Small time central limit theorems for semimartingales with applications

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13 Feb 2013

Overview

- $(X_t)_{t\geq 0}$ semimartingale with $X_0 = x_0$ a.s.
- f smooth
- We are interested in the limit

$$\lim_{t\to 0} \frac{f(X_t) - f(x_0)}{\sqrt{t}}$$

  in distribution
- Main motivation: Complement well known small time LDPs by a CLT
- There are connections to digital option prices and implied volatility slopes

Overview

- CLT for SDE solutions
- CLT for continuous semimartingales
- $\lim_{t \to 0} \mathbb{P}[X_t > x_0] = ?$
- Functional CLT for continuous semimartingales
- Jumps
- Digital options, implied volatility slope

CLT for SDE solutions

- Let X be a weak solution of the SDE

$$\mathrm{d}X_t^j = b_j(t, X_t)\,\mathrm{d}t + \sum_{k=1}^{d} \sigma_{jk}(t, X_t)\,\mathrm{d}B_t^k$$

where B is a standard d-dimensional Brownian motion, b is uniformly bounded in a neighborhood of $(0, x_0)$ and σ is continuous in $(0, x_0)$.

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  where B is a standard d-dimensional Brownian motion, b is uniformly bounded in a neighborhood of $(0, x_0)$ and σ is continuous in $(0, x_0)$.
- **Thm:** For every $f : \mathbb{R}^m \to \mathbb{R}^n$ s.t. there exists an open neighborhood U of x_0 with $f \in C^2(U, \mathbb{R}^n)$, there exists an a.s. positive stopping time τ such that

$$\frac{f(X_{t\wedge\tau}) - f(x_0)}{\sqrt{t}} \xrightarrow{d} N_f \text{ as } t \searrow 0,$$

  where N_f is a normal random vector with mean 0 and covariance matrix (with $L := \sigma(0, x_0)$)

$$V = (Df)(x_0)L\big(Df(x_0)L\big)^\top.$$

CLT for continuous semimartingales

- Assume: $X = x_0 + M + A$ continuous semimartingale
- M continuous local martingale, A locally finite variation,
- Conditions that ensure representation

$$X^j_{t\wedge\tau} = x_0 + \int_0^{t\wedge\tau} b^j_s \,\mathrm{d}s + \sum_{k=1}^{m} \int_0^{t\wedge\tau} \sigma^{jk}_s \,\mathrm{d}B^k_s$$

- some boundedness assumptions
- Then the small-time CLT is valid

CLT for continuous semimartingales: General Assumptions

1. $X_0 = x_0$ a.s.;
2. there exists an a.s. positive stopping time τ_A such that a.s.

$$A_t^j = \int_0^t b_s^j \, \mathrm{d}s, \quad t \in [0, \tau_A],$$

   for an adapted process b;
3. there exists a random variable C_b, such that $|b_t^j| \leq C_b < \infty$ for a.e. $t \in [0, \tau_A]$ a.s.;

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4. there exists an a.s. positive stopping time τ_M such that the covariation is a.s.

$$\langle M^j, M^k \rangle_t = \int_0^t \sum_{l=1}^m \sigma_s^{jl} \sigma_s^{kl} \, \mathrm{d}s, \quad t \in [0, \tau_M],$$

   for a progressive process σ;
5. there exists a deterministic constant $C_\sigma < \infty$, such that $|\sigma_t^{jk}| \leq C_\sigma$ for a.e. $t \in [0, \tau_M]$ a.s., $j, k \in \{1, \ldots, m\}$;
6. as $t \searrow 0$, $\sigma_t \to L$ a.s., where L is a deterministic matrix

CLT for continuous semimartingales

Thm: For every $f : \mathbb{R}^m \to \mathbb{R}^n$ s.t. there exists an open neighborhood U of x_0 with $f \in C^2(U, \mathbb{R}^n)$, there exists an a.s. positive stopping time τ such that

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CLT for continuous semimartingales: proof sketch (1/3)

- $n = 1$ (otherwise: Cramér-Wold)
- Choose open ball $\mathbf{B}$ with $\bar{\mathbf{B}} \subset U$
- $\tau := \tau_{\bar{\mathbf{B}}^c} \wedge \tau_A \wedge \tau_M$
- By Doob's integral representation theorem:

$$X^j_{t\wedge\tau} = x_0 + \int_0^{t\wedge\tau} b^j_s \,\mathrm{d}s + \sum_{k=1}^{m} \int_0^{t\wedge\tau} \sigma^{jk}_s \,\mathrm{d}B^k_s$$

CLT for continuous semimartingales: proof sketch (2/3)

By Itô's formula:

$$\frac{f(X_{t\wedge\tau}) - f(x_0)}{\sqrt{t}} = \frac{1}{\sqrt{t}} \int_0^{t\wedge\tau} (\mathcal{L}_s f)(X_s)\,\mathrm{d}s + \frac{1}{\sqrt{t}} \sum_{k,l=1}^{m} \int_0^{t\wedge\tau} \frac{\partial f}{\partial x_l}(X_s)\,\sigma_s^{lk}\,\mathrm{d}B_s^k,$$

where

$$(\mathcal{L}_s f)(u) := \frac{1}{2} \sum_{k,l=1}^{m} (\sigma_s \sigma_s^\top)_{kl} \frac{\partial^2 f}{\partial x_k \partial x_l}(u) + \sum_{k=1}^{m} b_s^k \frac{\partial f}{\partial x_k}(u)$$

CLT for continuous semimartingales: proof sketch (3/3)

- First term (use boundedness assumption on b, σ, choice of $\mathbf{B}$):

$$\frac{1}{\sqrt{t}} \int_0^{t\wedge\tau} (\mathcal{L}_s f)(X_s)\,\mathrm{d}s = O(\sqrt{t}), \quad t \to 0, \text{ a.s.}$$

- Second term:

$$\frac{1}{\sqrt{t}} \sum_{k,l=1}^{m} \int_0^{t\wedge\tau} \frac{\partial f}{\partial x_l}(X_s)\,\sigma_s^{lk}\,\mathrm{d}B_s^k$$

  Freeze integrand at $s = 0$, Cauchy-Schwarz, Itô's isometry
  $\longrightarrow$ converges in law to a Gaussian r.v.

- Slutsky's theorem $\square$

Heuristic CLT derivation from LDP (1/2)

- Recall: classical CLT can be heuristically derived from Cramér's theorem
- Suppose X satisfies

$$X_t = x_0 + \int_0^t \sigma(X_s)\,\mathrm{d}B_s, \quad t \geq 0$$

- Time-scaling: $X_t \stackrel{d}{=} X_1^{(\delta)}$, where

$$X_t^{(\delta)} = x_0 + \sqrt{\delta}\int_0^t \sigma(X_s^{(\delta)})\,\mathrm{d}B_s$$

- Small-noise LDP for diffusions (Freidlin-Wentzell) $\Longrightarrow X_t$ satisfies LDP as $t \to 0$ with rate function

$$I(x) = \frac{1}{2} \inf_{\substack{f \in H^1([0,1]):\\ f(0)=x_0,\\ f(1)=x}} \int_0^1 \frac{\dot{f}(s)^2}{\sigma(f(s))^2}\,\mathrm{d}s = \frac{1}{2}\left(\int_{x_0}^x \frac{\mathrm{d}u}{\sigma(u)}\right)^2$$

Heuristic CLT derivation from LDP (2/2)

- That is, for $\varepsilon > 0$ small and fixed, we have the asymptotics

$$\mathbb{P}(X_t \geq x_0 + \varepsilon) \simeq \exp(-I(x_0 + \varepsilon)/t)$$

- **Non-rigorous step:** replace ε by $z\sqrt{t}$:

$$I(x_0 + z\sqrt{t}) = \tfrac{1}{2} I''(x_0) z^2 t + o(t),$$

and so

$$\mathbb{P}\left(\frac{X_t - x_0}{\sqrt{t}} \geq z\right) \approx \exp\left(-\frac{z^2}{2\sigma(x_0)^2} + o(1)\right).$$

The limiting probability $\lim_{t\to 0}\mathbb{P}[X_t > x_0]$

- **Corollary:** If X satisfies our main assumptions, and the limit law is non-degenerate, then

$$\lim_{t\to 0}\mathbb{P}[X_t > x_0] = \frac{1}{2}.$$

- **Example** (degenerate limit law): $X_t = B_t^2$ (BM squared)

$$\lim_{t\to 0}\mathbb{P}[X_t > x_0] = 1, \quad \frac{X_t}{\sqrt{t}} \xrightarrow{d} 0$$

- **Example** (drift not abs. continuous): $X_t = \Phi^{-1}(p)\sqrt{t} + B_t$

$$\lim_{t\to 0}\mathbb{P}[X_t > x_0] = p \in (0, 1)$$

The limiting probability $\lim_{t\to 0} \mathbb{P}[X_t > x_0]$

- **Example** (All values $p \in [0, 1)$ can be realized by martingales):
- R_t^δ squared Bessel process of dimension $\delta \geq 0$

$$\mathrm{d}R_t^\delta = 2\sqrt{R_t^\delta}\,\mathrm{d}B_t + \delta\,\mathrm{d}t, \quad R_0^\delta = 0.$$

- $X_t^\delta := R_t^\delta - \delta t$ is a martingale
- For $\delta \in [0, \infty)$, $\lim_{t\to 0} \mathbb{P}[X_t^\delta > x_0]$ ranges over $[0, 1)$.
- Limit law: Dirac at zero; limit cdf not continuous at zero

The limiting probability $\lim_{t\to 0} \mathbb{P}[X_t > x_0]$: Refined asymptotics

- SDE

$$\mathrm{d}X_t = b(t,\cdot)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B_t$$

  b bounded predictable process, σ locally square integrable **deterministic** matrix function, smallest eigenvalue of $\sigma(\cdot)^\top \sigma(\cdot)$ uniformly bounded away from 0.

- **Thm:**

$$g_1(t) \le \mathbb{P}[X_t > x_0] \le g_2(t)$$

- g_1, g_2 explicit functions with

$$g_1(t) = \frac{1}{2} - \sqrt{\frac{\log 2}{2}} \|\sigma^{-1} b\|_{2,\infty}\, t^{1/2} + O(t),$$

$$g_2(t) = \frac{1}{2} + \sqrt{\frac{\log 2}{2}} \|\sigma^{-1} b\|_{2,\infty}\, t^{1/2} + O(t).$$

Functional CLT

Thm: Let X satisfy the general assumption. Then for every $f : \mathbb{R}^m \to \mathbb{R}^n$, such that there exists an open neighborhood U of x_0 with $f \in C^2(U, \mathbb{R}^n)$, the processes

$$Y^{f,u} := \left(\frac{f(X_{u(t\wedge\tau)}) - f(x_0)}{\sqrt{u}} \right)_{t\in[0,T]}, \quad u \in (0,1),$$

converge in law to a Brownian motion with variance-covariance matrix

$$V = (Df)(x_0)L(Df(x_0)L)^\top$$

as $u \searrow 0$.

Functional CLT: proof idea

- Convergence of finite-dimensional distributions ($u \to 0$):

$$(Y_{t_1}^{f,u}, \ldots, Y_{t_w}^{f,u}) \xrightarrow{d} (\widetilde{B}_{t_1}, \ldots, \widetilde{B}_{t_w}), \quad t_1, \ldots, t_w \in [0, T],$$

  $\widetilde{B}$ Brownian motion with variance-covariance matrix V
- Proof: Analogous to finite-dimensional CLT
- Tightness condition: $u_l \in (0, 1)$ arbitrary with $u_l \to 0$

$$\lim_{\delta \searrow 0} \overline{\lim_{l \to \infty}} \, \mathbb{P}\Big(\sup_{|s-t| \leq \delta} |Y_s^{f,u_l} - Y_t^{f,u_l}| > \varepsilon \Big) = 0, \quad \varepsilon > 0.$$

General Assumption with Jumps (1/2)

For $T > 0$, let $X = (X_t)_{t\in[0,T]}$ be an $\mathbb{R}^m$-valued semimartingale with decomposition

$$X = X^c + J,$$

such that:

1. The process X^c is a continuous semimartingale satisfying the general assumtion for continuous semimartingales.
2. The process J is given by

$$J_t = \int_0^t \int_{B_1} \psi(s,z)\big(\Pi(ds,dz) - \mu(ds,dz)\big) + \int_0^t \int_{\mathbb{R}^m\setminus B_1} \varphi(s,z)\Pi(ds,dz),$$

where B_1 denotes the unit ball in $\mathbb{R}^m$, Π is a Poisson random measure on $[0,T]\times\mathbb{R}^m$ with compensator μ.

General Assumption with Jumps (2/2)

The $\mathbb{R}^m$-valued processes ψ, φ are predictable with respect to the filtration generated by Π and

$$E\left[\int_0^T \int_{B_1} |\psi(s,z)|^2 \mu(ds,dz)\right] < \infty;$$

3 There exists an a.s. positive stopping time τ_J s.t.

$$E\Big[|\Pi - \mu|([0, t \wedge \tau_J] \times B_1)\Big] = o(t^{\frac{1}{2}}) \text{ as } t \searrow 0.$$

CLT with Jumps

Theorem

Let X satisfy the general assumption with jumps. Then for every $f : \mathbb{R}^m \to \mathbb{R}^n$ s.t. there exists an open neighborhood U of x_0 with $f \in C^2(U, \mathbb{R}^n)$, there exists an a.s. positive stopping time τ such that

$$\frac{f(X_{t\wedge\tau}) - f(x_0)}{\sqrt{t}} \xrightarrow{d} N_f \text{ as } t \searrow 0,$$

where N_f is a normal random vector with mean 0 and convariance matrix

$$V = (Df)(x_0)L\big(Df(x_0)L\big)^\top.$$

Functional CLT with Jumps

Theorem

Let X satisfy the general assumption with jumps. Then for every $f : \mathbb{R}^m \to \mathbb{R}^n$ s.t. there exists an open neighborhood U of x_0 with $f \in C^2(U, \mathbb{R}^n)$ there exists a positive stopping time τ such that

$$Y^{f,u} := \left(\frac{f(X_{u(t\wedge\tau)}) - f(x_0)}{\sqrt{u}} \right)_{t\in[0,T]}, \quad u \in (0,1)$$

converge in law to a Brownian motion with variance-covariance matrix given by

$$V = (Df)(x_0)L(Df(x_0)L)^\top, \quad u \searrow 0.$$

CLT for Lévy processes (Doney, Maller 2002)

- Diffusion coefficient σ, Lévy measure ν
- **Thm:** There are functions f, g with

$$\frac{X_t - f(t)}{g(t)} \to N(0,1) \quad \text{in distribution}$$

if and only if

$$\lim_{x \to 0} \frac{U(x)}{x^2 T(x)} = \infty$$

- $T(x) := \nu((x,\infty)) + \nu((-\infty,-x))$,
  $U(x) := \sigma^2 + 2\int_0^x yT(y)dy$
- $g(t) \sim c\sqrt{t}$ if and only if $\sigma \neq 0$, and then $c = \sigma$.
- Note: the typical pure jump processes of math. finance **do not** satisfy a CLT (Variance gamma, NIG, CGMY,...)

Application of CLTs: Digital options

- Underlying S_t, define $X_t = \log S_t$, let $\mathbb{P}$ be the pricing measure
- Price of a **digital call option** with log-strike k (with $r = 0$):

$$\mathbb{P}[X_T \geq k].$$

- Small-time asymptotics for $k \neq x_0$: Varadhan, Rüschendorf, Woerner, Forde, Jacquier, Figueroa-López, Houdré, Marchal, ...
- **At the money** ($k = x_0$): If our assumptions hold, then

$$\lim_{T \to 0} \mathbb{P}[X_T \geq k] = \frac{1}{2}.$$

Implied volatility slope

- Implied volatility $\sigma_{\text{imp}} = \sigma_{\text{imp}}(K, T)$:

$$C_{\text{BS}}(K, \sigma_{\text{imp}}, T) = C(K, T) := \mathbb{E}[(S_T - K)^+]$$

- Implied volatility slope:

$$\partial_K \sigma_{\text{imp}} = -\frac{\partial_K C_{\text{BS}} - \partial_K C}{\partial_\sigma C_{\text{BS}}}$$

- Under mild assumptions:

$$\partial_K \sigma_{\text{imp}} = -\frac{\partial_K C_{\text{BS}} + \mathbb{P}[S_T \geq K]}{\partial_\sigma C_{\text{BS}}}$$

- Well-known connection between implied vol slope and digitals

Implied volatility slope

- More explicitly:

$$\partial_K \sigma_{\mathrm{imp}} = \frac{\Phi(-\sigma_{\mathrm{imp}}\sqrt{T}/2) - \mathbb{P}[S_T \geq K]}{K\sqrt{T}\, n(\sigma_{\mathrm{imp}}\sqrt{T}/2)}$$

- Under mild assumptions we have $\sigma_{\mathrm{imp}}\sqrt{T} = o(1),\ T \to 0$, and so

$$\partial_K \sigma_{\mathrm{imp}} \sim \frac{\sqrt{2\pi}}{K\sqrt{T}} \left(\frac{1}{2} - \mathbb{P}[S_T \geq K] - \frac{\sigma_{\mathrm{imp}}\sqrt{T}}{2\sqrt{2\pi}} + O((\sigma_{\mathrm{imp}}\sqrt{T})^3) \right)$$

- **ATM asymptotics** of $\partial_K \sigma_{\mathrm{imp}}$ depend on **second order term** of $\mathbb{P}[S_T \geq K]$

ATM digital calls: Small-time expansions for Lévy models

- $S_t = \exp(X_t)$, X Lévy process with characteristic triplet (σ, ν, b)

$$\log E[e^{sX_1}] = \frac{\sigma^2 s^2}{2} + bs + \int_{-\infty}^{\infty} (e^{sx} - 1 - sx\mathbf{1}_{\{|x| \leq 1\}})\nu(dx).$$

- **Thm:** Suppose that $\sigma > 0$ and that there is $s_0 \in (1, s_+)$ such that

$$\phi(s) := \int_{-\infty}^{\infty} (e^{sx} - 1 - sx\mathbf{1}_{\{|x| \leq 1\}})\nu(dx)$$

is bounded for $\mathrm{Re}(s) = s_0$ fixed and $\mathrm{Im}(s) \in \mathbb{R}$. Then

$$\mathbb{P}[X_T > x_0] = \frac{1}{2} + \frac{b}{\sigma\sqrt{2\pi}}\sqrt{T} + O(T\log(1/T)), \quad T \to 0.$$

Implied vol slope for Lévy models

- **Thm:** Assumptions as before. Then

$$\lim_{T \to 0} \partial_K \sigma_{imp} = -(b/\sigma + \sigma/2).$$

- Examples:
  - Merton jump diffusion
  - Kou's double exponential jump diffusion
- Application:
  - get initial parameter values for calibration
  - qualitative influence of parameters on model behavior

Infinite activity Lévy processes

- Normal inverse Gaussian model:

$$\mathbb{P}[X_T > x_0] = \frac{1}{2} + \frac{\delta\beta}{\pi} T \log(1/T) + o(T \log(1/T))$$
$$\partial_K \sigma_{imp} \sim -\delta\sqrt{\tfrac{2}{\pi}}(\beta + \tfrac{1}{2}) T \log(1/T)$$

- Variance gamma model:

$$\mathbb{P}[X_T > x_0] = \frac{1}{2} - \frac{\log 2\nu\sigma^2}{2\nu} T + o(T)$$
$$\partial_K \sigma_{imp} \sim const \cdot \sqrt{T}$$

- Proof idea: Fourier representation, contour shift
- Question: criterion for $\lim_{t\to 0} \mathbb{P}[X_T > x_0] = \frac{1}{2}$?

References

- A. Bentata and R. Cont, *Short-time asymptotics for marginal distributions of semimartingales*, preprint, 2012.
- R. A. Doney and R. A. Maller, *Stability and attraction to normality for Lévy processes at zero and at infinity*, J. Theoret. Probab. 15 (2002).
- S. Gerhold, M. Kleinert, P. Porkert, M. Shkolnikov, *Small time central limit theorems for semimartingales with applications*, preprint, 2012.
- S. Gerhold, M. Shkolnikov, *Short maturity digitals and implied volatility slopes*, working paper, 2013.