On refined volatility smile expansion in the Heston model

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ICIAM, July 21 2011

Overview

- The Heston model
- Implied volatility asymptotics
- Local volatility asymptotics

Heston Model (1993)

- Dynamics

$$dS_t = S_t \sqrt{V_t} dW_t, \qquad S_0 = 1,$$
$$dV_t = (a + bV_t)\, dt + c\sqrt{V_t} dZ_t, \qquad V_0 = v_0 > 0,$$

- Correlated Brownian motions

$$d\langle W, Z\rangle_t = \rho dt, \qquad \rho \in [-1, 1]$$

- Parameters

$$a \geq 0, b \leq 0, c > 0$$

Option pricing by Fourier Transform

- Characteristic function, Fourier Inversion
- Numerical problems (oscillations) for extreme maturities and/or strikes
- Time consuming (e.g.: calibration, credit exposure)
- Possible remedy for calibration: Asymptotic approximations for prices and implied vol
- Can serve as initial values for optimization

Short maturity asymptotics

- Forde, Jacquier 2009: First order term (large deviations, Gärtner-Ellis theorem)
- Forde, Jacquier, Lee 2011:

$$\sigma^2_{BS}(k,T) = \sigma^2_0(k) + a(k)T + o(T), \qquad T \to 0$$

- $\sigma_0(k)$, $a(k)$ semi-explicit functions
- saddle point approximation

Lee's moment formula (2004)

- First order strike asymptotics
- Model-free result
- Relates critical moment to implied volatility

$$s^* := \sup\{s : E[S_T^s] < \infty\}$$

$$s^* =: \frac{1}{2\beta_1^2} + \frac{\beta_1^2}{8} + \frac{1}{2}$$

$$\limsup_{k\to\infty} \frac{\sigma_{BS}(k,T)\sqrt{T}}{\sqrt{k}} = \beta_1$$

- Refinements by Benaim, Friz (2008, 2009), Gulisashvili (2010), Gao and Lee (2011)

Heston Smile

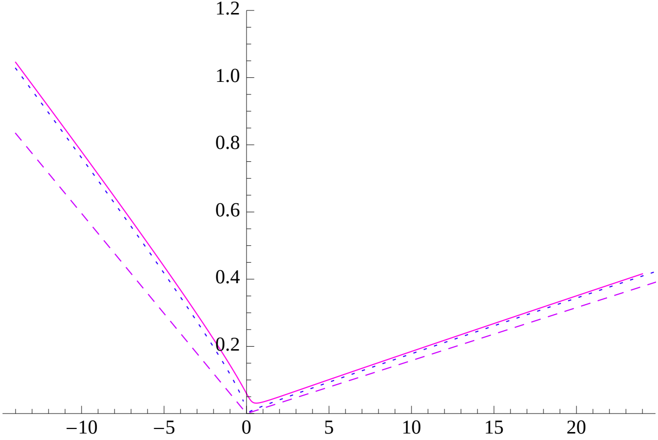


Figure: Heston model: Implied variance $\sigma_{BS}(k,1)^2$ in terms of log-strikes compared to the first order (dashed) and third order (dotted) approximations.

Heston density and implied vol expansion, SG et al. 2011

- Consider a fixed maturity $T > 0$.
- Density of spot for $x \to \infty$

$$D_T(x) = A_1 x^{-A_3} e^{A_2\sqrt{\log x}} (\log x)^{-3/4+a/c^2} \left(1+O((\log x)^{-1/2})\right)$$

- Implied volatility for $k = \log K \to \infty$

$$\sigma_{BS}(k, T)\sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{1}{k^{1/2}}\right)$$

Interpretation of smile expansion

- Implied volatility for $k = \log K \to \infty$

$$\sigma_{BS}(k, T)\sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{1}{k^{1/2}}\right)$$

- β_1 does not depend on $\sqrt{v_0}$
- β_2 depends linearly on $\sqrt{v_0}$
- Changes of $\sqrt{v_0}$ have second-order effects
- Increase $\sqrt{v_0}$: parallel shift, slope not affected
- Changes in mean-reversion level $\bar{v} = -a/b$ seen only in β_3

General remarks

- Constants depend on: critical moment, critical slope, critical curvature
- Critical moment etc. defined in a model-free manner
- Closed form of characteristic function not needed
- Work only with affine principles (Riccati equations)

Heston Model: Mgf of log-spot X_t

- Moment generating function

$$M(s) = E[e^{sX_t}] = \exp(\phi(s,t) + v_0\psi(s,t))$$

- Riccati equations

$$\begin{aligned} \partial_t \phi &= F(s,\psi), \;\; \phi(0) = 0, \\ \partial_t \psi &= R(s,\psi), \;\; \psi(0) = 0 \end{aligned}$$

$$\begin{aligned} F(s,v) &= av, \\ R(s,v) &= \frac{1}{2}(s^2 - s) + \frac{1}{2}c^2v^2 + bv + s\rho c v \end{aligned}$$

- Explicit solution possible, but cumbersome expression

Moment explosion

- Critical moment for time T

$$s^* := \sup \{s \geq 1 : E[S_T^s] < \infty\}$$

- Explosion time for moment of order s

$$T^*(s) = \sup \{t \geq 0 : E[S_t^s] < \infty\}$$

- Critical slope, critical curvature:

$$\sigma := -\partial_s T^*|_{s^*} \geq 0 \qquad \text{and} \qquad \kappa := \partial_s^2 T^*|_{s^*}$$

Explicit Explosion time for the Heston model

- Explosion time for moment of order s is explicit:

$$T^*(s) = \frac{2}{\sqrt{-\Delta(s)}} \left(\arctan \frac{\sqrt{-\Delta(s)}}{s\rho c + b} + \pi \right),$$

$$\Delta(s) := (s\rho c + b)^2 - c^2 \left(s^2 - s \right)$$

- Critical moment s^*: Find numerically from

$$T^*(s^*) = T.$$

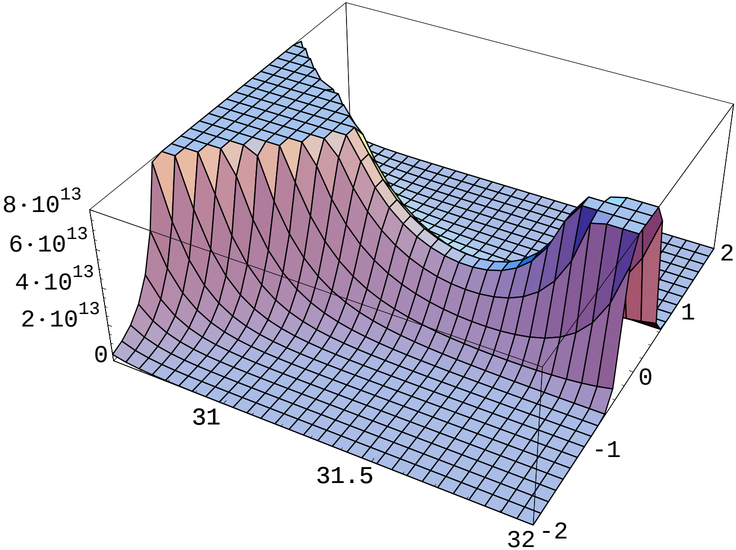
Saddle point method

- Moment generating function of log-spot: $M(s) = E[e^{sX_T}]$
- Density of S_T by Fourier inversion:

$$D_T(x) = \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} x^{-s-1} M(s) ds$$

- Integrand has singularity at critical moment
- Shift contour through a saddle point of the integrand.
- For large x, the integral is concentrated around the saddle.
- Local expansion of integrand yields expansion of whole integral.

The surface $|x^{-s-1}M(s)|$



Asymptotics of ψ and ϕ near critical moment

- Recall $M(s) = \exp(\phi(s,t) + v_0 \psi(s,t))$
- For $s \to s^*$ we have (with $\beta := \sqrt{2v_0}/c\sqrt{\sigma}$)

$$\psi(s,T) = \frac{\beta^2}{s^* - s} + const + O(s^* - s),$$
$$\phi(s,T) = \frac{2a}{c^2} \log \frac{1}{s^* - s} + const + O(s^* - s)$$

- Found from Riccati equations

Saddle point method

- Finding the saddle point: $0 =$ derivative of integrand
- Use only first order expansion:

$$0 = \frac{\partial}{\partial s} x^{-s-1} \exp\left(\frac{\beta^2}{s^* - s}\right)$$

- Approximate saddle point at

$$\hat{s}(x) = s^* - \beta/\sqrt{\log x}$$

Tail estimate

- Finding saddle point + local expansion fairly routine
- Problem: Verify concentration
- Needs some insight into behaviour of function away from saddle point
- Show exponential decay by ODE comparison (Riccati ODEs)

Result of saddle point method

- Density asymptotics for $x \to \infty$

$$D_T(x) = A_1 x^{-A_3} e^{A_2 \sqrt{\log x}} (\log x)^{-3/4 + a/c^2} \left(1 + O((\log x)^{-1/2})\right)$$

- Constants in terms of critical moment and critical slope:

$$A_3 = s^* + 1 \qquad \text{and} \qquad A_2 = 2\frac{\sqrt{2v_0}}{c\sqrt{\sigma}}$$

- Easily extended to full asymptotic expansion

Call prices and Smile asymptotics

- Gulisashvili (2010): Assumes that density of spot varies regularly at infinity

$$D_T(x) = x^{-\gamma} h(x),$$

  h varies slowly at infinity, $\gamma > 2$
- Expansions of call prices and implied volatility
- Similarly for left tail ($x \to 0$, $k = \log K \to -\infty$)

Call prices

- Call price for strike $K \to \infty$

$$C(K) = \frac{A_1}{(-A_3+1)(-A_3+2)} K^{-A_3+2} e^{A_2\sqrt{\log K}} (\log K)^{-\frac{3}{4}+\frac{a}{c^2}} \times \left(1 + O\left((\log K)^{-\frac{1}{4}}\right)\right)$$

Smile asymptotics

- Implied volatility for log-strike $k \to \infty$

$$\sigma_{BS}(k, T)\sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{1}{k^{1/2}}\right)$$

- Gao, Lee (2011): refinement to $O(k^{-3/4})$.
  They use a higher order transfer result (call price $\to$ implied vol)

Local volatility

- Given call price surface

$$C = C(K, T) = C_{\mathrm{BS}}(K, T; \sigma_{\mathrm{BS}}(K, T))$$

- Reproduced by local volatility model

$$dS_t/S_t = \sigma_{\mathrm{loc}}(S_t, t)dW_t$$

- Dupire's formula (1994)

$$\sigma^2_{\mathrm{loc}}(K, T) = \frac{2\partial_T C}{K^2 \partial_{KK} C}$$

Heston local vol for large state variable

- Suppose that call prices are given by Heston model
- The resulting local vol satisfies

$$\sigma^2_{\mathrm{loc}}(e^k, T) \sim const \times k, \qquad k \to \infty.$$

- Proof: Dupire's formula, Riccati equations, saddle point method

General wing formula for local vol

- log moment generating function ($X_T =$ log-spot)

$$m(s,T) = \log E[\exp(s, X_T)]$$

- saddle point $\hat{s}(k,T)$

$$\left.\frac{\partial}{\partial s} m(s,T)\right|_{s=\hat{s}} = k$$

- Wing formula for $k \to \infty$:

$$\sigma^2_{\mathrm{loc}}(e^k, T) \approx \left.\frac{2\frac{\partial}{\partial T} m(s,T)}{s(s-1)}\right|_{s=\hat{s}(k,T)}$$

Local vol wing formula: Proof idea

- Dupire's formula + Fourier inversion

$$\sigma_{\mathrm{loc}}^2(K,T) = \frac{2\partial_T C}{K^2\partial_{KK} C} = \frac{2\int_{-i\infty}^{i\infty} \frac{\partial_T m(s,T)}{s(s-1)} e^{-ks} M(s,T)\,ds}{\int_{-i\infty}^{i\infty} e^{-ks} M(s,T)\,ds}$$

- saddle point method $\longrightarrow$ cancellation

Heston local vol

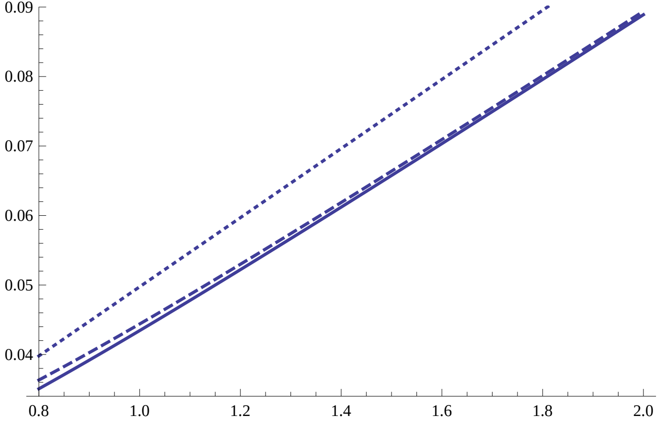


Figure: Heston model: approximation of $\sigma^2_{\mathrm{loc}}(e^k, T)$ for large k. Dashed: wing formula; dotted: *const* $\times$ k

Possible applications of local vol asymptotics

- Extrapolate market data to non-liquid ranges, in a way consistent with Heston or other chosen model
- Local vol asymptotics for jump models: Test given call price surface for jump behavior

References

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