Small time central limit theorems for semimartingales with applications

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13 Feb 2013
Overview

- \((X_t)_{t \geq 0}\) semimartingale with \(X_0 = x_0\) a.s.
- \(f\) smooth
- We are interested in the limit

\[
\lim_{t \to 0} \frac{f(X_t) - f(x_0)}{\sqrt{t}}
\]

in distribution

- Main motivation: Complement well known small time LDPs by a CLT
- There are connections to digital option prices and implied volatility slopes
Overview

- CLT for SDE solutions
- CLT for continuous semimartingales
- \( \lim_{t \to 0} \mathbb{P}[X_t > x_0] = ? \)
- Functional CLT for continuous semimartingales
- Jumps
- Digital options, implied volatility slope
CLT for SDE solutions

- Let $X$ be a weak solution of the SDE

$$
\begin{align*}
\frac{dX_t^j}{dt} &= b_j(t, X_t) \, dt + \sum_{k=1}^{d} \sigma_{jk}(t, X_t) \, dB_t^k \\
\end{align*}
$$

where $B$ is a standard $d$-dimensional Brownian motion, $b$ is uniformly bounded in a neighborhood of $(0, x_0)$ and $\sigma$ is continuous in $(0, x_0)$. 
CLT for SDE solutions

Let \( X \) be a weak solution of the SDE

\[
dX_t^j = b_j(t, X_t) \, dt + \sum_{k=1}^{d} \sigma_{jk}(t, X_t) \, dB_t^k
\]

where \( B \) is a standard \( d \)-dimensional Brownian motion, \( b \) is uniformly bounded in a neighborhood of \((0, x_0)\) and \( \sigma \) is continuous in \((0, x_0)\).

**Thm:** For every \( f : \mathbb{R}^m \to \mathbb{R}^n \) s.t. there exists an open neighborhood \( U \) of \( x_0 \) with \( f \in C^2(U, \mathbb{R}^n) \), there exists an a.s. positive stopping time \( \tau \) such that

\[
\frac{f(X_{t \wedge \tau}) - f(x_0)}{\sqrt{t}} \xrightarrow{d} N_f \text{ as } t \downarrow 0,
\]

where \( N_f \) is a normal random vector with mean 0 and covariance matrix (with \( L := \sigma(0, x_0) \))

\[
V = (Df)(x_0)L(Df(x_0)L)^\top.
\]
CLT for continuous semimartingales

- Assume: $X = x_0 + M + A$ continuous semimartingale
- $M$ continuous local martingale, $A$ locally finite variation,
- Conditions that ensure representation

$$X_{t \wedge \tau}^j = x_0 + \int_0^{t \wedge \tau} b_s^j \, ds + \sum_{k=1}^m \int_0^{t \wedge \tau} \sigma_s^{jk} \, dB_s^k$$

- some boundedness assumptions
- Then the small-time CLT is valid
CLT for continuous semimartingales: General Assumptions

1. $X_0 = x_0$ a.s.;
2. there exists an a.s. positive stopping time $\tau_A$ such that a.s.

$$A_t^i = \int_0^t b_s^i \, ds, \quad t \in [0, \tau_A],$$

for an adapted process $b$;
3. there exists a random variable $C_b$, such that $|b_t^i| \leq C_b < \infty$ for a.e. $t \in [0, \tau_A]$ a.s.;
CLT for continuous semimartingales: General Assumptions

1. \( X_0 = x_0 \) a.s.;

2. there exists an a.s. positive stopping time \( \tau_A \) such that a.s.
   \[
   A^i_t = \int_0^t b^i_s \, ds, \quad t \in [0, \tau_A],
   \]
   for an adapted process \( b \);

3. there exists a random variable \( C_b \), such that \( |b^i_t| \leq C_b < \infty \)
   for a.e. \( t \in [0, \tau_A] \) a.s.;

4. there exists an a.s. positive stopping time \( \tau_M \) such that the
   covariation is a.s.
   \[
   \langle M^j, M^k \rangle_t = \int_0^t \sum_{l=1}^m \sigma^j_l \sigma^k_l \, ds, \quad t \in [0, \tau_M],
   \]
   for a progressive process \( \sigma \);

5. there exists a deterministic constant \( C_\sigma < \infty \), such that
   \( |\sigma^j_k_t| \leq C_\sigma \) for a.e. \( t \in [0, \tau_M] \) a.s., \( j, k \in \{1, \ldots, m\} \);

6. as \( t \downarrow 0 \), \( \sigma_t \to L \) a.s., where \( L \) is a deterministic matrix
**Thm:** For every $f : \mathbb{R}^m \to \mathbb{R}^n$ s.t. there exists an open neighborhood $U$ of $x_0$ with $f \in C^2(U, \mathbb{R}^n)$, there exists an a.s. positive stopping time $\tau$ such that

$$\frac{f(X_{t \wedge \tau}) - f(x_0)}{\sqrt{t}} \xrightarrow{d} N_f \text{ as } t \downarrow 0,$$

where $N_f$ is a normal random vector with mean 0 and covariance matrix

$$V = (Df)(x_0)L(Df(x_0)L)^\top.$$
n = 1 (otherwise: Cramér-Wold)

Choose open ball $B$ with $\bar{B} \subset U$

$\tau := \tau_{\bar{B}^c} \wedge \tau_A \wedge \tau_M$

By Doob’s integral representation theorem:

$$X_{t \wedge \tau}^j = x_0 + \int_0^{t \wedge \tau} b_s^j \, ds + \sum_{k=1}^m \int_0^{t \wedge \tau} \sigma_{s}^{jk} \, dB_s^k$$
By Itô’s formula:

\[
\frac{f(X_{t\wedge \tau}) - f(x_0)}{\sqrt{t}} = \frac{1}{\sqrt{t}} \int_0^{t\wedge \tau} (\mathcal{L}_s f)(X_s) \, ds + \frac{1}{\sqrt{t}} \sum_{k,l=1}^m \int_0^{t\wedge \tau} \frac{\partial f}{\partial x_l}(X_s) \sigma_s^{lk} \, dB_s^k,
\]

where

\[
(\mathcal{L}_s f)(u) := \frac{1}{2} \sum_{k,l=1}^m (\sigma_s \sigma_s^\top)_{kl} \frac{\partial^2 f}{\partial x_k \partial x_l}(u) + \sum_{k=1}^m b_s^k \frac{\partial f}{\partial x_k}(u)
\]
First term (use boundedness assumption on $b, \sigma$, choice of $B$):

$$\frac{1}{\sqrt{t}} \int_0^{t \wedge \tau} (\mathcal{L}_s f)(X_s) \, ds = O(\sqrt{t}), \quad t \to 0, \text{ a.s.}$$

Second term:

$$\frac{1}{\sqrt{t}} \sum_{k,l=1}^{m} \int_0^{t \wedge \tau} \frac{\partial f}{\partial x_l}(X_s) \sigma_{s}^{lk} \, dB_s^k$$

Freeze integrand at $s = 0$, Cauchy-Schwarz, Itô’s isometry

$$\rightarrow \text{converges in law to a Gaussian r.v.}$$

Slutsky’s theorem
Heuristic CLT derivation from LDP (1/2)

- Recall: classical CLT can be heuristically derived from Cramér’s theorem
- Suppose $X$ satisfies

$$X_t = x_0 + \int_0^t \sigma(X_s) \, dB_s, \quad t \geq 0$$

- Time-scaling: $X_t \overset{d}{=} X_1^{(\delta)}$, where

$$X_t^{(\delta)} = x_0 + \sqrt{\delta} \int_0^t \sigma(X_s^{(\delta)}) \, dB_s$$

- Small-noise LDP for diffusions (Freidlin-Wentzell) $\implies X_t$ satisfies LDP as $t \to 0$ with rate function

$$I(x) = \frac{1}{2} \inf_{f \in H^1([0,1])} \begin{array}{c} \\ \int_0^1 \frac{\dot{f}(s)^2}{\sigma(f(s))^2} \, ds = \frac{1}{2} \left( \int_{x_0}^x \frac{du}{\sigma(u)} \right)^2 \end{array}$$

\quad f(0) = x_0, \quad f(1) = x$$
Heuristic CLT derivation from LDP (2/2)

- That is, for $\varepsilon > 0$ small and fixed, we have the asymptotics

\[ \mathbb{P}(X_t \geq x_0 + \varepsilon) \approx \exp\left(-I(x_0 + \varepsilon)/t\right) \]

- **Non-rigorous step:** replace $\varepsilon$ by $z \sqrt{t}$:

\[ I(x_0 + z \sqrt{t}) = \frac{1}{2} I''(x_0) z^2 t + o(t), \]

and so

\[ \mathbb{P}\left(\frac{X_t - x_0}{\sqrt{t}} \geq z\right) \approx \exp\left(-\frac{z^2}{2\sigma(x_0)^2} + o(1)\right). \]
The limiting probability $\lim_{t \to 0} \mathbb{P}[X_t > x_0]$

- **Corollary**: If $X$ satisfies our main assumptions, and the limit law is non-degenerate, then

  $$\lim_{t \to 0} \mathbb{P}[X_t > x_0] = \frac{1}{2}.$$ 

- **Example** (degenerate limit law): $X_t = B_t^2$ (BM squared)

  $$\lim_{t \to 0} \mathbb{P}[X_t > x_0] = 1, \quad \frac{X_t}{\sqrt{t}} \xrightarrow{d} 0$$

- **Example** (drift not abs. continuous): $X_t = \Phi^{-1}(p)\sqrt{t} + B_t$

  $$\lim_{t \to 0} \mathbb{P}[X_t > x_0] = p \in (0, 1)$$
The limiting probability $\lim_{t \to 0} \mathbb{P}[X_t > x_0]$

- **Example** (All values $p \in [0, 1)$ can be realized by martingales):
  - $R^\delta_t$ squared Bessel process of dimension $\delta \geq 0$
    \[ dR^\delta_t = 2 \sqrt{R^\delta_t} \, dB_t + \delta \, dt, \quad R^\delta_0 = 0. \]
  - $X^\delta_t := R^\delta_t - \delta t$ is a martingale
  - For $\delta \in [0, \infty)$, $\lim_{t \to 0} \mathbb{P}[X^\delta_t > x_0]$ ranges over $[0, 1)$.
  - Limit law: Dirac at zero; limit cdf not continuous at zero
The limiting probability \( \lim_{t \to 0} \mathbb{P}[X_t > x_0] \): Refined asymptotics

- **SDE**

\[
    dX_t = b(t, \cdot) \, dt + \sigma(t) \, dB_t
\]

\( b \) bounded predictable process, \( \sigma \) locally square integrable **deterministic** matrix function, smallest eigenvalue of \( \sigma(\cdot) \top \sigma(\cdot) \) uniformly bounded away from 0.

- **Thm:**

\[
    g_1(t) \leq \mathbb{P}[X_t > x_0] \leq g_2(t)
\]

- \( g_1, g_2 \) explicit functions with

\[
    g_1(t) = \frac{1}{2} - \sqrt{\frac{\log 2}{2}} \| \sigma^{-1} b \|_{2, \infty} t^{1/2} + O(t),
\]

\[
    g_2(t) = \frac{1}{2} + \sqrt{\frac{\log 2}{2}} \| \sigma^{-1} b \|_{2, \infty} t^{1/2} + O(t).
\]
**Thm:** Let $X$ satisfy the general assumption. Then for every $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$, such that there exists an open neighborhood $U$ of $x_0$ with $f \in C^2(U, \mathbb{R}^n)$, the processes

$$Y^{f,u} := \left( \frac{f(X_{u(t\wedge \tau)}) - f(x_0)}{\sqrt{u}} \right)_{t \in [0,T]}, \quad u \in (0,1),$$

converge in law to a Brownian motion with variance-covariance matrix

$$V = (Df)(x_0)L(Df(x_0)L)^\top$$

as $u \downarrow 0$. 
Functional CLT: proof idea

- Convergence of finite-dimensional distributions \((u \to 0)\):

\[
(Y_{t_1}^{f,u}, \ldots, Y_{t_w}^{f,u}) \overset{d}{\to} (\tilde{B}_{t_1}, \ldots, \tilde{B}_{t_w}), \quad t_1, \ldots, t_w \in [0, T],
\]

\(\tilde{B}\) Brownian motion with variance-covariance matrix \(V\)

- Proof: Analogous to finite-dimensional CLT

- Tightness condition: \(u_l \in (0, 1)\) arbitrary with \(u_l \to 0\)

\[
\lim_{\delta \searrow 0} \lim_{l \to \infty} \mathbb{P}\left( \sup_{|s-t| \leq \delta} |Y_{s}^{f,u_l} - Y_{t}^{f,u_l}| > \varepsilon \right) = 0, \quad \varepsilon > 0.
\]
General Assumption with Jumps (1/2)

For $T > 0$, let $X = (X_t)_{t \in [0, T]}$ be an $\mathbb{R}^m$-valued semimartingale with decomposition

$$X = X^c + J,$$

such that:

1. The process $X^c$ is a continuous semimartingale satisfying the general assumption for continuous semimartingales.
2. The process $J$ is given by

$$J_t = \int_0^t \int_{B_1} \psi(s, z)(\Pi(ds, dz) - \mu(ds, dz))$$

$$+ \int_0^t \int_{\mathbb{R}^m \setminus B_1} \varphi(s, z)\Pi(ds, dz),$$

where $B_1$ denotes the unit ball in $\mathbb{R}^m$, $\Pi$ is a Poisson random measure on $[0, T] \times \mathbb{R}^m$ with compensator $\mu$. 
General Assumption with Jumps (2/2)

The $\mathbb{R}^m$-valued processes $\psi, \varphi$ are predictable with respect to the filtration generated by $\Pi$ and

$$E \left[ \int_0^T \int_{B_1} |\psi(s, z)|^2 \mu(ds, dz) \right] < \infty;$$

3. There exists an a.s. positive stopping time $\tau_J$ s.t.

$$E \left[ |\Pi - \mu|( [0, t \wedge \tau_J] \times B_1 ) \right] = o(t^{\frac{1}{2}}) \text{ as } t \searrow 0.$$
Theorem
Let $X$ satisfy the general assumption with jumps. Then for every $f : \mathbb{R}^m \to \mathbb{R}^n$ s.t. there exists an open neighborhood $U$ of $x_0$ with $f \in C^2(U, \mathbb{R}^n)$, there exists an a.s. positive stopping time $\tau$ such that

$$\frac{f(X_{t\wedge\tau}) - f(x_0)}{\sqrt{t}} \overset{d}{\to} N_f \text{ as } t \searrow 0,$$

where $N_f$ is a normal random vector with mean 0 and covariance matrix

$$V = (Df)(x_0)L(Df(x_0)L)^\top.$$
Theorem
Let $X$ satisfy the general assumption with jumps. Then for every $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ s.t. there exists an open neighborhood $U$ of $x_0$ with $f \in C^2(U, \mathbb{R}^n)$ there exists a positive stopping time $\tau$ such that
\[
Y^{f,u} := \left( \frac{f(X_u(t\wedge\tau)) - f(x_0)}{\sqrt{u}} \right)_{t\in[0,T]}, \quad u \in (0, 1)
\]
converge in law to a Brownian motion with variance-covariance matrix given by
\[
V = (Df)(x_0)L(Df(x_0)L)^\top, \quad u \searrow 0.
\]
CLT for Lévy processes (Doney, Maller 2002)

- Diffusion coefficient $\sigma$, Lévy measure $\nu$

- **Thm:** There are functions $f, g$ with

$$\frac{X_t - f(t)}{g(t)} \to N(0, 1) \quad \text{in distribution}$$

if and only if

$$\lim_{x \to 0} \frac{U(x)}{x^2 T(x)} = \infty$$

- $T(x) := \nu((x, \infty)) + \nu((\infty, -x))$,
  $U(x) := \sigma^2 + 2 \int_0^x yT(y)dy$

- $g(t) \sim c \sqrt{t}$ if and only if $\sigma \neq 0$, and then $c = \sigma$.

- Note: the typical pure jump processes of math. finance **do not** satisfy a CLT (Variance gamma, NIG, CGMY,...)
Application of CLTs: Digital options

- Underlying $S_t$, define $X_t = \log S_t$, let $\mathbb{P}$ be the pricing measure.
- Price of a digital call option with log-strike $k$ (with $r = 0$):
  \[
  \mathbb{P}[X_T \geq k].
  \]
- Small-time asymptotics for $k \neq x_0$: Varadhan, Rüschendorf, Woerner, Forde, Jacquier, Figueroa-López, Houdré, Marchal, ...
- At the money ($k = x_0$): If our assumptions hold, then
  \[
  \lim_{T \to 0} \mathbb{P}[X_T \geq k] = \frac{1}{2}.
  \]
Implied volatility slope

- Implied volatility $\sigma_{\text{imp}} = \sigma_{\text{imp}}(K, T)$:
  
  $$\text{C}_{\text{BS}}(K, \sigma_{\text{imp}}, T) = \text{C}(K, T) := \mathbb{E}[(S_T - K)^+]$$

- Implied volatility slope:
  
  $$\partial_K\sigma_{\text{imp}} = -\frac{\partial_K \text{C}_{\text{BS}} - \partial_K \text{C}}{\partial_\sigma \text{C}_{\text{BS}}}$$

- Under mild assumptions:
  
  $$\partial_K\sigma_{\text{imp}} = -\frac{\partial_K \text{C}_{\text{BS}} + \mathbb{P}[S_T \geq K]}{\partial_\sigma \text{C}_{\text{BS}}}$$

- Well-known connection between implied vol slope and digitals
Implied volatility slope

More explicitly:

\[ \partial_K \sigma_{\text{imp}} = \frac{\Phi(-\sigma_{\text{imp}}\sqrt{T}/2) - \mathbb{P}[S_T \geq K]}{K\sqrt{T} n(\sigma_{\text{imp}}\sqrt{T}/2)} \]

Under mild assumptions we have \( \sigma_{\text{imp}}\sqrt{T} = o(1), \; T \to 0 \), and so

\[ \partial_K \sigma_{\text{imp}} \sim \frac{\sqrt{2\pi}}{K\sqrt{T}} \left( \frac{1}{2} - \mathbb{P}[S_T \geq K] - \frac{\sigma_{\text{imp}}\sqrt{T}}{2\sqrt{2\pi}} + O((\sigma_{\text{imp}}\sqrt{T})^3) \right) \]

ATM asymptotics of \( \partial_K \sigma_{\text{imp}} \) depend on second order term of \( \mathbb{P}[S_T \geq K] \)
ATM digital calls: Small-time expansions for Lévy models

- $S_t = \exp(X_t)$, $X$ Lévy process with characteristic triplet $(\sigma, \nu, b)$

\[
\log E[e^{sX_1}] = \frac{\sigma^2 s^2}{2} + bs + \int_{-\infty}^{\infty} (e^{sx} - 1 - sx1_{\{|x|\leq 1\}})\nu(dx).
\]

- **Thm:** Suppose that $\sigma > 0$ and that there is $s_0 \in (1, s_+)$ such that
\[
\phi(s) := \int_{-\infty}^{\infty} (e^{sx} - 1 - sx1_{\{|x|\leq 1\}})\nu(dx)
\]
is bounded for $\Re(s) = s_0$ fixed and $\Im(s) \in \mathbb{R}$. Then
\[
P[X_T > x_0] = \frac{1}{2} + \frac{b}{\sigma \sqrt{2\pi}} \sqrt{T} + O(T \log(1/T)), \quad T \to 0.
\]
Implied vol slope for Lévy models

- **Thm:** Assumptions as before. Then

\[
\lim_{T \to 0} \partial_K \sigma_{imp} = -(b/\sigma + \sigma/2).
\]

- **Examples:**
  - Merton jump diffusion
  - Kou’s double exponential jump diffusion

- **Application:**
  - get initial parameter values for calibration
  - qualitative influence of parameters on model behavior
Infinite activity Lévy processes

- Normal inverse Gaussian model:
  \[ \mathbb{P}[X_T > x_0] = \frac{1}{2} + \frac{\delta \beta}{\pi} T \log(1/T) + o(T \log(1/T)) \]
  \[ \partial_K \sigma_{imp} \sim -\delta \sqrt{\frac{2}{\pi}} \left( \beta + \frac{1}{2} \right) T \log(1/T) \]

- Variance gamma model:
  \[ \mathbb{P}[X_T > x_0] = \frac{1}{2} - \frac{\log 2\nu\sigma^2}{2\nu} T + o(T) \]
  \[ \partial_K \sigma_{imp} \sim \text{const} \cdot \sqrt{T} \]

- Proof idea: Fourier representation, contour shift
- Question: criterion for \( \lim_{t \to 0} \mathbb{P}[X_T > x_0] = \frac{1}{2} \)?
References