Local Volatility Models: Approximation and Regularization

Stefan Gerhold

Vienna UT

Joint work with P. Friz and M. Yor

June 2013

Overview

- Local vol model: recreates marginals of a given diffusion (via call price surface)
- Inconsistent with jumps in the underlying
- Local Lévy models
- We propose a simple truncation to improve the classical diffusion local vol model
- Quantifying the blowup of local vol for small time

Local volatility ($r = 0$ for simplicity)

- If an underlying satisfies

$$dS_t/S_t = \sigma(t, S_t)dW_t,$$

then call prices $C(K, T)$ satisfy the forward PDE

$$\partial_T C(K, T) = \frac{1}{2}K^2\sigma(K, T)^2\partial_{KK}C(K, T).$$

- Conversely: The *local volatility model*

$$dS_t/S_t = \sigma_{\mathrm{loc}}(S_t, t)dW_t$$

reproduces a given smooth call price surface C, where:

- Dupire's formula (1994)

$$\sigma^2_{\mathrm{loc}}(K, T) = \frac{2\partial_T C}{K^2\partial_{KK}C}$$

Using Dupire's formula for models with jumps

- Suppose that $C(K, T)$ is generated from a model with jumps
- Variance gamma model: Call price not C^2 w.r.t. strike (but works for T large)
- Jumps cause blowup of local vol as $T \to 0$, hence local vol model may be ill-defined (Cont, Gu 2012).
- Even if Dupire's formula is well-defined, the local vol model may not match the marginals of the jump process.

Local Lévy models

- Carr, Geman, Madan, Yor 2004
- Dynamics

$$dS_t = \sigma(S_{t-}, t) S_{t-} dW_t + \int_{-\infty}^{\infty} (e^x - 1)(m_{(S_{s-},s)}(dx, du) - \mu_{(S_{s-},s)}(dx, du))$$

- m is an integer valued random measure independent of W
- μ is its compensator
- σ and m are chosen to reproduce a given call price surface

Local Lévy models

- Local speed function a_0:

$$\mu_{(S_{s-},s)}(dx, dt) = a_0(S_{t-}, t)\nu(dx)dt$$

- Call price PIDE:

$$C_T = \frac{1}{2}\sigma^2(K, T)K^2 C_{KK} + \int_0^\infty y C_{KK}(y, T) a_0(y, T)\psi(\log \frac{K}{y})dy$$

- ψ is double exponential tail of Lévy measure:

$$\psi(z) = \begin{cases} \int_{-\infty}^{z}(e^z - e^x)\nu(dx) & z < 0 \\ \int_z^\infty (e^x - e^z)\nu(dx) & z > 0 \end{cases}$$

Calculating the model parameters from the call price surface

- PIDE parameter identification: get σ, a_0, ν from call price surface
- Ill-posed inverse problem
- Kindermann, Albrecher, Mayer, Engl 2008: Tikhonov regularization for a_0 (speed function), given σ and ν
- Kindermann, Mayer 2011: Tikhonov regularization for all parameters

Results by Cont, Gu (2012)

- Local vol models and local jump diffusion models are incompatible
- The sets of call price surfaces they generate are disjoint
- Hence using local vol is questionable if one believes that the underlying has jumps
- Local vol surface blows up as $T \to 0$ if the underlying has jumps

Our approach: regularization of local vol (Friz, G., Yor 2013)

- Local vol models: Inconsistent with jumps in underlying, non-robust (recalibration!)
- Local Lévy models: Theoretically more sound, but harder to implement
- We propose a “poor man’s approach”: Retain local vol dynamics, but with stochastic initial value
- Consistent with jumps in underlying
- Original call price surface recovered with arbitrary precision
- Calibration simpler than for local Lévy models

Regularization of local vol: Idea

- Pick a small $\varepsilon > 0$
- Get law of underlying at time ϵ from market data
- Let the (properly shifted) local vol dynamics run from time ϵ on
- Gives a diffusion process S^{ε} on the time interval $[\varepsilon, \infty)$
- As $\varepsilon \to 0$, the given call price surface is recovered

Regularization of local vol: a trivial observation

- **Assumption:** Suppose that the given C is such that $dS/S = \sigma_{\mathrm{loc}}(S,t)dW_t$ has a well-defined solution.
- Define ε-shifted local volatility

$$\sigma_\varepsilon^2(K,T) = \frac{2\partial_T C(K, T+\varepsilon)}{K^2 \partial_{KK} C(K, T+\varepsilon)}.$$

  Then the solution of $dS^\varepsilon/S^\varepsilon = \sigma_\varepsilon(S^\varepsilon, t)dW$, started at randomized spot S_0^ε with distribution

$$\mathbb{P}[S_0^\varepsilon \in dK]/dK = \partial_{KK} C(K, \varepsilon),$$

  satisfies

$$\mathbb{E}[(S_T^\varepsilon - K)^+] = C(K, T+\varepsilon) \to C(K,T) \quad \text{as} \quad \varepsilon \to 0.$$

- Key point: the **assumption** is not necessary!

Regularization of local vol

Theorem (Friz, G., Yor 2013): Assume that (S_t) is a martingale (possibly with jumps) with associated smooth call price surface C:

$$C(K,T) = \mathbb{E}[(S_T - K)^+],$$

such that $\partial_T C > 0$ and $\partial_{KK} C > 0$, i.e. (strict) absence of calendar and butterfly spreads.
Define ε-shifted local volatility

$$\sigma_\varepsilon^2(K,T) = \frac{2\partial_T C(K, T+\varepsilon)}{K^2 \partial_{KK} C(K, T+\varepsilon)}.$$

Then $dS^\varepsilon / S^\varepsilon = \sigma_\varepsilon(S^\varepsilon, t) dW$, started at randomized spot S_0^ε with distribution

$$\mathbb{P}[S_0^\varepsilon \in dK]/dK = \partial_{KK} C(K, \varepsilon),$$

admits a unique, non-explosive strong SDE solution such that

$$\forall K, T \geq 0 : \mathbb{E}[(S_T^\varepsilon - K)^+] \to C(K,T) \quad \text{as} \quad \varepsilon \to 0.$$

Regularization of local vol: Proof idea 1/4

- Let $q^\varepsilon(dS,T)$ be the law of S_T^ε, and $p^\varepsilon(S,T)$ be the density of $S_{T+\varepsilon}$
- Calculate

$$
\begin{aligned}
\mathbb{E}[(S_T^\varepsilon - K)^+] &= \int (S-K)^+ \, q^\varepsilon(dS,T) \\
&\overset{!}{=} \int (S-K)^+ \, p^\varepsilon(S,T) dS \\
&= \mathbb{E}[(S_{T+\varepsilon} - K)^+] \\
&= C(K, T+\varepsilon).
\end{aligned}
$$

  Then let $\varepsilon \to 0$.
- Need to show $S_T^\varepsilon \overset{d}{=} S_{T+\varepsilon}$

Regularization of local vol: Proof idea 2/4

- Define

$$a^{\varepsilon}(K,T) := \frac{\partial_T C(K, T+\varepsilon)}{p^{\varepsilon}(K,T)}$$

- p^{ε} satisfies the Fokker-Plack equation

$$\partial_{KK}(a^{\varepsilon} p^{\varepsilon}) = \partial_T p^{\varepsilon}$$

- Note: $a^{\varepsilon}(S,t)\partial_{SS}$ is the generator of S^{ε}.

Regularization of local vol: Proof idea 3/4

- For any test function,

$$\varphi(S_t^\varepsilon) - \varphi(S_0^\varepsilon) - \int_0^t a^\varepsilon(S_t^\varepsilon, t)\partial_{SS}\varphi(S_s^\varepsilon)ds$$

  is a martingale.

- Take expectation:

$$\int \varphi(S)q^\varepsilon(dS,t) = \int \varphi(S)q^\varepsilon(dS,0) + \int_0^t \int a^\varepsilon(S,s)\varphi''(S)q^\varepsilon(dS,s)$$

  for any smooth φ with compact support.

- Hence q^ε is also a (weak) solution of the Fokker-Planck equation

Regularization of local vol: Proof idea 4/4

- So our result is a corollary of the following uniqueness theorem (Pierre 2012):
- $U := (0, \infty) \times \mathbb{R}$
- Let $a : (t, x) \in \bar{U} \to a(t, x) \in \mathbb{R}_+$ be a continuous function with $a(t, x) > 0$ for $(t, x) \in U$, and let μ be a probability measure with $\int |x| \mu(dx) < \infty$.
- Then there exists at most one family of probability measures $(p(t, dx), t \geq 0)$ such that
  - $t \geq 0 \to p(t, dx)$ is weakly continuous
  - $p(0, dx) = \mu(dx)$ and

$$\partial_t p - \partial_{xx}(ap) = 0 \quad \text{in } \mathcal{D}'(U)$$

(i.e., in the sense of Schwartz distributions on the open set U.)

Brief side remark about peacocks

- A peacock (PCOC=processus croissant pour l'ordre convexe) is an integrable process (X_t) such that

$$t \mapsto E[\psi(X_t)] \quad \text{increases for every convex } \psi.$$

- If X has the same one-dimensional marginals as some martingale, then X is a peacock (Jensen's inequality).
- Kellerer's theorem (1972): The converse is also true.
- Hirsch, Roynette, Yor (2012): New proof + extension.
- Part of the proof resembles our construction. In particular, Pierre's uniqueness theorem is used.

Quantifying the blowup of local vol in jump models

- Recall Dupire's formula:

$$\sigma_{\mathrm{loc}}^2(K,T) = \frac{2\partial_T C}{K^2 \partial_{KK} C}$$

- Example: NIG model. Density $\partial_{KK} C$ explicit,and $\approx T$ for small T.
  $\partial_T C$ tends to a constant (by forward PIDE).
- Hence the blowup in the NIG model:

$$\sigma_{\mathrm{loc}}^2(K,T) \approx \frac{1}{T}, \quad K \neq S_0, T \to 0.$$

Quantifying the blowup of local vol in jump models

- More examples for off-the-money blowup ($K \neq S_0$ fixed):

$$\sigma^2_{\mathrm{loc}}(K,T) \approx 1/T \quad \text{(Merton jump diffusion)}$$
$$\sigma^2_{\mathrm{loc}}(K,T) \approx 1/\sqrt{T} \quad \text{(Kou's diffusion)}$$
$$\sigma^2_{\mathrm{loc}}(K,T) \approx 1/T \quad \text{(Normal inverse Gaussian)}$$

General asymptotic formula for local vol (De Marco, Friz, G. 2013)

- log moment generating function ($X_T = \log S_T$)

$$m(s,T) = \log E[\exp(sX_T)]$$

- saddle point $\hat{s}(k,T)$

$$\frac{\partial}{\partial s} m(s,T)\bigg|_{s=\hat{s}} = k$$

- Asymptotic approximation for "extreme" K or T:

$$\sigma^2_{\mathrm{loc}}(K,T) \approx \frac{2\frac{\partial}{\partial T} m(s,T)}{s(s-1)}\bigg|_{s=\hat{s}(k,T)}$$

General asymptotic formula for local vol: proof idea

- Moment generating function ($X_T = \log S_T$):

$$M(s, T) := E[\exp(sX_T)], \qquad m(s, T) := \log M(s, T)$$

- Dupire's formula + Fourier inversion

$$\sigma^2_{\mathrm{loc}}(K, T) = \frac{2\partial_T C}{K^2 \partial_{KK} C} = \frac{2\int_{-i\infty}^{i\infty} \frac{\partial_T m(s,T)}{s(s-1)} e^{-ks} M(s, T) ds}{\int_{-i\infty}^{i\infty} e^{-ks} M(s, T) ds}$$

- Saddle point method: Leading terms are integrands evaluated at saddle point $\longrightarrow$ cancellation

Summary

- Small time shift in local vol allows to accommodate jumps
- Small-maturity smile (usually steep) from market data; no need for steep wings of local vol function
- Asymptotic consistency proof by Pierre's uniquenes theorem for Fokker-Planck equations
- Future work: numerical tests (robust recalibration?)

References

- P. Carr, H. Geman, D. Madan, M. Yor, *From local volatility to local Lévy models*, 2004.
- R. Cont Y. Gu, *Local vs. non-local forward equations for option pricing*, 2012.
- S. De Marco, P. Friz, S. Gerhold, *Rational shapes of local volatility*, 2013.
- P. Friz, S. Gerhold, M. Yor, *How to make Dupire's local volatility work with jumps*, 2013.
- F. Hirsch, B. Roynette, M. Yor, *Kellerer's theorem revisited*, 2012.