Extrapolation analytics for Dupire's local volatility

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Implied vol and local vol

- **Implied volatility surface**
  - Asymptotics well understood
  - Berestycki-Busca-Florent formula (short maturity)
  - Lee's moment formula (large strike)
- **Local volatility surface**
  - Few asymptotic results
  - We give a new wing approximation
  - General heuristic; rigorous for Heston model
  - Applications: parametrization design, toxicity index (model risk)

Implied vol: Lee's moment formula (2004)

- Given call price surface

$$C = C(K,T) = C_{\mathrm{BS}}(K,T;\sigma_{\mathrm{imp}}(K,T))$$

- First order strike asymptotics ($k = \log K$)

$$\limsup_{k\to\infty} \frac{\sigma_{\mathrm{imp}}(K,T)\sqrt{T}}{\sqrt{k}} = const$$

- Applications: Calibration; parametrization design (must grow no faster than $\sqrt{k}$)
- Refinements by Benaim, Friz (2008, 2009), Gulisashvili (2010), Gao and Lee (2011)

Local volatility

- Given call price surface $C(K, T)$
- $\sigma_{\mathrm{loc}}(\cdot, \cdot)$ is a function such that the diffusion

$$dS_t/S_t = \sigma_{\mathrm{loc}}(S_t, t)dW_t$$

reproduces the given call prices:

$$\mathbb{E}[(S_T - K)^+] = C(K, T)$$

- Dupire's formula (1994)

$$\sigma^2_{\mathrm{loc}}(K, T) = \frac{2\partial_T C}{K^2 \partial_{KK} C}$$

Typical use of local volatility

- Observe **market call prices** $C(K, T)$ for a finite set of strikes K and maturities T
- Interpolate smoothly
- Calibrate a parametric local vol surface
- Use the resulting local vol model to price exotic options by Monte Carlo
- We assume instead: Call prices $C(K, T)$ are **generated by a a model** (e.g., Heston)

Heston Model

- Consider Call price surface $C_{\text{Hes}}(K, T)$ generated by Heston:

$$dS_t = S_t \sqrt{V_t} dW_t, \qquad S_0 = 1,$$
$$dV_t = (a + bV_t)\, dt + c\sqrt{V_t} dZ_t, \qquad V_0 = v_0 > 0,$$

- Correlated Brownian motions

$$d\langle W, Z\rangle_t = \rho dt, \qquad \rho \in [-1, 1]$$

- Parameters

$$a \geq 0, b \leq 0, c > 0$$

Local vol in the Heston model

- Heston dynamics $\Longrightarrow$ Call prices $\Longrightarrow$ local vol surface
- Dupire's formula

$$\sigma_{\mathrm{loc}}^2(K,T) = \frac{2\partial_T C_{\mathrm{Hes}}}{K^2 \partial_{KK} C_{\mathrm{Hes}}}$$

- New wing asymptotics ($k = \log K$)

$$\begin{aligned} \sigma_{\mathrm{loc}}^2(K,T) &\sim \mathit{const} \times k, \qquad && K \to \infty \\ \sigma_{\mathrm{loc}}^2(K,T) &\sim \mathit{const} \times |k|, \qquad && K \to 0 \end{aligned}$$

- Similarly for the Stein-Stein model (Friz, De Marco 2012; large deviations)

Local vol in the Heston model

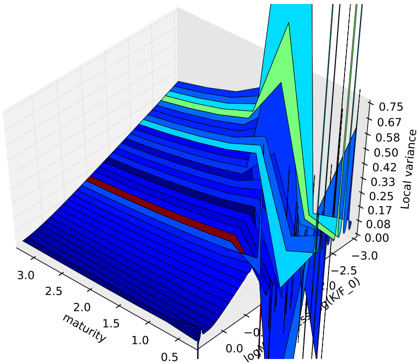

Figure: Local variance for Heston model computed with Dupire's formula. Call price derivatives computed via 1D integration of Heston characteristic function on a fixed integration contour.

Local vol in the Heston model

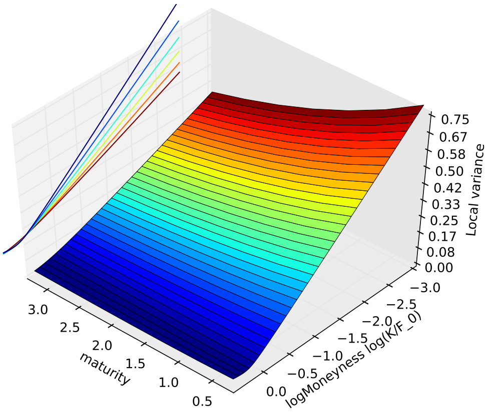

Figure: Local variance for Heston model computed with Dupire's formula. Adaptive contour with shift into saddle point. Note the linear increase.

Application 1: Design local vol parametrizations

- Example: Gatheral's SVI parametrization
- Popular parametrization of the **implied vol** surface

$$\sigma_{\mathrm{imp}}(K,T)^2 T \approx SVI(k; a, b, c, m, s)$$
$$k \mapsto a + b\left((-m+k)c + \sqrt{(-m+k)^2 + s}\right)$$

- Gatheral, Jacquier 2011: Heston, $T \to \infty \Rightarrow$ SVI
- Wings ($k \to \pm\infty$) compatible with Lee's formula
- Our asymptotic result motivates SVI parametrization also for **local vol** $\sigma_{\mathrm{loc}}(K,T)$
- There exist arbitrage-free call price surfaces whose local vol has this wing behavior

Application 2: Model risk

- Consider a path-dependent exotic
- SV = price under **stochastic vol model**
- LV = price under associated **local vol model**
- Note: local vol model recreates marginals of stoch vol model, but not the full law $\Longrightarrow$ in general $SV \neq LV$
- Similar price: low model risk (e.g., variance swap)
- Different price: high model risk (e.g, volatility swap)
- **Toxicity index** (Reghai 2011)

$$I = \frac{|SV - LV|}{|SV + LV|}$$

Application 2: Model risk

- How to calculate local vol of a stochastic vol model?
- We need $\sigma_{\text{loc}}(K, T)$ in particular for large/small K (Monte Carlo requires it)
- Dupire's formula + Fourier inversion: unstable for large/small K
- Conditioning:

$$\sigma^2_{\text{loc}}(K, T) = E[\sigma^2_{\text{stoch}}(T) | S_T = K]$$

  Difficult for $K \gg S_0$ (condition on unlikely events)
- $\Rightarrow$ Wing approximation useful for computation

Towards a general wing approximation of local vol

- Moment generating function ($X_T = \log S_T$):

$$M(s,T) = E[\exp(s, X_T)], \qquad m(s,T) = \log M(s,T)$$

- Dupire's formula + Fourier inversion

$$\sigma^2_{\mathrm{loc}}(K,T) = \frac{2\partial_T C}{K^2 \partial_{KK} C} = \frac{2\int_{-i\infty}^{i\infty} \frac{\partial_T m(s,T)}{s(s-1)} e^{-ks} M(s,T)ds}{\int_{-i\infty}^{i\infty} e^{-ks} M(s,T)ds}$$

- Saddle point method: Leading terms are integrands evaluated at saddle point $\longrightarrow$ cancellation

General wing formula for local vol

- log moment generating function ($X_T = \log S_T$)

$$m(s,T) = \log E[\exp(s, X_T)]$$

- saddle point $\hat{s}(k,T)$

$$\left.\frac{\partial}{\partial s} m(s,T)\right|_{s=\hat{s}} = k$$

- "Lee type" wing formula for $k \to \infty$:

$$\boxed{\sigma^2_{\mathrm{loc}}(K,T) \approx \left.\frac{2\frac{\partial}{\partial T} m(s,T)}{s(s-1)}\right|_{s=\hat{s}(k,T)}}$$

Saddle point approximation of the numerator

$$
\begin{aligned}
&2\int_{-i\infty}^{i\infty} \frac{\partial_T m(s,T)}{s(s-1)} e^{-ks} M(s,T)\,ds \\
&\sim 2\int_{\hat{s}-ih(k)}^{\hat{s}+ih(k)} \frac{\partial_T m(s,T)}{s(s-1)} e^{-ks} M(s,T)\,ds \\
&\sim 2e^{m(\hat{s},T)-k\hat{s}} \int_{\hat{s}-ih(k)}^{\hat{s}+ih(k)} \frac{\partial_T m(\hat{s},T)}{\hat{s}(\hat{s}-1)} \exp\left(\tfrac{1}{2} m''(\hat{s},T)(s-\hat{s})^2\right) ds \\
&\sim 2\frac{\partial_T m(\hat{s},T)}{\hat{s}(\hat{s}-1)} e^{m(\hat{s},T)-k\hat{s}} \int_{\hat{s}-ih(k)}^{\hat{s}+ih(k)} \exp\left(\tfrac{1}{2} m''(\hat{s},T)(s-\hat{s})^2\right) ds.
\end{aligned}
$$

Approximation of the denominator: Same, but without the factor $2\frac{\partial_T m(\hat{s},T)}{\hat{s}(\hat{s}-1)}$

Two ways to use the formula

- As it is (numerically very accurate, but not quite explicit):

$$\boxed{\sigma^2_{\mathrm{loc}}(K,T) \approx \left. \frac{2\frac{\partial}{\partial T} m(s,T)}{s(s-1)} \right|_{s=\hat{s}(k,T)}}$$

- Use asymptotics of saddle point $\hat{s}(k,T)$ and mgf $\Longrightarrow$ explicit formula (model-dependent)
- E.g., *const* $\times\, k$ for Heston. Explicit, but model-dependent and less accurate.

Heston model: Numerical example (left wing)

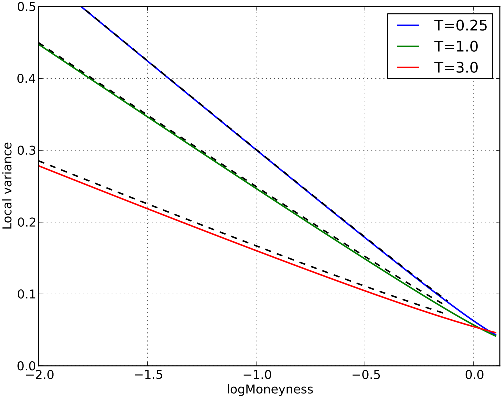

Figure: Local variance $\sigma^2_{\text{loc}}(k, T)$ and our approximation in the Heston model.

Heston model: Numerical example (right wing)

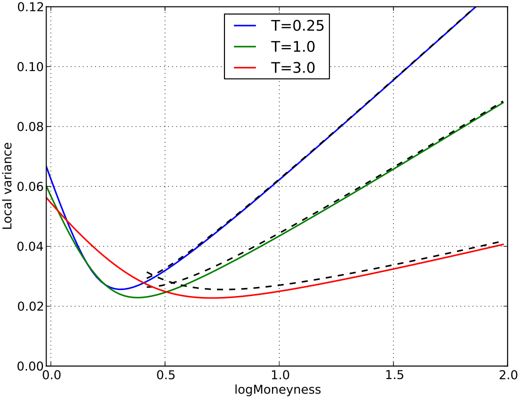

Figure: Local variance $\sigma^2_{\mathrm{loc}}(k, T)$ and our approximation in the Heston model.

Heston model: Accuracy by strike and maturity

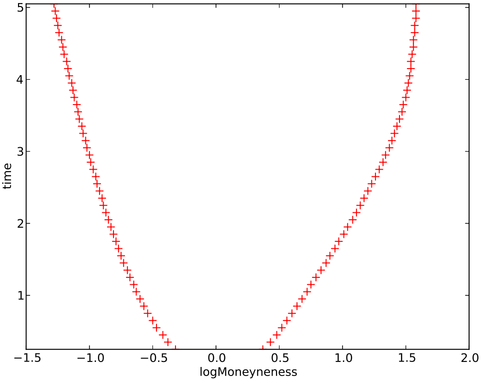

Figure: Boundaries of the region where the relative error of our approximation is less than 5%.

Heston model: Implied volatility, $T = 0.25$

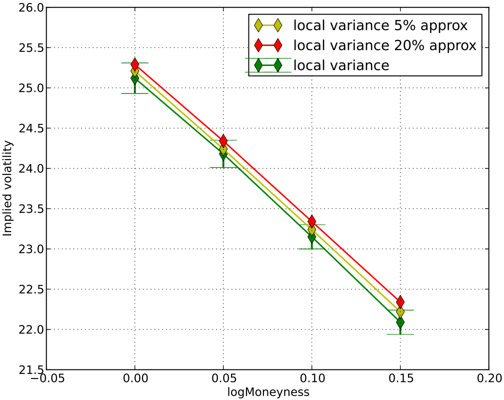

Figure: Green: Local vol computed by Dupire's formula. Red: Use our approximation, as soon as its accuracy is over 20%. Yellow: Same, with 5%.

Heston model: Implied volatility, $T = 1$

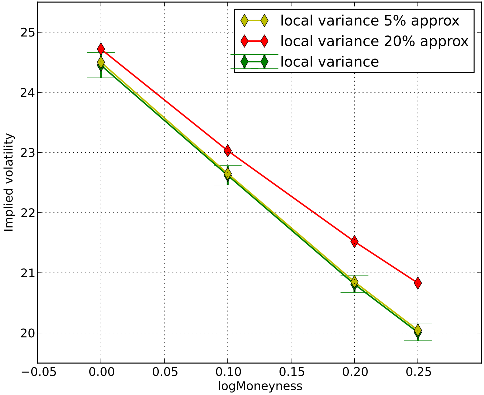

Figure: Green: Local vol computed by Dupire's formula. Red: Use our approximation, as soon as its accuracy is over 20%. Yellow: Same, with 5%.

Heston model: rigorous proof

- Finding saddle point + local expansion of integrands fairly routine
- Problem: Verify concentration
- Needs some insight into behaviour of integrand away from saddle point
- Show exponential decay of integrands by ODE comparison (Riccati ODEs, similar to Friz, SG, Gulisashvili, Sturm, Quantitative Finance 2011)

Using Dupire's formula for models with jumps

- Variance gamma model: Call price not smooth enough for Dupire's formula (but works for T large)
- Even if Dupire's formula is well-defined, the local vol model may not match the marginals of the jump process.
- Our wing approximation works also for jump models:

$$\sigma^2_{\mathrm{loc}}(K,T) \sim c_T k^{1/2} \qquad \text{Kou model}$$
$$\sigma^2_{\mathrm{loc}}(K,T) \sim c_T \log k \qquad \text{Variance gamma model}$$

References

- De Marco, Friz, SG: *Rational Shapes of the Local Volatility Surface* (submitted to RISK, 2012).
- De Marco, Friz: *Large deviations for diffusions and local volatilities,* working paper, 2012.
- Friz, SG: *Don't stay local – extrapolation analytics for Dupire's local volatility,* arXiv preprint, 2011.

- Work in progress: **Prove** wing formula in good generality (not just Heston and Stein-Stein).