Transaction Costs, Trading Volume, and the Liquidity Premium

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Introduction

Optimal portfolios in frictionless markets

Merton (1971, 1973): Consider a safe and a risky asset, following

$$dS_t^0/S_t^0 = r dt, \quad dS_t/S_t = (\mu + r)dt + \sigma dW_t, \quad \mu, \sigma > 0$$

For an investor with constant relative risk aversion γ and arbitrary time horizon:

- Optimal policy: constant risky fraction $\pi_* = \mu/\gamma\sigma^2$, only depends on (r, μ, σ) through **mean-variance ratio** μ/σ^2
- Utility grows at certainty equivalent rate $\beta = r + \mu^2/2\gamma\sigma^2$
- Hence, assets are ranked by **Sharpe ratio** μ/σ
- Same optimal strategy for utility from consumption
- Explicit solution in terms of model parameters and preferences
- Results shape much of our intuition, but are not robust w.r.t. transaction costs since the implied trading volume is infinite

Introduction

Optimal portfolios with transaction costs

- Black-Scholes model for S^0, S as before
- But now, buy at **ask price** S, sell at **bid price** $(1-\epsilon)S$, where $\epsilon \in (0,1)$ is the width of the bid-ask spread.

Simplest possible benchmark optimization problem?

- Investment depends on time horizon $\Rightarrow$ infinite horizon
- Utility from consumption (Magill and Constantinidis (1976), Davis and Norman (1990), Shreve and Soner (1994)): Keep risky fraction in **no-trade region** $[\pi_-, \pi_+]$ around π_*
- π_-, π_+ characterized by a free boundary problem
- Janeček and Shreve (2004): First-order asymptotics via viscosity approach
- Further simplification: Drop consumption!

Introduction

Long-run optimal portfolios with transaction costs

Maximize the **certainty equivalent rate**

$$\liminf_{T\to\infty} \frac{1}{T}\frac{1}{1-\gamma}\log E\left[\left(\varphi_T^0 + \varphi_T^+(1-\epsilon)S_T - \varphi_T^- S_T\right)^{1-\gamma}\right] \to \text{max!}$$

With transaction costs:

- Simplest case: log-utility (γ "=" 1). Solution in Taksar et al. (1989), asymptotics in SG, Muhle-Karbe & Schachermayer (2011)
- No trade region $[\pi_-, \pi_+]$, optimal growth rate known via the solution of a one-dimensional equation
- $\gamma \neq 1$: Similar heuristic results of Dumas and Luciano (1991)
- Starting point of our analysis

Main results and Implications

Definition of the key extra parameter, the gap

Lemma

For small $\epsilon > 0$, there is a unique $\lambda > 0$ for which the solution to

$$w'(x) + (1-\gamma)w(x)^2 + \left(\frac{2\mu}{\sigma^2} - 1\right) w(x) - \gamma \left(\frac{\mu-\lambda}{\gamma\sigma^2}\right)\left(\frac{\mu+\lambda}{\gamma\sigma^2}\right) = 0$$

$$w(0) = \frac{\mu-\lambda}{\gamma\sigma^2},$$

also satisfies the following terminal condition:

$$w(\log(u_\lambda/l_\lambda)) = \frac{\mu+\lambda}{\gamma\sigma^2} \quad \text{where} \quad u_\lambda/l_\lambda = \frac{1}{(1-\epsilon)}\frac{(\mu+\lambda)(\mu-\lambda-\gamma\sigma^2)}{(\mu-\lambda)(\mu+\lambda-\gamma\sigma^2)}$$

- Scalar equation for λ, implicit function theorem yields:

$$\lambda = \gamma\sigma^2 \left(\frac{3}{4\gamma}\pi_*^2 (1-\pi_*)^2\right)^{1/3} \epsilon^{1/3} + O(\epsilon).$$

- λ/σ^2 only depends on μ, σ through μ/σ^2

Main results and Implications

Optimal policy

Theorem

It is a long-run optimal policy to keep the risky weight within the buy and sell bounds

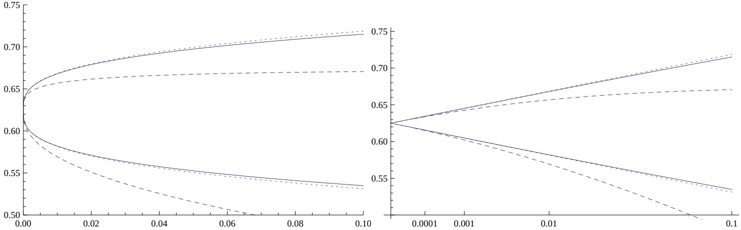
$$\pi_\pm = \frac{\mu \pm \lambda}{\gamma\sigma^2} = \pi_* \pm \left(\frac{3}{4\gamma}\pi_*^2 (1-\pi_*)^2\right)^{1/3} \epsilon^{1/3} + O(\epsilon).$$

- By evaluating π_-, π_+ in terms of the trading prices, the no-trade region becomes symmetric around $\pi_* = \mu/\gamma\sigma^2$
- Boundaries known explicitly in terms of μ, σ, γ and the gap λ
- At the first order, same result as in Janeček and Shreve (2004), i.e., investment and consumption separate
- As without transaction costs, policy only depends on investment opportunities μ, σ via mean-variance ratio μ/σ^2

Main results and Implications

Optimal policy

Buy and sell boundaries as functions of the spread ε:



- $\mu = 8\%, \sigma = 16\%, \gamma = 5$, zero discount rate for consumption
- First term of the expansion almost perfectly matches exact optimal weights
- Boundaries of Davis and Norman (1990) with consumption also close for $\epsilon < 1\%$, but start diverging for larger values

Main results

Welfare

Theorem

The certainty equivalent rate is given by $\beta = r + (\mu^2 - \lambda^2)/2\gamma\sigma^2$

- Long-run power investor is indifferent between
  - trading the risky asset with transaction costs
  - trading a frictionless asset with the same volatility σ but with lower expected excess return $\sqrt{\mu^2 - \lambda^2}$
- Hence, $\mu - \sqrt{\mu^2 - \lambda^2}$ is the liquidity premium introduced by Constantinidis (1986)
- Alternative interpretation: Need additional uncorrelated frictionless asset with volatility σ and excess return λ to offset transaction costs
- Certainty equivalent rate no longer solely depends on the Sharpe ratio μ/σ, due to different trading costs

Main results and Implications

Trading volume

Theorem

Relative turnover, *i.e., the number $d||\varphi||_t$ of shares traded as a fraction of the number $|\varphi_t|$ of shares held, has long-term average*

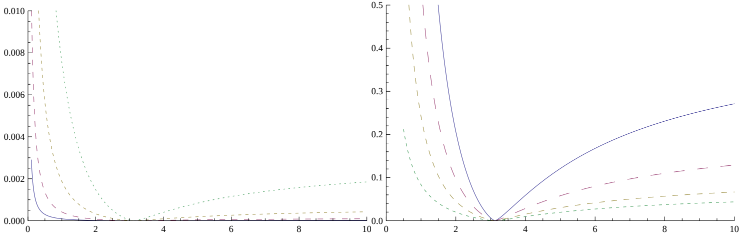
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{d\|\varphi\|_t}{|\varphi_t|} = \left(1-\frac{\mu-\lambda}{\gamma\sigma^2}\right)\frac{\sigma^2}{2}\left(\frac{\frac{2\mu}{\sigma^2}-1}{(u_\lambda/l_\lambda)^{\frac{2\mu}{\sigma^2}-1}-1}\right) + \left(1-\frac{\mu+\lambda}{\gamma\sigma^2}\right)\frac{\sigma^2}{2}\left(\frac{1-\frac{2\mu}{\sigma^2}}{(u_\lambda/l_\lambda)^{1-\frac{2\mu}{\sigma^2}}-1}\right)$$

- Proposed in the empirical literature to measure trading activity
- Also determined completely in terms of μ, σ, γ and the gap λ
- In our model, real-world trading volume is a puzzle for the opposite reason as the equity premium: The implied risk aversion is too low

Main results and Implications

Trading volume

Liquidity premium and share turnover as functions of risk aversion:



- $\mu = 8\%$, $\sigma = 16\%$, and $\epsilon = 0.01\%$, 0.1%, 1%, 10%
- Small liquidity premium (as in Constantinides (1986)) and trading volume without leverage ($\gamma > \mu/\sigma^2$)
- No trading and hence no liquidity premium for full investment
- Much larger liquidity premia and trading volume with leverage
- Similar behaviour of both quantities. Connection?

Main results and Implications

Trading volume

- Asymptotic expansion of relative share turnover:

$$\mathrm{ShT} = \frac{\sigma^2}{2}(1-\pi_*)^2\pi_*\left(\frac{3}{4\gamma}\pi_*^2(1-\pi_*)^2\right)^{-1/3}\epsilon^{-1/3} + O(\epsilon^{1/3})$$

- Asymptotic expansion of the liquidity premium:

$$\mathrm{LiP} = \frac{\mu}{2\pi_*^2}\left(\frac{3}{4\gamma}\pi_*^2(1-\pi_*)^2\right)^{2/3}\epsilon^{2/3} + O(\epsilon^{4/3})$$

- Universal relation, independent of model parameters and preferences:

$$\mathrm{LiP} = \frac{3}{4}\epsilon\,\mathrm{ShT} + O(\epsilon^{4/3}),$$

- As $\gamma \uparrow \infty$, relative turnover decreases due to lower risky weight, but increases due to smaller no-trade region
- Effects offset, giving a finite limit. Interpretation?

Heuristics and Proofs

Shadow Price

- Determine a "shadow price" $\tilde{S}$ lying in the bid-ask spread, whose long-run optimal portfolio only buys stocks when $\tilde{S} = S$ resp. sells when $\tilde{S} = (1-\epsilon)S$
- Optimizer for $\tilde{S}$ is then also optimal for $(\underline{S}, \overline{S})$
- Previously used for log-utility in Kallsen and M-K (2010), SG, Muhle-Karbe, and Schachermayer (2011)

Heuristics and Proofs

Shadow Price

Theorem

*The same growth rate **and** portfolio are long-run optimal in the BS model with transaction costs and in a frictionless market with **shadow price***

$$d\tilde{S}_t/\tilde{S}_t = (\tilde{\mu}(Y_t) + r)dt + \tilde{\sigma}(Y_t)dW_t$$

- $\tilde{\mu}(\cdot)$ and $\tilde{\sigma}(\cdot)$ are functions known explicitly in terms of λ
- Y is reflected BM with drift on an interval, corresponds to log-stock-cash ratio
- Tracks position of the portfolio in the no-trade region
- Constant coefficient market with transaction costs corresponds to frictionless market with stochastic coefficients
- Key tool for verification as in Kallsen & Muhle-Karbe (2010), SG, Muhle-Karbe & Schachermayer (2011) for log-utility

Heuristics and Proofs

Heuristics

Standard control arguments:

- Long-run ansatz $V(t,x,y) = x^{1-\gamma} v(y/x) e^{-(1-\gamma)(r+\beta)t}$ formally yields linear reduced form HJB equation:

$$\frac{\sigma^2}{2} z^2 v''(z) + \mu z v'(z) - (1-\gamma)\beta v(z) = 0 \qquad \text{if } l < z < u,$$
$$((1+\epsilon) + l) v'(l) - (1-\gamma) v(l) = 0$$
$$(1/(1+\epsilon) + u) v'(u) - (1-\gamma) v(u) = 0$$

- Growth rate β, free boundaries l, u unknown, therefore add smooth-pasting conditions:

$$((1+\epsilon) + l) v''(l) + \gamma v'(l) = 0$$
$$(1/(1+\epsilon) + u) v''(u) + \gamma v'(u) = 0$$

Heuristics and Proofs

Heuristics

As in Dumas and Luciano (1991), plug the boundary conditions into the ODE and simplify. But reparametrize using...

- the risky fraction $\pi_- = \frac{l}{1+l}$ of wealth in terms of the ask price S when it is time to buy
- the risky fraction $\pi_+ = \frac{(1-\epsilon)u}{1+(1-\epsilon)u}$ of wealth in terms of the bid price $(1-\epsilon)S$ when it is time to sell

Then:

$$\pi_\pm = \frac{\mu}{\gamma\sigma^2} \pm \frac{\sqrt{\mu^2 - 2\beta\gamma\sigma^2}}{\gamma\sigma^2},$$

and

$$\beta =: \frac{\mu^2 - \lambda^2}{2\gamma\sigma^2} \Rightarrow \pi_\pm = \frac{\mu \pm \lambda}{\gamma\sigma^2}$$

- Remains to determine λ

Heuristics and Proofs

Heuristics

- Plug u, l in terms of λ into the boundary conditions and do some substitutions for the HJB equation
- Find that λ has to be chosen so that the solution to the Riccati initial value problem

  $$w'(x) + (1-\gamma)w(x)^2 + \left(\frac{2\mu}{\sigma^2} - 1\right) w(x) - \gamma \left(\frac{\mu-\lambda}{\gamma\sigma^2}\right)\left(\frac{\mu+\lambda}{\gamma\sigma^2}\right) = 0$$
  $$w(0) = \frac{\mu-\lambda}{\gamma\sigma^2},$$

  also satisfies the second boundary condition

  $$w(\log(u_\lambda/l_\lambda)) = \frac{\mu+\lambda}{\gamma\sigma^2} \quad \text{where} \quad u_\lambda/l_\lambda = \frac{1}{(1-\epsilon)}\frac{(\mu+\lambda)(\mu-\lambda-\gamma\sigma^2)}{(\mu-\lambda)(\mu+\lambda-\gamma\sigma^2)}$$

- This is the characterization for λ from above, asymptotics follow from implicit function theorem

Heuristics and Proofs

Proofs

- Reduced value function with transaction costs is a function of the stock-cash ratio, $e^Y = \varphi S/\varphi^0$, normalized to lie in $[1, u/l]$
- Assume the same holds for the ratio $\tilde{S}/S$
- Since S is a multiple of e^Y in the no-trade region, this is most conveniently written as $\tilde{S} = Se^{-Y}g(e^Y)$, as in SG, Muhle-Karbe, and Schachermayer (2011)
- Y follows Brownian motion in the no-trade region
- To keep it in $[0, \log(u/l)]$, complement this by reflection at the boundaries:

$$dY_t = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t + dL_t - dU_t,$$

where L, U are the local times at 0 and $\log(u/l)$

Heuristics and Proofs

Proofs

So how to determine g in the ansatz $\tilde{S} = Se^{-Y}g(e^Y)$?

- To ensure $\tilde{S} = \overline{S}$ resp. $\tilde{S} = \underline{S}$ at the boundaries and make $\tilde{S}$ an Itô process, impose:

$$g(1) = 1,\ g'(1) = 1,\ g(u/l) = (1-\epsilon)u/l,\ g'(u/l) = 1-\epsilon$$

- Using long-run ansatz for frictionless markets, derive reduced HJB equation for generic g
- Then, to find the shadow price, match the corresponding value function to its counterpart in the transaction cost problem
- This gives a (complicated) ODE for g
- But we are lucky...

Heuristics and Proofs

Proofs

- There is an explicit solution for g in terms of λ, namely

$$g(e^y) =$$
$$\begin{cases} \frac{1}{1-\epsilon}\left(1+\frac{\gamma\sigma^2}{\mu-\lambda}\left(\left(1+\frac{\mu-\lambda}{\gamma\sigma^2}\left(\frac{b}{b^2-a^2}-\frac{a}{b^2-a^2}\tanh\left[\tanh^{-1}\left(\frac{b}{a}\right)-ay\right]\right)\right)^{-1}-1\right)\right) \\ \frac{1}{1-\epsilon}\left(1+\frac{\gamma\sigma^2}{\mu-\lambda}\left(\left(1+\frac{\mu-\lambda}{\gamma\sigma^2}\left(\frac{b}{b^2+a^2}-\frac{a}{a^2+b^2}\tan\left[\tan^{-1}\left(\frac{b}{a}\right)+ay\right]\right)\right)^{-1}-1\right)\right) \\ \frac{1}{1-\epsilon}\left(1+\frac{\gamma\sigma^2}{\mu-\lambda}\left(\left(1+\frac{\mu-\lambda}{\gamma\sigma^2}\left(\frac{b}{b^2-a^2}-\frac{a}{b^2-a^2}\coth\left[\coth^{-1}\left(\frac{b}{a}\right)-ay\right]\right)\right)^{-1}-1\right)\right) \end{cases}$$

where the different cases depend on μ, σ, γ, and

$$a = \frac{1}{2}\sqrt{\left|4(\gamma-1)\frac{\mu^2-\lambda^2}{\gamma\sigma^4} - \left(1-\frac{2\mu}{\sigma^2}\right)^2\right|}, \quad b = \frac{1}{2} - \frac{\mu}{\sigma^2} + (\gamma-1)\frac{\mu-\lambda}{\gamma\sigma^2}$$

- This function satisfies the smooth pasting conditions and the ODE derived above

Heuristics and Proofs

Proofs

- $\tilde{S}$ is an Itô process, its drift and diffusion coefficients are known functions of reflected BM with Drift Y
- Solution to the ergodic HJB equation for $\tilde{S}$ known explicitly by construction, yields optimal growth rate β and portfolio for $\tilde{S}$
- Straightforward calculations: Corresponding number φ of stocks is

$$\frac{d\varphi_t}{\varphi_t} = \left(1 - \frac{\mu - \lambda}{\gamma\sigma^2}\right) dL_t - \left(1 - \frac{\mu + \lambda}{\gamma\sigma^2}\right) dU_t$$

- Local times L and U only increase when $Y = 0$ resp. when $Y = \log(u/l)$, i.e., when $\tilde{S} = S$ resp. $\tilde{S} = (1 - \epsilon)S$, hence $\tilde{S}$ is a shadow price
- Same portfolio and growth rate are also long-run optimal with transaction costs

Summary

Results

Portfolio choice problem of a CRRA investor with long horizon:

- Black-Scholes model or, more generally, models with constant mean-variance ratio
- No-trade region, certainty equivalent rate, liquidity premium, and trading volume determined in terms of model parameters and the gap λ, which is the solution to a scalar equation
- For small transaction costs $\epsilon \downarrow 0$: Asymptotic expansions of arbitrary order for all quantities
- Rigorous verification by determining a shadow price

Summary

Implications

- The no-trade region is symmetric around the frictionless solution, when evaluated using trading prices
- Trading boundaries only depend on the model parameters through the mean-variance ratio
- At the first order, trading volume, liquidity premium, and spread are linked independent of preference and model parameters, which leads to testable implications
- With leverage: Much bigger impact of transaction costs
- In our model, real-world trading volume is a puzzle for the opposite reason as the equity premium: The implied risk aversion is too low