Local volatility asymptotics and small-time central limit theorems

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Talk consists of two parts:

- PART I: Local volatility approximations
- PART II: Small-time central limit theorems for semimartingales

PART I: Overview

- Local vol model: recreates marginals of a given diffusion (via call price surface)
- Question: Given market dynamics, what does the associated local vol surface look like?
- Asymptotics for small time, larke strike
- Applications: model risk; parametrization design
- Short end of the local vol surface: regularization

Local volatility

- Given call price surface

$$C = C(K, T)$$

- Reproduced by local volatility model

$$dS_t/S_t = \sigma_{\mathrm{loc}}(S_t, t)dW_t$$

- Dupire's formula (1994)

$$\sigma_{\mathrm{loc}}^2(K, T) = \frac{2\partial_T C}{K^2 \partial_{KK} C}$$

Heston Model

- Consider Call price surface $C_{\mathrm{Hes}}(K, T)$ generated by Heston:

$$dS_t = S_t\sqrt{V_t}dW_t, \qquad S_0 = 1,$$
$$dV_t = (a + bV_t)\,dt + c\sqrt{V_t}dZ_t, \qquad V_0 = v_0 > 0,$$

- Correlated Brownian motions

$$d\langle W, Z\rangle_t = \rho dt, \qquad \rho \in [-1, 1]$$

- Parameters

$$a \geq 0, b \leq 0, c > 0$$

Local vol in the Heston model (De Marco, Friz, SG 2013)

- Heston dynamics $\Longrightarrow$ Call prices $\Longrightarrow$ local vol surface
- Dupire's formula

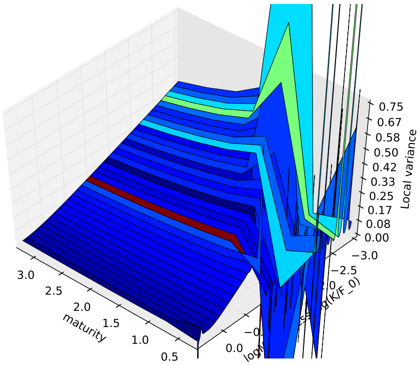
$$\sigma^2_{\mathrm{loc}}(K, T) = \frac{2\partial_T C_{\mathrm{Hes}}}{K^2 \partial_{KK} C_{\mathrm{Hes}}}$$

- New wing asymptotics ($k = \log K$)

$$\sigma^2_{\mathrm{loc}}(K, T) \sim \mathit{const} \times k, \qquad K \to \infty$$
$$\sigma^2_{\mathrm{loc}}(K, T) \sim \mathit{const} \times |k|, \qquad K \to 0$$

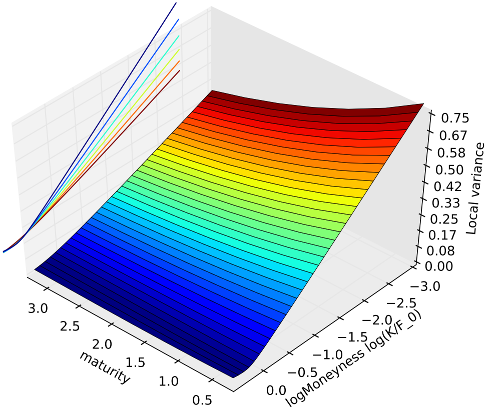
- Looks like Lee's moment formula for *implied vol*
- Similarly for the Stein-Stein model (Friz, De Marco 2012; large deviations)

Local vol in the Heston model



Local variance for Heston model computed with Dupire's formula. Call price derivatives computed via 1D integration of Heston characteristic function on a fixed integration contour.

Local vol in the Heston model



Local variance for Heston model computed with Dupire's formula. Adaptive contour with shift into saddle point. Note the linear increase.

Application 1: Design local vol parametrizations

- Example: Gatheral's SVI parametrization
- Popular parametrization of the **implied vol** surface

$$\sigma_{\mathrm{imp}}(K,T)^2 T \approx SVI(k; a, b, c, m, s)$$
$$k \mapsto a + b\left((-m+k)c + \sqrt{(-m+k)^2 + s}\right)$$

- Gatheral, Jacquier 2011: Heston, $T \to \infty \Rightarrow$ SVI
- Wings ($k \to \pm\infty$) compatible with Lee's formula
- Our asymptotic result motivates SVI parametrization also for **local vol** $\sigma_{\mathrm{loc}}(K,T)$

Application 2: Model risk

- Consider a path-dependent exotic
- SV = price under **stochastic vol model**
- LV = price under associated **local vol model**
- Note: local vol model recreates marginals of stoch vol model, but not the full law $\Longrightarrow$ in general $SV \neq LV$
- Similar price: low model risk (e.g., variance swap)
- Different price: high model risk (e.g, volatility swap)
- **Toxicity index** (Reghai 2011)

$$I = \frac{|SV - LV|}{|SV + LV|}$$

Application 2: Model risk

- How to calculate local vol of a stochastic vol model?
- We need $\sigma_{\text{loc}}(K, T)$ in particular for large/small K (Monte Carlo requires it)
- Dupire's formula + Fourier inversion: unstable for large/small K
- Conditioning:

$$\sigma^2_{\text{loc}}(K, T) = E[\sigma^2_{\text{stoch}}(T) | S_T = K]$$

  Difficult for $K \gg S_0$ (condition on unlikely events)
- $\Rightarrow$ Wing approximation useful for computation

Towards a general wing approximation of local vol

- Moment generating function ($X_T = \log S_T$):

$$M(s, T) := E[\exp(sX_T)], \qquad m(s, T) := \log M(s, T)$$

- Dupire's formula + Fourier inversion

$$\sigma^2_{\mathrm{loc}}(K, T) = \frac{2\partial_T C}{K^2 \partial_{KK} C} = \frac{2\int_{-i\infty}^{i\infty} \frac{\partial_T m(s,T)}{s(s-1)} e^{-ks} M(s, T) ds}{\int_{-i\infty}^{i\infty} e^{-ks} M(s, T) ds}$$

- Saddle point method: Leading terms are integrands evaluated at saddle point $\longrightarrow$ cancellation

General wing formula for local vol (De Marco, Friz, SG 2013)

- log moment generating function ($X_T = \log S_T$)

$$m(s, T) = \log E[\exp(sX_T)]$$

- saddle point $\hat{s}(k, T)$

$$\frac{\partial}{\partial s} m(s, T)\Big|_{s=\hat{s}} = k$$

- "Lee type" wing formula for $k \to \infty$:

$$\boxed{\sigma^2_{\mathrm{loc}}(K, T) \approx \frac{2\frac{\partial}{\partial T} m(s, T)}{s(s-1)}\Bigg|_{s=\hat{s}(k,T)}}$$

Two ways to use the formula

- As it is (numerically very accurate, but not quite explicit):

$$\sigma^2_{\mathrm{loc}}(K,T) \approx \left. \frac{2\frac{\partial}{\partial T} m(s,T)}{s(s-1)} \right|_{s=\hat{s}(k,T)}$$

- Use asymptotics of saddle point $\hat{s}(k,T)$ and mgf $\Longrightarrow$ explicit formula (model-dependent)
- E.g., *const* $\times$ k for Heston. Explicit, but model-dependent and less accurate.

Heston model: Numerical example (left wing)

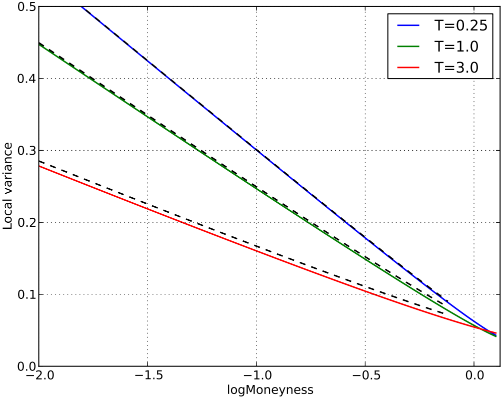


Abbildung: Local variance $\sigma^2_{\mathrm{loc}}(k, T)$ and our approximation in the Heston model.

Heston model: Numerical example (right wing)

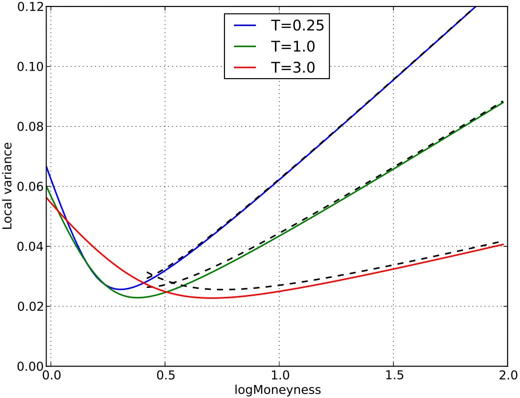


Abbildung: Local variance $\sigma^2_{\mathrm{loc}}(k, T)$ and our approximation in the Heston model.

Heston model: Accuracy by strike and maturity

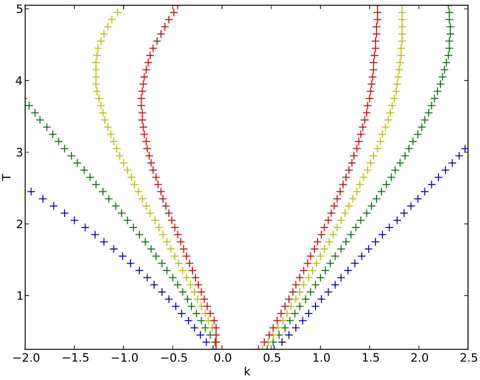


Abbildung: Boundaries of the region where the relative error of our approximation is less than 2% (blue), 3% (green), 4% (yellow), and 5% (red).

Heston model: Implied volatility, $T = 0.25$

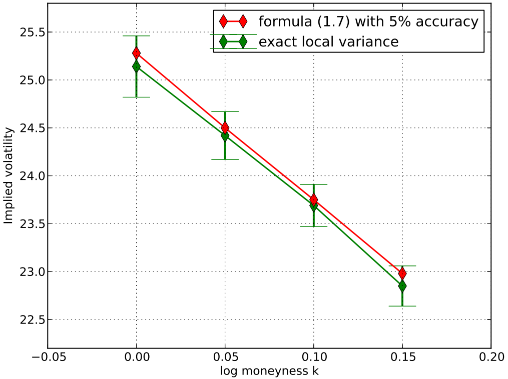


Abbildung: Green: Local vol computed by Dupire's formula. Red: Use our approximation, as soon as its accuracy is over 5%.

Heston model: Implied volatility, $T = 5$

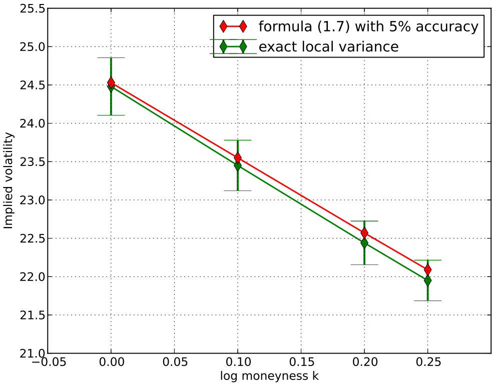


Abbildung: Green: Local vol computed by Dupire's formula. Red: Use our approximation, as soon as its accuracy is over 5%.

Heston model: rigorous proof

- Finding saddle point + local expansion of integrands fairly routine
- Problem: Verify concentration
- Needs some insight into behaviour of integrand away from saddle point
- Show exponential decay of integrands by ODE comparison (Riccati ODEs, similar to Friz, SG, Gulisashvili, Sturm, Quantitative Finance 2011)
- [Korenblum's ratio Tauberian theorem?]

Using Dupire's formula for models with jumps

- Variance gamma model: Call price not C^2 w.r.t. strike (but works for T large)
- Jumps $\Longrightarrow$ Blowup of local vol as $T \to 0$, hence local vol model may be ill-defined.
  Dupire's formula: $\sigma^2_{\mathrm{loc}}(K, T) = 2\partial_T C / (K^2 \partial_{KK} C)$.
  Call PIDE:

$$\partial_T C = \frac{1}{2} K^2 \sigma^2 \partial_{KK} C \\ + \int_{-\infty}^{\infty} \nu(dz)(C(Ke^{-z}, T) - C(K, T) - K(e^z - 1)\partial_K C)$$

- Even if Dupire's formula is well-defined, the local vol model may not match the marginals of the jump process.

Jumps: blowup if local vol as $T \to 0$

- Our saddle point formula works well

$$\sigma_{\mathrm{loc}}^2(K,T) \approx \left. \frac{2\frac{\partial}{\partial T} m(s,T)}{s(s-1)} \right|_{s=\hat{s}(k,T)}$$

- Examples for off-the-money blowup ($K \neq S_0$ fixed):

$$\sigma_{\mathrm{loc}}^2(K,T) \approx 1/T \quad \text{(Merton jump diffusion)}$$
$$\sigma_{\mathrm{loc}}^2(K,T) \approx 1/\sqrt{T} \quad \text{(Kou's diffusion)}$$
$$\sigma_{\mathrm{loc}}^2(K,T) \approx 1/T \quad \text{(Normal inverse Gaussian)}$$

Regularization of local vol

- A realistic market model should have jumps (especially for pricing short-maturity products)
- But then local vol model is ill-defined (blowup of σ_{loc} for $T \to 0$)
- Question: is there a truncation of the local vol surface that always gives a well-defined local vol model?

Regularization of local vol

Theorem (Friz, SG, Yor 2013): Assume that (S_t) is a martingale (possibly with jumps) with associated smooth call price surface C,

$$\forall K, T \geq 0 : C(K, T) = \mathbb{E}[(S_T - K)^+],$$

such that $\partial_T C > 0$ and $\partial_{KK} C > 0$, i.e. (strict) absence of calendar and butterfly spreads. Define ε-shifted local volatility

$$\sigma_\varepsilon^2(K, T) = \frac{2\partial_T C(K, T + \varepsilon)}{K^2 \partial_{KK} C(K, T + \varepsilon)}.$$

Then $dS^\varepsilon / S^\varepsilon = \sigma_\varepsilon(S^\varepsilon, t) dW$, started at randomized spot S_0^ε with distribution

$$\mathbb{P}[S_0^\varepsilon \in dK]/dK = \partial_{KK} C(K, \varepsilon),$$

admits a unique, non-explosive strong SDE solution such that

$$\forall K, T \geq 0 : \mathbb{E}[(S_T^\varepsilon - K)^+] \to C(K, T) \quad \text{as} \quad \varepsilon \to 0.$$

Regularization of local vol: Proof

- Let $q^\varepsilon(dS, T)$ be the law of S_T^ε, and $p^\varepsilon(S, T)$ be the density of $S_{T+\varepsilon}$
- Calculate

$$\begin{aligned}\mathbb{E}[(S_T^\varepsilon - K)^+] &= \int (S-K)^+ \, q^\varepsilon(dS, T) \\ &\stackrel{!}{=} \int (S-K)^+ \, p^\varepsilon(S, T) dS \\ &= \mathbb{E}[(S_{T+\varepsilon} - K)^+] \\ &= C(K, T+\varepsilon).\end{aligned}$$

  Then let $\varepsilon \to 0$.
- Need to show $S_T^\varepsilon \stackrel{d}{=} S_{T+\varepsilon}$

Regularization of local vol: Proof

- Define

$$a^{\varepsilon}(K,T) := \frac{\partial_T C(K, T+\varepsilon)}{p^{\varepsilon}(K,T)}$$

- p^{ε} satisfies the Fokker-Plack equation

$$\partial_{KK}(a^{\varepsilon} p^{\varepsilon}) = \partial_T p^{\varepsilon}$$

- But q^{ε} is also a (weak) solution, in the sense that

$$\int \varphi(S) q^{\varepsilon}(dS, t) = \int \varphi(S) q^{\varepsilon}(dS, 0) + \int_0^t \int a^{\varepsilon}(S,s)\varphi''(S) q^{\varepsilon}(dS, s)$$

for any smooth φ with compact support. (Since $a^{\varepsilon}(S,t)\partial_{SS}$ is the generator of S^{ε}.)

Regularization of local vol: Proof

- So our result is a corollary of the following uniqueness theorem (Pierre 2012):
- $U := (0, \infty) \times \mathbb{R}$
- Let $a : (t, x) \in \bar{U} \to a(t, x) \in \mathbb{R}_+$ be a continuous function with $a(t, x) > 0$ for $(t, x) \in U$, and let μ be a probability measure with $\int |x|\mu(dx) < \infty$. Then there exists at most one family of probability measures $(p(t, dx), t \geq 0)$ such that
  - $t \geq 0 \to p(t, dx)$ is weakly continuous
  - $p(0, dx) = \mu(dx)$ and

$$\partial_t p - \partial_{xx}(ap) = 0 \quad \text{in } \mathcal{D}'(U)$$

(i.e., in the sense of Schwartz distributions on the open set U.)

PART II: Small-time central limit theorems for semimartingales

- $(X_t)_{t \geq 0}$ semimartingale with $X_0 = x_0$ a.s.
- f smooth
- We are interested in the limit

$$\lim_{t \to 0} \frac{f(X_t) - f(x_0)}{\sqrt{t}}$$

  in distribution
- Main motivation: Complement well known small time LDPs by a CLT
- There are connections to digital option prices and implied volatility slopes

PART II: Overview

- CLT for SDE solutions
- CLT for continuous semimartingales
- $\lim_{t \to 0} \mathbb{P}[X_t > x_0] = ?$
- Functional CLT for continuous semimartingales
- Digital options, implied volatility slope

CLT for SDE solutions

- Let X be a weak solution of the SDE

$$\mathrm{d}X_t^j = b_j(t, X_t)\,\mathrm{d}t + \sum_{k=1}^{d} \sigma_{jk}(t, X_t)\,\mathrm{d}B_t^k$$

  where B is a standard d-dimensional Brownian motion, b is uniformly bounded in a neighborhood of $(0, x_0)$ and σ is continuous in $(0, x_0)$.

CLT for SDE solutions

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$$\mathrm{d}X_t^j = b_j(t, X_t)\,\mathrm{d}t + \sum_{k=1}^{d} \sigma_{jk}(t, X_t)\,\mathrm{d}B_t^k$$

  where B is a standard d-dimensional Brownian motion, b is uniformly bounded in a neighborhood of $(0, x_0)$ and σ is continuous in $(0, x_0)$.
- **Thm:** For every $f : \mathbb{R}^m \to \mathbb{R}^n$ s.t. there exists an open neighborhood U of x_0 with $f \in C^2(U, \mathbb{R}^n)$, there exists an a.s. positive stopping time τ such that

$$\frac{f(X_{t\wedge\tau}) - f(x_0)}{\sqrt{t}} \xrightarrow{d} N_f \text{ as } t \searrow 0,$$

  where N_f is a normal random vector with mean 0 and covariance matrix (with $L := \sigma(0, x_0)$)

$$V = (Df)(x_0)L\big(Df(x_0)L\big)^\top.$$

CLT for continuous semimartingales

- Assume: $X = x_0 + M + A$ continuous semimartingale
- M continuous local martingale, A locally finite variation,
- Conditions that ensure representation

$$X^j_{t\wedge\tau} = x_0 + \int_0^{t\wedge\tau} b^j_s \,\mathrm{d}s + \sum_{k=1}^{m} \int_0^{t\wedge\tau} \sigma^{jk}_s \,\mathrm{d}B^k_s$$

- some boundedness assumptions
- Then the small-time CLT is valid

CLT for continuous semimartingales: General Assumptions

1. $X_0 = x_0$ a.s.;
2. there exists an a.s. positive stopping time τ_A such that a.s.

$$A_t^j = \int_0^t b_s^j \, \mathrm{d}s, \quad t \in [0, \tau_A],$$

   for an adapted process b;
3. there exists a random variable C_b, such that $|b_t^j| \leq C_b < \infty$ for a.e. $t \in [0, \tau_A]$ a.s.;

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4. there exists an a.s. positive stopping time τ_M such that the covariation is a.s.

$$\langle M^j, M^k \rangle_t = \int_0^t \sum_{l=1}^{m} \sigma_s^{jl} \sigma_s^{kl} \, \mathrm{d}s, \quad t \in [0, \tau_M],$$

   for a progressive process σ;
5. there exists a deterministic constant $C_\sigma < \infty$, such that $|\sigma_t^{jk}| \leq C_\sigma$ for a.e. $t \in [0, \tau_M]$ a.s., $j, k \in \{1, \dots, m\}$;
6. as $t \searrow 0$, $\sigma_t \to L$ a.s., where L is a deterministic matrix

CLT for continuous semimartingales

Thm: For every $f : \mathbb{R}^m \to \mathbb{R}^n$ s.t. there exists an open neighborhood U of x_0 with $f \in C^2(U, \mathbb{R}^n)$, there exists an a.s. positive stopping time τ such that

$$\frac{f(X_{t\wedge\tau}) - f(x_0)}{\sqrt{t}} \xrightarrow{d} N_f \text{ as } t \searrow 0,$$

where N_f is a normal random vector with mean 0 and covariance matrix

$$V = (Df)(x_0)L\big(Df(x_0)L\big)^\top.$$

CLT for continuous semimartingales: proof sketch (1/3)

- $n = 1$ (otherwise: Cramér-Wold)
- Choose open ball $\mathbf{B}$ with $\bar{\mathbf{B}} \subset U$
- $\tau := \tau_{\bar{\mathbf{B}}^c} \wedge \tau_A \wedge \tau_M$
- By Doob's integral representation theorem:

$$X^j_{t\wedge\tau} = x_0 + \int_0^{t\wedge\tau} b^j_s \,\mathrm{d}s + \sum_{k=1}^{m} \int_0^{t\wedge\tau} \sigma^{jk}_s \,\mathrm{d}B^k_s$$

CLT for continuous semimartingales: proof sketch (2/3)

By Itô's formula:

$$\frac{f(X_{t\wedge\tau}) - f(x_0)}{\sqrt{t}} = \frac{1}{\sqrt{t}} \int_0^{t\wedge\tau} (\mathcal{L}_s f)(X_s)\,\mathrm{d}s + \frac{1}{\sqrt{t}} \sum_{k,l=1}^{m} \int_0^{t\wedge\tau} \frac{\partial f}{\partial x_l}(X_s)\,\sigma_s^{lk}\,\mathrm{d}B_s^k,$$

where

$$(\mathcal{L}_s f)(u) := \frac{1}{2} \sum_{k,l=1}^{m} (\sigma_s \sigma_s^\top)_{kl} \frac{\partial^2 f}{\partial x_k \partial x_l}(u) + \sum_{k=1}^{m} b_s^k \frac{\partial f}{\partial x_k}(u)$$

CLT for continuous semimartingales: proof sketch (3/3)

- First term (use boundedness assumption on b, σ, choice of $\mathbf{B}$):

$$\frac{1}{\sqrt{t}} \int_0^{t\wedge\tau} (\mathcal{L}_s f)(X_s)\,\mathrm{d}s = O(\sqrt{t}), \quad t \to 0, \text{ a.s.}$$

- Second term:

$$\frac{1}{\sqrt{t}} \sum_{k,l=1}^{m} \int_0^{t\wedge\tau} \frac{\partial f}{\partial x_l}(X_s)\,\sigma_s^{lk}\,\mathrm{d}B_s^k$$

  Freeze integrand at $s = 0$, Cauchy-Schwarz, Itô's isometry
  $\longrightarrow$ converges in law to a Gaussian r.v.

- Slutsky's theorem $\square$

Heuristic CLT derivation from LDP in the elliptic case (1/2)

- Recall: classical CLT can be heuristically derived from Cramér's theorem
- Suppose X satisfies

$$X_t = x_0 + \int_0^t \sigma(X_s)\,\mathrm{d}B_s, \quad t \geq 0$$

- Time-scaling: $X_t \overset{d}{=} X_1^{(\delta)}$, where

$$X_t^{(\delta)} = x_0 + \sqrt{\delta} \int_0^t \sigma(X_s^{(\delta)})\,\mathrm{d}B_s$$

- Small-noise LDP for diffusions (Freidlin-Wentzell) $\Longrightarrow X_t$ satisfies LDP as $t \to 0$ with rate function

$$I(x) = \frac{1}{2} \inf_{\substack{f \in H^1([0,1]): \\ f(0)=x_0, \\ f(1)=x}} \int_0^1 \frac{\dot{f}(s)^2}{\sigma(f(s))^2}\,\mathrm{d}s = \frac{1}{2}\left(\int_{x_0}^x \frac{\mathrm{d}u}{\sigma(u)}\right)^2$$

Heuristic CLT derivation from LDP in the elliptic case (2/2)

- That is, for $\varepsilon > 0$ small and fixed, we have the asymptotics

$$\mathbb{P}(X_t \geq x_0 + \varepsilon) \simeq \exp(-I(x_0 + \varepsilon)/t)$$

- **Non-rigorous step:** replace ε by $z\sqrt{t}$:

$$I(x_0 + z\sqrt{t}) = \frac{1}{2} I''(x_0) z^2 t + o(t),$$

and so

$$\mathbb{P}\left(\frac{X_t - x_0}{\sqrt{t}} \geq z\right) \approx \exp\left(-\frac{z^2}{2\sigma(x_0)^2} + o(1)\right).$$

The limiting probability $\lim_{t\to 0}\mathbb{P}[X_t > x_0]$

- **Corollary:** If X satisfies our main assumptions, and the limit law is non-degenerate, then

$$\lim_{t\to 0}\mathbb{P}[X_t > x_0] = \lim_{t\to 0}\mathbb{P}[\frac{X_t - x_0}{\sqrt{t}} > 0] = \frac{1}{2}.$$

- **Example** (degenerate limit law): $X_t = B_t^2$ (BM squared)

$$\lim_{t\to 0}\mathbb{P}[X_t > x_0] = 1, \quad \frac{X_t}{\sqrt{t}} \xrightarrow{d} 0$$

- **Example** (drift not abs. continuous): $X_t = \Phi^{-1}(p)\sqrt{t} + B_t$

$$\lim_{t\to 0}\mathbb{P}[X_t > x_0] = p \in (0,1)$$

The limiting probability $\lim_{t\to 0}\mathbb{P}[X_t > x_0]$

- **Example** (All values $p \in [0,1)$ can be realized by martingales):
- R_t^δ squared Bessel process of dimension $\delta \geq 0$

$$\mathrm{d}R_t^\delta = 2\sqrt{R_t^\delta}\,\mathrm{d}B_t + \delta\,\mathrm{d}t, \quad R_0^\delta = 0.$$

- $X_t^\delta := R_t^\delta - \delta t$ is a martingale
- For $\delta \in [0,\infty)$, $\lim_{t\to 0}\mathbb{P}[X_t^\delta > x_0]$ ranges over $[0,1)$.
- Limit law: Dirac at zero; limit cdf not continuous at zero

The limiting probability $\lim_{t\to 0}\mathbb{P}[X_t > x_0]$: Refined asymptotics

- SDE
$$\mathrm{d}X_t = b(t,\cdot)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B_t$$
b bounded predictable process, σ locally square integrable **deterministic** matrix function, smallest eigenvalue of $\sigma(\cdot)^\top\sigma(\cdot)$ uniformly bounded away from 0.
- **Thm:**
$$g_1(t) \le \mathbb{P}[X_t > x_0] \le g_2(t)$$
- g_1, g_2 explicit functions with

$$g_1(t) = \frac{1}{2} - \sqrt{\frac{\log 2}{2}}\|\sigma^{-1}b\|_{2,\infty}\, t^{1/2} + O(t),$$
$$g_2(t) = \frac{1}{2} + \sqrt{\frac{\log 2}{2}}\|\sigma^{-1}b\|_{2,\infty}\, t^{1/2} + O(t).$$

Functional CLT

Thm: Let X satisfy the general assumption. Then for every $f : \mathbb{R}^m \to \mathbb{R}^n$, such that there exists an open neighborhood U of x_0 with $f \in C^2(U, \mathbb{R}^n)$, the processes

$$Y^{f,u} := \left(\frac{f(X_{u(t\wedge\tau)}) - f(x_0)}{\sqrt{u}} \right)_{t\in[0,T]}, \quad u \in (0,1),$$

converge in law to a Brownian motion with variance-covariance matrix

$$V = (Df)(x_0)L(Df(x_0)L)^\top$$

as $u \searrow 0$.

Functional CLT: proof idea

- Convergence of finite-dimensional distributions ($u \to 0$):

$$(Y_{t_1}^{f,u}, \dots, Y_{t_w}^{f,u}) \xrightarrow{d} (\widetilde{B}_{t_1}, \dots, \widetilde{B}_{t_w}), \quad t_1, \dots, t_w \in [0, T],$$

  $\widetilde{B}$ Brownian motion with variance-covariance matrix V

- Proof: Analogous to finite-dimensional CLT
- Tightness condition: $u_l \in (0,1)$ arbitrary with $u_l \to 0$

$$\lim_{\delta \searrow 0} \overline{\lim_{l \to \infty}} \, \mathbb{P}\Big(\sup_{|s-t| \leq \delta} |Y_s^{f,u_l} - Y_t^{f,u_l}| > \varepsilon \Big) = 0, \quad \varepsilon > 0.$$

CLT for Lévy processes (Doney, Maller 2002)

- Diffusion coefficient σ, Lévy measure ν
- **Thm:** There are functions f, g with

$$\frac{X_t - f(t)}{g(t)} \to N(0,1) \quad \text{in distribution}$$

if and only if

$$\lim_{x \to 0} \frac{\frac{1}{x} U(x)}{x T(x)} = \infty$$

- $T(x) := \nu((x,\infty)) + \nu((-\infty,-x))$,
  $U(x) := \sigma^2 + 2\int_0^x yT(y)dy$
- $g(t) \sim c\sqrt{t}$ if and only if $\sigma \neq 0$, and then $c = \sigma$.
- Note: the typical pure jump processes of math. finance **do not** satisfy a CLT (Variance gamma, NIG, CGMY,...)

Application of CLTs: Digital options

- Underlying S_t, define $X_t = \log S_t$, let $\mathbb{P}$ be the pricing measure
- Price of a **digital call option** with log-strike k (with $r = 0$):

$$\mathbb{P}[X_T \geq k].$$

- Small-time asymptotics for $k \neq x_0$: Varadhan, Rüschendorf, Woerner, Forde, Jacquier, Figueroa-López, Houdré, Marchal, ...
- **At the money** ($k = x_0$): If our assumptions hold, then

$$\lim_{T \to 0} \mathbb{P}[X_T \geq k] = \frac{1}{2}.$$

Implied volatility slope

- Implied volatility $\sigma_{\text{imp}} = \sigma_{\text{imp}}(K, T)$:

$$C_{\text{BS}}(K, \sigma_{\text{imp}}, T) = C(K, T) := \mathbb{E}[(S_T - K)^+]$$

- Implied volatility slope:

$$\partial_K \sigma_{\text{imp}} = -\frac{\partial_K C_{\text{BS}} - \partial_K C}{\partial_\sigma C_{\text{BS}}}$$

- Under mild assumptions:

$$\partial_K \sigma_{\text{imp}} = -\frac{\partial_K C_{\text{BS}} + \mathbb{P}[S_T \geq K]}{\partial_\sigma C_{\text{BS}}}$$

- Well-known connection between implied vol slope and digitals

Implied volatility slope

- More explicitly:

$$\partial_K \sigma_{\mathrm{imp}} = \frac{\Phi(-\sigma_{\mathrm{imp}}\sqrt{T}/2) - \mathbb{P}[S_T \geq K]}{K\sqrt{T}\, n(\sigma_{\mathrm{imp}}\sqrt{T}/2)}$$

- Under mild assumptions we have $\sigma_{\mathrm{imp}}\sqrt{T} = o(1),\ T \to 0$, and so

$$\partial_K \sigma_{\mathrm{imp}} \sim \frac{\sqrt{2\pi}}{K\sqrt{T}} \left(\frac{1}{2} - \mathbb{P}[S_T \geq K] - \frac{\sigma_{\mathrm{imp}}\sqrt{T}}{2\sqrt{2\pi}} + O((\sigma_{\mathrm{imp}}\sqrt{T})^3) \right)$$

- **ATM asymptotics** of $\partial_K \sigma_{\mathrm{imp}}$ depend on **second order term** of $\mathbb{P}[S_T \geq K]$

ATM digital calls: Small-time expansions for Lévy models

- $S_t = \exp(X_t)$, X Lévy process with characteristic triplet (σ, ν, b)

$$\log E[e^{sX_1}] = \frac{\sigma^2 s^2}{2} + bs + \int_{-\infty}^{\infty} (e^{sx} - 1 - sx\mathbf{1}_{\{|x|\leq 1\}})\nu(dx).$$

- **Thm:** Suppose that $\sigma > 0$ and that there is $s_0 \in (1, s_+)$ such that

$$\phi(s) := \int_{-\infty}^{\infty} (e^{sx} - 1 - sx\mathbf{1}_{\{|x|\leq 1\}})\nu(dx)$$

is bounded for $\mathrm{Re}(s) = s_0$ fixed and $\mathrm{Im}(s) \in \mathbb{R}$. Then

$$\mathbb{P}[X_T > x_0] = \frac{1}{2} + \frac{b}{\sigma\sqrt{2\pi}}\sqrt{T} + O(T \log(1/T)), \quad T \to 0.$$

Implied vol slope for Lévy models

- **Thm:** Assumptions as before. Then

$$\lim_{T \to 0} \partial_K \sigma_{imp} = -(b/\sigma + \sigma/2).$$

- Examples:
  - Merton jump diffusion
  - Kou's double exponential jump diffusion
- Application:
  - get initial parameter values for calibration
  - qualitative influence of parameters on model behavior

Infinite activity Lévy processes

- Normal inverse Gaussian model:

$$\mathbb{P}[X_T > x_0] = \frac{1}{2} + \frac{\delta\beta}{\pi} T \log(1/T) + o(T \log(1/T))$$
$$\partial_K \sigma_{imp} \sim -\delta \sqrt{\tfrac{2}{\pi}} (\beta + \tfrac{1}{2}) T \log(1/T)$$

- Variance gamma model:

$$\mathbb{P}[X_T > x_0] = \frac{1}{2} - \frac{\log 2\nu\sigma^2}{2\nu} T + o(T)$$
$$\partial_K \sigma_{imp} \sim const \cdot \sqrt{T}$$

- Proof idea: Fourier representation, contour shift
- Question: criterion for $\lim_{t\to 0} \mathbb{P}[X_T > x_0] = \frac{1}{2}$?

References

- S. De Marco, P. Friz, S. Gerhold, *Rational shapes of local volatility,* Risk, February 2013.
- R. A. Doney and R. A. Maller, *Stability and attraction to normality for Lévy processes at zero and at infinity,* J. Theoret. Probab. 15 (2002).
- P. Friz, S. Gerhold, M. Yor, *How to make Dupire's local volatility work with jumps,* submitted, 2013.
- S. Gerhold, M. Kleinert, P. Porkert, M. Shkolnikov, *Small time central limit theorems for semimartingales with applications,* submitted, 2012.
- S. Gerhold, M. Shkolnikov, *Short maturity digitals and implied volatility slopes,* working paper, 2013.