loc}}^2(K, T) \approx \frac{2 \frac{\partial}{\partial T} m(s, T)}{s(s - 1)} \bigg|_{s=\hat{s}(k, T)}$$
Two ways to use the formula

- As it is (numerically very accurate, but not quite explicit):

\[
\sigma^2_{\text{loc}}(K, T) \approx 2 \left. \frac{\partial}{\partial T} m(s, T) \right|_{s=\hat{s}(k, T)} \frac{s(s-1)}{s(s-1)}
\]

- Use asymptotics of saddle point \( \hat{s}(k, T) \) and mgf \( \Rightarrow \) explicit formula (model-dependent)

- E.g., \( \text{const} \times k \) for Heston. Explicit, but model-dependent and less accurate.
**Heston model: Numerical example (left wing)**

Abbildung: Local variance $\sigma^2_{loc}(k, T)$ and our approximation in the Heston model.
Abbildung: Local variance $\sigma^2_{\text{loc}}(k, T)$ and our approximation in the Heston model.
Abbildung: Boundaries of the region where the relative error of our approximation is less than 2% (blue), 3% (green), 4% (yellow), and 5% (red).
Heston model: Implied volatility, $T = 0.25$

Abbildung: Green: Local vol computed by Dupire’s formula. Red: Use our approximation, as soon as its accuracy is over 5%.
Heston model: Implied volatility, $T = 5$

**Abbildung**: Green: Local vol computed by Dupire's formula. Red: Use our approximation, as soon as its accuracy is over 5%.
Heston model: rigorous proof

- Finding saddle point + local expansion of integrands fairly routine
- Problem: Verify concentration
- Needs some insight into behaviour of integrand away from saddle point
- Show exponential decay of integrands by ODE comparison (Riccati ODEs, similar to Friz, SG, Gulisashvili, Sturm, Quantitative Finance 2011)
- [Korenblum’s ratio Tauberian theorem?]
Using Dupire’s formula for models with jumps

- Variance gamma model: Call price not $C^2$ w.r.t. strike (but works for $T$ large)

- Jumps $\Rightarrow$ Blowup of local vol as $T \to 0$, hence local vol model may be ill-defined.

Dupire’s formula: $\sigma_{loc}^2(K, T) = 2\partial_T C/(K^2 \partial_K K C)$.

Call PIDE:

$$\partial_T C = \frac{1}{2} K^2 \sigma^2 \partial_{KK} C$$

$$+ \int_{-\infty}^{\infty} \nu(dz)(C(Ke^{-z}, T) - C(K, T) - K(e^z - 1)\partial_K C)$$

- Even if Dupire’s formula is well-defined, the local vol model may not match the marginals of the jump process.
Jumps: blowup if local vol as $T \to 0$

- Our saddle point formula works well

$$\sigma_{\text{loc}}^2(K, T) \approx \left. \frac{2}{s(s - 1)} \frac{\partial m(s, T)}{\partial T} \right|_{s = \hat{s}(k, T)}$$

- Examples for off-the-money blowup ($K \neq S_0$ fixed):

  $$\sigma_{\text{loc}}^2(K, T) \approx \frac{1}{T} \quad \text{(Merton jump diffusion)}$$
  $$\sigma_{\text{loc}}^2(K, T) \approx \frac{1}{\sqrt{T}} \quad \text{(Kou’s diffusion)}$$
  $$\sigma_{\text{loc}}^2(K, T) \approx \frac{1}{T} \quad \text{(Normal inverse Gaussian)}$$
Regularization of local vol

- A realistic market model should have jumps (especially for pricing short-maturity products)
- But then local vol model is ill-defined (blowup of $\sigma_{loc}$ for $T \to 0$)
- Question: is there a truncation of the local vol surface that always gives a well-defined local vol model?
Regularization of local vol

**Theorem** (Friz, SG, Yor 2013): Assume that $(S_t)$ is a martingale (possibly with jumps) with associated smooth call price surface $C$, such that $\partial_T C > 0$ and $\partial_{KK} C > 0$, i.e. (strict) absence of calendar and butterfly spreads. Define $\varepsilon$-shifted local volatility

$$\sigma^2_\varepsilon(K, T) = \frac{2 \partial_T C(K, T + \varepsilon)}{K^2 \partial_{KK} C(K, T + \varepsilon)}.$$

Then $dS^\varepsilon/S^\varepsilon = \sigma_\varepsilon(S^\varepsilon, t)dW$, started at randomized spot $S^\varepsilon_0$ with distribution

$$\mathbb{P}[S^\varepsilon_0 \in dK]/dK = \partial_{KK} C(K, \varepsilon),$$

admits a unique, non-explosive strong SDE solution such that

$$\forall K, T \geq 0 : \mathbb{E}[(S^\varepsilon_T - K)^+] \to C(K, T) \quad \text{as} \quad \varepsilon \to 0.$$
Regularization of local vol: Proof

- Let \( q^\varepsilon(dS, T) \) be the law of \( S_T^\varepsilon \), and \( p^\varepsilon(S, T) \) be the density of \( S_{T+\varepsilon} \).
- Calculate

\[
\mathbb{E}[(S_T^\varepsilon - K)^+] = \int (S - K)^+ q^\varepsilon(dS, T) \\
\equiv \int (S - K)^+ p^\varepsilon(S, T) dS \\
= \mathbb{E}[(S_{T+\varepsilon} - K)^+] \\
= C(K, T + \varepsilon).
\]

Then let \( \varepsilon \to 0 \).
- Need to show \( S_T^\varepsilon \overset{d}{=} S_{T+\varepsilon} \)
Regularization of local vol: Proof

- Define

\[ a^\varepsilon(K, T) := \frac{\partial_TC(K, T + \varepsilon)}{p^\varepsilon(K, T)} \]

- \( p^\varepsilon \) satisfies the Fokker-Plack equation

\[ \partial_{KK}(a^\varepsilon p^\varepsilon) = \partial_T p^\varepsilon \]

- But \( q^\varepsilon \) is also a (weak) solution, in the sense that

\[ \int \varphi(S)q^\varepsilon(dS, t) = \int \varphi(S)q^\varepsilon(dS, 0) + \int_0^t \int a^\varepsilon(S, s)\varphi''(S)q^\varepsilon(dS, s) \]

for any smooth \( \varphi \) with compact support. (Since \( a^\varepsilon(S, t)\partial_{SS} \) is the generator of \( S^\varepsilon \).)
So our result is a corollary of the following uniqueness theorem (Pierre 2012):

\[ U := (0, \infty) \times \mathbb{R} \]

Let \( a : (t, x) \in \bar{U} \rightarrow a(t, x) \in \mathbb{R}_+ \) be a continuous function with \( a(t, x) > 0 \) for \((t, x) \in U\), and let \( \mu \) be a probability measure with \( \int |x| \mu(dx) < \infty \). Then there exists at most one family of probability measures \((p(t, dx), t \geq 0)\) such that

- \( t \geq 0 \rightarrow p(t, dx) \) is weakly continuous
- \( p(0, dx) = \mu(dx) \) and

\[
\partial_t p - \partial_{xx}(ap) = 0 \quad \text{in} \ D'(U)
\]

(i.e., in the sense of Schwartz distributions on the open set \( U \)).
PART II: Small-time central limit theorems for semimartingales

- $(X_t)_{t \geq 0}$ semimartingale with $X_0 = x_0$ a.s.
- $f$ smooth
- We are interested in the limit

$$\lim_{t \to 0} \frac{f(X_t) - f(x_0)}{\sqrt{t}}$$

in distribution
- Main motivation: Complement well known small time LDPs by a CLT
- There are connections to digital option prices and implied volatility slopes
PART II: Overview

- CLT for SDE solutions
- CLT for continuous semimartingales
- $\lim_{t \to 0} \mathbb{P}[X_t > x_0] = ?$
- Functional CLT for continuous semimartingales
- Digital options, implied volatility slope
CLT for SDE solutions

Let $X$ be a weak solution of the SDE

$$dX_t^j = b_j(t, X_t) \, dt + \sum_{k=1}^{d} \sigma_{jk}(t, X_t) \, dB_t^k$$

where $B$ is a standard $d$-dimensional Brownian motion, $b$ is uniformly bounded in a neighborhood of $(0, x_0)$ and $\sigma$ is continuous in $(0, x_0)$. 
CLT for SDE solutions

Let $X$ be a weak solution of the SDE

$$dX^j_t = b_j(t, X_t) \, dt + \sum_{k=1}^{d} \sigma_{jk}(t, X_t) \, dB_t^k$$

where $B$ is a standard $d$-dimensional Brownian motion, $b$ is uniformly bounded in a neighborhood of $(0, x_0)$ and $\sigma$ is continuous in $(0, x_0)$.

**Thm:** For every $f : \mathbb{R}^m \to \mathbb{R}^n$ s.t. there exists an open neighborhood $U$ of $x_0$ with $f \in C^2(U, \mathbb{R}^n)$, there exists an a.s. positive stopping time $\tau$ such that

$$\frac{f(X_{t \wedge \tau}) - f(x_0)}{\sqrt{t}} \xrightarrow{d} N_f \text{ as } t \searrow 0,$$

where $N_f$ is a normal random vector with mean 0 and covariance matrix (with $L := \sigma(0, x_0)$)

$$V = (Df)(x_0) L (Df(x_0) L)^\top.$$
CLT for continuous semimartingales

- Assume: $X = x_0 + M + A$ continuous semimartingale
- $M$ continuous local martingale, $A$ locally finite variation,
- Conditions that ensure representation

$$X_{t^\wedge \tau}^j = x_0 + \int_0^{t^\wedge \tau} b_s^j \, ds + \sum_{k=1}^m \int_0^{t^\wedge \tau} \sigma_{s}^{jk} \, dB_s^k$$

- some boundedness assumptions
- Then the small-time CLT is valid
CLT for continuous semimartingales: General Assumptions

1. $X_0 = x_0$ a.s.;

2. there exists an a.s. positive stopping time $\tau_A$ such that a.s.

$$A^j_t = \int_0^t b^j_s \, ds, \quad t \in [0, \tau_A],$$

for an adapted process $b$;

3. there exists a random variable $C_b$, such that $|b^j_t| \leq C_b < \infty$
   for a.e. $t \in [0, \tau_A]$ a.s.;
CLT for continuous semimartingales: General Assumptions

1. $X_0 = x_0$ a.s.;
2. there exists an a.s. positive stopping time $\tau_A$ such that a.s.

$$A_t^j = \int_0^t b_s^j \, ds, \quad t \in [0, \tau_A],$$

for an adapted process $b$;
3. there exists a random variable $C_b$, such that $|b_t^j| \leq C_b < \infty$ for a.e. $t \in [0, \tau_A]$ a.s.;
4. there exists an a.s. positive stopping time $\tau_M$ such that the covariation is a.s.

$$\langle M^j, M^k \rangle_t = \int_0^t \sum_{l=1}^m \sigma_s^{jl} \sigma_s^{kl} \, ds, \quad t \in [0, \tau_M],$$

for a progressive process $\sigma$;
5. there exists a deterministic constant $C_\sigma < \infty$, such that $|\sigma_t^{jk}| \leq C_\sigma$ for a.e. $t \in [0, \tau_M]$ a.s., $j, k \in \{1, \ldots, m\}$;
6. as $t \searrow 0$, $\sigma_t \to L$ a.s., where $L$ is a deterministic matrix.
CLT for continuous semimartingales

**Thm:** For every $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ s.t. there exists an open neighborhood $U$ of $x_0$ with $f \in C^2(U, \mathbb{R}^n)$, there exists an a.s. positive stopping time $\tau$ such that

$$
\frac{f(X_{t\wedge\tau}) - f(x_0)}{\sqrt{t}} \xrightarrow{d} N_f \text{ as } t \downarrow 0,
$$

where $N_f$ is a normal random vector with mean 0 and covariance matrix

$$V = (Df)(x_0)L(Df(x_0)L)^\top.$$
n = 1 (otherwise: Cramér-Wold)

Choose open ball $B$ with $\bar{B} \subset U$

$\tau := \tau_{B^c} \wedge \tau_A \wedge \tau_M$

By Doob’s integral representation theorem:

$$X_{t \wedge \tau}^j = x_0 + \int_0^{t \wedge \tau} b_s^j \, ds + \sum_{k=1}^m \int_0^{t \wedge \tau} \sigma_s^{jk} \, dB^k_s$$
By Itô’s formula:

$$\frac{f(X_{t\land \tau}) - f(x_0)}{\sqrt{t}} = \frac{1}{\sqrt{t}} \int_0^{t\land \tau} (L_s f)(X_s) \, ds$$

$$+ \frac{1}{\sqrt{t}} \sum_{k,l=1}^m \int_0^{t\land \tau} \frac{\partial f}{\partial x_l}(X_s) \sigma_{s}^{lk} \, dB^k_s,$$

where

$$(L_s f)(u) := \frac{1}{2} \sum_{k,l=1}^m (\sigma_s \sigma_s^\top)_{kl} \frac{\partial^2 f}{\partial x_k \partial x_l}(u) + \sum_{k=1}^m b^k_s \frac{\partial f}{\partial x_k}(u)$$
CLT for continuous semimartingales: proof sketch (3/3)

- First term (use boundedness assumption on $b, \sigma$, choice of $B$):

$$\frac{1}{\sqrt{t}} \int_0^{t \wedge \tau} (\mathcal{L}_s f)(X_s) \, ds = O(\sqrt{t}), \quad t \to 0, \text{ a.s.}$$

- Second term:

$$\frac{1}{\sqrt{t}} \sum_{k, l=1}^m \int_0^{t \wedge \tau} \frac{\partial f}{\partial x_l}(X_s) \sigma_{s}^{lk} \, dB^k_s$$

Freeze integrand at $s = 0$, Cauchy-Schwarz, Itô’s isometry \(\to\) converges in law to a Gaussian r.v.

- Slutsky’s theorem
Heuristic CLT derivation from LDP in the elliptic case (1/2)

- Recall: classical CLT can be heuristically derived from Cramér’s theorem
- Suppose \( X \) satisfies
  \[
  X_t = x_0 + \int_0^t \sigma(X_s) \, dB_s, \quad t \geq 0
  \]
- Time-scaling: \( X_t \overset{d}{=} X_1^{(\delta)} \), where
  \[
  X_t^{(\delta)} = x_0 + \sqrt{\delta} \int_0^t \sigma(X_s^{(\delta)}) \, dB_s
  \]
- Small-noise LDP for diffusions (Freidlin-Wentzell) \( \implies X_t \)
satisfies LDP as \( t \to 0 \) with rate function
  \[
  I(x) = \frac{1}{2} \inf_{\substack{f \in H^1([0,1]): \ f(0) = x_0, \\ f(1) = x}} \int_0^1 \frac{\dot{f}(s)^2}{\sigma(f(s))^2} \, ds = \frac{1}{2} \left( \int_{x_0}^x \frac{du}{\sigma(u)} \right)^2
  \]
That is, for $\varepsilon > 0$ small and fixed, we have the asymptotics

$$\mathbb{P}(X_t \geq x_0 + \varepsilon) \approx \exp(-I(x_0 + \varepsilon)/t)$$

**Non-rigorous step:** replace $\varepsilon$ by $z\sqrt{t}$:

$$I(x_0 + z\sqrt{t}) = \frac{1}{2} I''(x_0) z^2 t + o(t),$$

and so

$$\mathbb{P}\left(\frac{X_t - x_0}{\sqrt{t}} \geq z\right) \approx \exp\left(-\frac{z^2}{2\sigma(x_0)^2} + o(1)\right).$$
The limiting probability $\lim_{t \to 0} \mathbb{P}[X_t > x_0]$

- **Corollary:** If $X$ satisfies our main assumptions, and the limit law is non-degenerate, then

$$\lim_{t \to 0} \mathbb{P}[X_t > x_0] = \lim_{t \to 0} \mathbb{P}\left[ \frac{X_t - x_0}{\sqrt{t}} > 0 \right] = \frac{1}{2}.$$

- **Example** (degenerate limit law): $X_t = B_t^2$ (BM squared)

$$\lim_{t \to 0} \mathbb{P}[X_t > x_0] = 1, \quad \frac{X_t}{\sqrt{t}} \xrightarrow{d} 0$$

- **Example** (drift not abs. continuous): $X_t = \Phi^{-1}(p)\sqrt{t} + B_t$

$$\lim_{t \to 0} \mathbb{P}[X_t > x_0] = p \in (0, 1)$$
The limiting probability \( \lim_{t \to 0} \mathbb{P}[X_t > x_0] \)

- **Example** (All values \( p \in [0, 1] \) can be realized by martingales):
  - \( R_t^\delta \) squared Bessel process of dimension \( \delta \geq 0 \)
    \[
dR_t^\delta = 2 \sqrt{R_t^\delta} \, dB_t + \delta \, dt, \quad R_0^\delta = 0.
\]
  - \( X_t^\delta := R_t^\delta - \delta t \) is a martingale
  - For \( \delta \in [0, \infty) \), \( \lim_{t \to 0} \mathbb{P}[X_t^\delta > x_0] \) ranges over \([0, 1)\).
  - Limit law: Dirac at zero; limit cdf not continuous at zero
The limiting probability $\lim_{t \to 0} \mathbb{P}[X_t > x_0]$: Refined asymptotics

- **SDE**

  $$dX_t = b(t, \cdot) \, dt + \sigma(t) \, dB_t$$

  $b$ bounded predictable process, $\sigma$ locally square integrable **deterministic** matrix function, smallest eigenvalue of $\sigma(\cdot)^\top \sigma(\cdot)$ uniformly bounded away from 0.

- **Thm:**

  $$g_1(t) \leq \mathbb{P}[X_t > x_0] \leq g_2(t)$$

- $g_1, g_2$ explicit functions with

  $$g_1(t) = \frac{1}{2} - \sqrt{\frac{\log 2}{2}} \|\sigma^{-1} b\|_{2,\infty} t^{1/2} + O(t),$$

  $$g_2(t) = \frac{1}{2} + \sqrt{\frac{\log 2}{2}} \|\sigma^{-1} b\|_{2,\infty} t^{1/2} + O(t).$$
**Thm:** Let $X$ satisfy the general assumption. Then for every $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$, such that there exists an open neighborhood $U$ of $x_0$ with $f \in C^2(U, \mathbb{R}^n)$, the processes

$$Y^{f,u} := \left( \frac{f(X_{u(t \wedge \tau)}) - f(x_0)}{\sqrt{u}} \right)_{t \in [0,T]}, \quad u \in (0,1),$$

converge in law to a Brownian motion with variance-covariance matrix

$$V = (Df)(x_0)L(Df(x_0)L)^\top$$

as $u \searrow 0$. 

**Functional CLT**
Functional CLT: proof idea

- Convergence of finite-dimensional distributions ($u \to 0$):
  \[
  (Y_{t_1}^{f,u}, \ldots, Y_{t_w}^{f,u}) \xrightarrow{d} (\tilde{B}_{t_1}, \ldots, \tilde{B}_{t_w}), \quad t_1, \ldots, t_w \in [0, T],
  \]
  $\tilde{B}$ Brownian motion with variance-covariance matrix $V$

- Proof: Analogous to finite-dimensional CLT

- Tightness condition: $u_l \in (0, 1)$ arbitrary with $u_l \to 0$

  \[
  \lim_{\delta \searrow 0} \lim_{l \to \infty} \mathbb{P}\left( \sup_{|s-t| \leq \delta} |Y_s^{f,u_l} - Y_t^{f,u_l}| > \varepsilon \right) = 0, \quad \varepsilon > 0.
  \]
CLT for Lévy processes (Doney, Maller 2002)

- Diffusion coefficient $\sigma$, Lévy measure $\nu$
- **Thm:** There are functions $f, g$ with

\[
\frac{X_t - f(t)}{g(t)} \to N(0, 1) \quad \text{in distribution}
\]

if and only if

\[
\lim_{x \to 0} \frac{1}{x} U(x) \frac{1}{x T(x)} = \infty
\]

- $T(x) := \nu((x, \infty)) + \nu((\infty, -x))$
- $U(x) := \sigma^2 + 2 \int_0^x y T(y) dy$

- $g(t) \sim c \sqrt{t}$ if and only if $\sigma \neq 0$, and then $c = \sigma$.
- Note: the typical pure jump processes of math. finance **do not** satisfy a CLT (Variance gamma, NIG, CGMY,...)
Application of CLTs: Digital options

- Underlying $S_t$, define $X_t = \log S_t$, let $\mathbb{P}$ be the pricing measure
- Price of a **digital call option** with log-strike $k$ (with $r = 0)$:
  \[ \mathbb{P}[X_T \geq k]. \]
- Small-time asymptotics for $k \neq x_0$: Varadhan, Rüschendorf, Woerner, Forde, Jacquier, Figueroa-López, Houdré, Marchal, ...
- **At the money** ($k = x_0$): If our assumptions hold, then
  \[ \lim_{T \to 0} \mathbb{P}[X_T \geq k] = \frac{1}{2}. \]
Implied volatility slope

- Implied volatility $\sigma_{\text{imp}} = \sigma_{\text{imp}}(K, T)$:
  
  $$C_{\text{BS}}(K, \sigma_{\text{imp}}, T) = C(K, T) := \mathbb{E}[(S_T - K)^+]$$

- Implied volatility slope:
  
  $$\partial_K \sigma_{\text{imp}} = -\frac{\partial_K C_{\text{BS}} - \partial_K C}{\partial_{\sigma} C_{\text{BS}}}$$

- Under mild assumptions:
  
  $$\partial_K \sigma_{\text{imp}} = -\frac{\partial_K C_{\text{BS}} + \mathbb{P}[S_T \geq K]}{\partial_{\sigma} C_{\text{BS}}}$$

- Well-known connection between implied vol slope and digitals
Implied volatility slope

More explicitly:

$$\partial_K \sigma_{\text{imp}} = \frac{\Phi(-\sigma_{\text{imp}} \sqrt{T}/2) - \mathbb{P}[S_T \geq K]}{K \sqrt{T} \cdot n(\sigma_{\text{imp}} \sqrt{T}/2)}$$

Under mild assumptions we have $\sigma_{\text{imp}} \sqrt{T} = o(1)$, $T \to 0$, and so

$$\partial_K \sigma_{\text{imp}} \sim \frac{\sqrt{2\pi}}{K \sqrt{T}} \left( \frac{1}{2} - \mathbb{P}[S_T \geq K] - \frac{\sigma_{\text{imp}} \sqrt{T}}{2 \sqrt{2\pi}} + O((\sigma_{\text{imp}} \sqrt{T})^3) \right)$$

ATM asymptotics of $\partial_K \sigma_{\text{imp}}$ depend on second order term of $\mathbb{P}[S_T \geq K]$
ATM digital calls: Small-time expansions for Lévy models

- $S_t = \exp(X_t)$, $X$ Lévy process with characteristic triplet $(\sigma, \nu, b)$

$$\log E[e^{sx}] = \frac{\sigma^2 s^2}{2} + bs + \int_{-\infty}^{\infty} \left(e^{sx} - 1 - sx1_{\{|x|\leq 1\}}\right)\nu(dx).$$

- **Thm**: Suppose that $\sigma > 0$ and that there is $s_0 \in (1, s_+)$ such that

  \[ \phi(s) := \int_{-\infty}^{\infty} \left(e^{sx} - 1 - sx1_{\{|x|\leq 1\}}\right)\nu(dx) \]

  is bounded for $\Re(s) = s_0$ fixed and $\Im(s) \in \mathbb{R}$. Then

  $$\mathbb{P}[X_T > x_0] = \frac{1}{2} + \frac{b}{\sigma \sqrt{2\pi}} \sqrt{T} + O(T \log(1/T)), \quad T \to 0.$$
Implied vol slope for Lévy models

- **Thm:** Assumptions as before. Then

\[
\lim_{T \to 0} \partial_K \sigma_{imp} = -(b/\sigma + \sigma/2).
\]

- **Examples:**
  - Merton jump diffusion
  - Kou’s double exponential jump diffusion

- **Application:**
  - get initial parameter values for calibration
  - qualitative influence of parameters on model behavior
Infinite activity Lévy processes

- Normal inverse Gaussian model:

\[ P[X_T > x_0] = \frac{1}{2} + \frac{\delta \beta}{\pi} T \log(1/ T) + o(T \log(1/ T)) \]

\[ \partial_K \sigma_{\text{imp}} \sim -\delta \sqrt{\frac{2}{\pi}} (\beta + \frac{1}{2}) T \log(1/ T) \]

- Variance gamma model:

\[ P[X_T > x_0] = \frac{1}{2} - \frac{\log 2 \nu \sigma^2}{2 \nu} T + o(T) \]

\[ \partial_K \sigma_{\text{imp}} \sim \text{const} \cdot \sqrt{T} \]

- Proof idea: Fourier representation, contour shift

- Question: criterion for \( \lim_{t \to 0} P[X_T > x_0] = \frac{1}{2} \)?
References