On refined volatility smile expansion in the Heston model

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Overview

- The Heston model
- Implied volatility asymptotics
- Local volatility asymptotics
Heston Model (1993)

- **Dynamics**

\[
    dS_t = S_t \sqrt{V_t} dW_t, \quad S_0 = 1,
\]
\[
    dV_t = (a + bV_t) \, dt + c \sqrt{V_t} \, dZ_t, \quad V_0 = v_0 > 0,
\]

- **Correlated Brownian motions**

\[
    d\langle W, Z \rangle_t = \rho \, dt, \quad \rho \in [-1, 1]
\]

- **Parameters**

\[
    a \geq 0, \quad b \leq 0, \quad c > 0
\]
Option pricing by Fourier Transform

- Characteristic function, Fourier Inversion
- Numerical problems (oscillations) for extreme maturities and/or strikes
- Time consuming (e.g.: calibration, credit exposure)
- Possible remedy for calibration: Asymptotic approximations for prices and implied vol
- Can serve as initial values for optimization
Short maturity asymptotics

- Forde, Jacquier 2009: First order term (large deviations, Gärtner-Ellis theorem)
- Forde, Jacquier, Lee 2011:
  \[ \sigma_{BS}^2(k, T) = \sigma_0^2(k) + a(k) T + o(T), \quad T \to 0 \]
- \( \sigma_0(k), a(k) \) semi-explicit functions
- saddle point approximation
Lee’s moment formula (2004)

- First order strike asymptotics
- Model-free result
- Relates critical moment to implied volatility

\[ s^* := \sup \{ s : E[S^s_T] < \infty \} \]

\[ s^* =: \frac{1}{2\beta_1^2} + \frac{\beta_1^2}{8} + \frac{1}{2} \]

\[ \limsup_{k \to \infty} \frac{\sigma_{BS}(k, T)\sqrt{T}}{\sqrt{k}} = \beta_1 \]

Figure: Heston model: Implied variance $\sigma_{BS}(k,1)^2$ in terms of log-strikes compared to the first order (dashed) and third order (dotted) approximations.
Consider a fixed maturity $T > 0$.

Density of spot for $x \to \infty$

$$D_T(x) = A_1 x^{-A_3} e^{A_2 \sqrt{\log x}} (\log x)^{-3/4+a/c^2} \left(1 + O((\log x)^{-1/2})\right)$$

Implied volatility for $k = \log K \to \infty$

$$\sigma_{BS}(k, T) \sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{1}{k^{1/2}}\right)$$
Interpretation of smile expansion

- Implied volatility for $k = \log K \to \infty$

\[
\sigma_{BS}(k, T) \sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O \left( \frac{1}{k^{1/2}} \right)
\]

- $\beta_1$ does not depend on $\sqrt{v_0}$
- $\beta_2$ depends linearly on $\sqrt{v_0}$
- Changes of $\sqrt{v_0}$ have second-order effects
- Increase $\sqrt{v_0}$: parallel shift, slope not affected
- Changes in mean-reversion level $\bar{v} = -a/b$ seen only in $\beta_3$
General remarks

- Constants depend on: critical moment, critical slope, critical curvature
- Critical moment etc. defined in a model-free manner
- Closed form of characteristic function not needed
- Work only with affine principles (Riccati equations)
Moment generating function
\[ M(s) = E[e^{sX_t}] = \exp(\phi(s, t) + \nu_0\psi(s, t)) \]

Riccati equations
\[ \partial_t\phi = F(s, \psi), \quad \phi(0) = 0, \]
\[ \partial_t\psi = R(s, \psi), \quad \psi(0) = 0 \]
\[ F(s, \nu) = av, \]
\[ R(s, \nu) = \frac{1}{2}(s^2 - s) + \frac{1}{2}c^2\nu^2 + bv + s\rho c\nu \]

Explicit solution possible, but cumbersome expression
Critical moment for time $T$

$$s^* := \sup\{ s \geq 1 : E[S_t^s] < \infty \}$$

Explosion time for moment of order $s$

$$T^*(s) = \sup\{ t \geq 0 : E[S_t^s] < \infty \}$$

Critical slope, critical curvature:

$$\sigma := -\partial_s T^*|_{s^*} \geq 0 \quad \text{and} \quad \kappa := \partial^2_s T^*|_{s^*}$$
Explicit Explosion time for the Heston model

- Explosion time for moment of order $s$ is explicit:

$$T^*(s) = \frac{2}{\sqrt{-\Delta(s)}} \left( \arctan \frac{\sqrt{-\Delta(s)}}{s\rho c + b} + \pi \right),$$

$$\Delta(s) := (s\rho c + b)^2 - c^2 (s^2 - s)$$

- Critical moment $s^*$: Find numerically from

$$T^*(s^*) = T.$$
Saddle point method

- Moment generating function of log-spot: $M(s) = E[e^{sX_T}]$
- Density of $S_T$ by Fourier inversion:

$$D_T(x) = \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} x^{-s-1} M(s) ds$$

- Integrand has singularity at critical moment
- Shift contour through a saddle point of the integrand.
- For large $x$, the integral is concentrated around the saddle.
- Local expansion of integrand yields expansion of whole integral.
The surface $|x^{-s-1}M(s)|$
Asymptotics of \( \psi \) and \( \phi \) near critical moment

- Recall \( M(s) = \exp(\phi(s, t) + v_0 \psi(s, t)) \)
- For \( s \to s^* \) we have (with \( \beta := \sqrt{2v_0/c\sqrt{\sigma}} \))

\[
\psi(s, T) = \frac{\beta^2}{s^* - s} + \text{const} + O(s^* - s),
\]
\[
\phi(s, T) = \frac{2a}{c^2} \log \frac{1}{s^* - s} + \text{const} + O(s^* - s)
\]

- Found from Riccati equations
Finding the saddle point: $0 = \text{derivative of integrand}$

Use only first order expansion:

$$0 = \frac{\partial}{\partial s} x^{-s-1} \exp \left( \frac{\beta^2}{s^* - s} \right)$$

Approximate saddle point at

$$\hat{s}(x) = s^* - \frac{\beta}{\sqrt{\log x}}$$
Tail estimate

- Finding saddle point + local expansion fairly routine
- Problem: Verify concentration
- Needs some insight into behaviour of function away from saddle point
- Show exponential decay by ODE comparison (Riccati ODEs)
Result of saddle point method

- Density asymptotics for $x \to \infty$

$$D_T(x) = A_1 x^{-A_3} e^{A_2 \sqrt{\log x}} (\log x)^{-3/4 + a/c^2} (1 + O((\log x)^{-1/2}))$$

- Constants in terms of critical moment and critical slope:

$$A_3 = s^* + 1 \quad \text{and} \quad A_2 = 2 \frac{\sqrt{2v_0}}{c \sqrt{\sigma}}$$

- Easily extended to full asymptotic expansion
Call prices and Smile asymptotics

- Gulisashvili (2010): Assumes that density of spot varies regularly at infinity

\[ D_T(x) = x^{-\gamma} h(x), \]

- \( h \) varies slowly at infinity, \( \gamma > 2 \)
- Expansions of call prices and implied volatility
- Similarly for left tail \( (x \to 0, \ k = \log K \to -\infty) \)
Call prices

• Call price for strike $K \to \infty$

$$C(K) = \frac{A_1}{(-A_3 + 1)(-A_3 + 2)} K^{-A_3 + 2} e^{A_2 \sqrt{\log K}} (\log K)^{-\frac{3}{4}} + \frac{a}{c^2}$$

$$\times \left( 1 + O \left( (\log K)^{-\frac{1}{4}} \right) \right)$$
Smile asymptotics

- Implied volatility for log-strike $k \to \infty$
  \[
  \sigma_{BS}(k, T)\sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{1}{k^{1/2}}\right)
  \]

- Gao, Lee (2011): refinement to $O(k^{-3/4})$. They use a higher order transfer result (call price $\to$ implied vol)
Local volatility

- Given call price surface

\[ C = C(K, T) = C_{BS}(K, T; \sigma_{BS}(K, T)) \]

- Reproduced by local volatility model

\[ \frac{dS_t}{S_t} = \sigma_{loc}(S_t, t) dW_t \]

- Dupire’s formula (1994)

\[ \sigma^2_{loc}(K, T) = \frac{2\partial_T C}{K^2 \partial_{KK} C} \]
Suppose that call prices are given by Heston model.

The resulting local vol satisfies

\[ \sigma^2_{loc}(e^k, T) \sim \text{const} \times k, \quad k \to \infty. \]

Proof: Dupire’s formula, Riccati equations, saddle point method.
General wing formula for local vol

- log moment generating function \((X_T = \log\text{-spot})\)
  \[ m(s, T) = \log E[\exp(s, X_T)] \]

- saddle point \(\hat{s}(k, T)\)
  \[ \frac{\partial}{\partial s} m(s, T) \bigg|_{s=\hat{s}} = k \]

- Wing formula for \(k \to \infty\):
  \[ \sigma^2_{\text{loc}}(e^k, T) \approx \frac{2 \frac{\partial}{\partial T} m(s, T)}{s(s-1)} \bigg|_{s=\hat{s}(k, T)} \]
Local vol wing formula: Proof idea

- Dupire’s formula + Fourier inversion

\[
\sigma^2_{\text{loc}}(K, T) = \frac{2 \partial_T C}{K^2 \partial_{KK} C} \\
= \frac{2 \int_{-i\infty}^{i\infty} \frac{\partial_T m(s, T)}{s(s-1)} e^{-ks} M(s, T) ds}{\int_{-i\infty}^{i\infty} e^{-ks} M(s, T) ds}
\]

- saddle point method \(\rightarrow\) cancellation
Figure: Heston model: approximation of $\sigma_{loc}^2(e^k, T)$ for large $k$. Dashed: wing formula; dotted: const $\times k$
Possible applications of local vol asymptotics

- Extrapolate market data to non-liquid ranges, in a way consistent with Heston or other chosen model
- Local vol asymptotics for jump models: Test given call price surface for jump behavior


