Special Functions: From Lindelöf Integrals to Volatility Smiles

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Habilitationskolloquium, TU Wien

June 28, 2011

Contents of the thesis

[1] J. P. Bell, S. Gerhold, M. Klazar, and F. Luca, *Non-holonomicity of sequences defined via elementary functions*, Annals of Combinatorics, 12 (2008), pp. 1–16.

[2] S. Gerhold, *Asymptotic estimates for some number-theoretic power series*, Acta Arith., 142 (2010), pp. 187–196.

[3] P. Flajolet, S. Gerhold, and B. Salvy, *Lindelöf representations and (non-)holonomic sequences*, Electronic Journal of Combinatorics, 17 (2010).

[4] P. Friz, S. Gerhold, A. Gulisashvili, and S. Sturm, *On refined volatility smile expansion in the Heston model*. To appear in Quantitative Finance, 2011.

[5] S. Gerhold, *The Longstaff-Schwartz algorithm for Lévy models: Results on fast and slow convergence*, Ann. Appl. Probab., 21(2), pp. 589–608, 2011.

Part I

S. Gerhold, *Asymptotic estimates for some number-theoretic power series*, Acta Arith., 142 (2010), pp. 187–196.

- Möbius μ-function:

$$\mu(n) = \begin{cases} (-1)^k & n \text{ square-free, with } k \text{ prime factors} \\ 0 & \text{otherwise} \end{cases}$$

- Dirichlet generating function

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

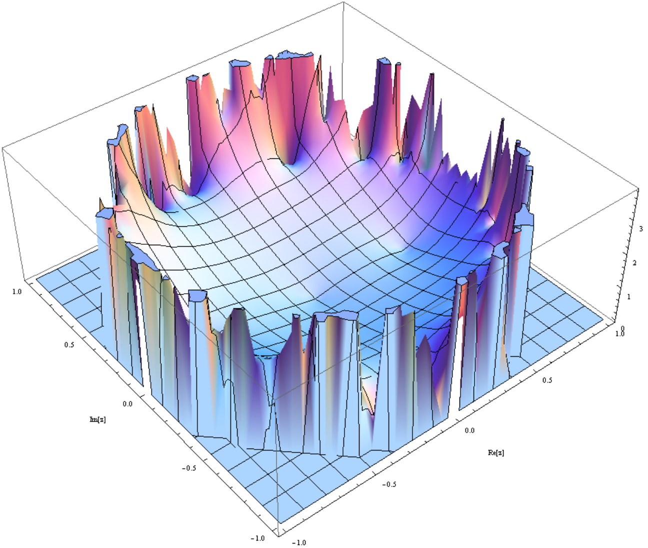
with $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ the Riemann zeta function

- We are interested in the *ordinary* generating function

$$\sum_{n=1}^{\infty} \mu(n) z^n$$

Plot of $(\Re(z), \Im(z)) \mapsto \left| \sum_{n \geq 1} \mu(n) z^n \right|$

Unit circle is a natural boundary (by Thm of Carlson; integral coefficients, convergence radius 1, not a rational function of z)



Asymptotic behavior as $z \to 1$

- Best previously known bound (from Walfisz 1963; based on estimate for the Mertens function $M(x) = \sum_{n \le x} \mu(n)$):

$$\sum_{n=1}^{\infty} \mu(n) z^n = \mathrm{O}\left(\frac{1}{t} \exp\left(-\frac{c(\log 1/t)^{3/5}}{(\log\log 1/t)^{1/5}} \right) \right),$$

$$t = -\log z \sim 1 - z \to 0.$$

- SG (2010):

$$\sum_{n=1}^{\infty} \mu(n) z^n = \mathrm{O}\left(\frac{1}{t} \exp\left(-\frac{0.0203 \times \log(1/t)}{(\log\log 1/t)^{2/3}(\log\log\log 1/t)^{1/3}} \right) \right)$$

- Riemann Hypothesis "$\Longrightarrow$"

$$\sum_{n=1}^{\infty} \mu(n) z^n = \mathrm{O}\left(\frac{1}{\sqrt{t}} \right)$$

Proof idea

Relation between Dirichlet generating function

$$D(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

and ordinary generating function ($z = \mathrm{e}^{-t}$):

$$\sum_{n=1}^{\infty} \mu(n)\mathrm{e}^{-nt} = \frac{1}{2\pi\mathrm{i}} \int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty} \frac{\Gamma(s)}{\zeta(s)} t^{-s} \mathrm{d}s, \qquad \kappa > 1.$$

Proof idea

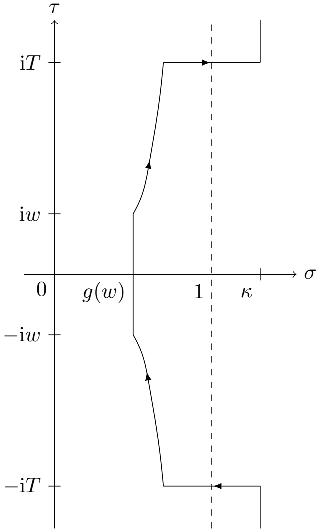
- $$\sum_{n=1}^{\infty} \mu(n) \mathrm{e}^{-nt} = \frac{1}{2\pi \mathrm{i}} \int_{\kappa - \mathrm{i}\infty}^{\kappa + \mathrm{i}\infty} \frac{\Gamma(s)}{\zeta(s)} t^{-s} \mathrm{d}s, \qquad \kappa > 1$$

  Shift the integration contour to the left. Attention: Zeros of ζ
- Best known zero-free region (Korobov-Vinogradov; $s = \sigma + \mathrm{i}\tau$):

  $$\sigma \geq g(\tau) := \begin{cases} 1 - b(\log|\tau|)^{-\alpha}(\log\log|\tau|)^{-\beta} & |\tau| \geq w \\ 1 - b(\log w)^{-\alpha}(\log\log w)^{-\beta} & |\tau| \leq w \end{cases}$$

  for certain $\alpha, \beta, b, w > 0$.

Deformed integration contour



$T = (\log 1/|t|)(\log\log 1/|t|)^{-\alpha}$ balances the contributions of the various contour parts

MR2601060 (2011d:11231) 11N37 (11M26 30B10)
Gerhold, Stefan (A-TUWN-FV)
Asymptotic estimates for some number-theoretic power series.
Acta Arith. **142** (2010), *no. 2,* 187–196.

The author proves a general estimate for a power series with complex coefficients $\sum_{n\geq 1} a_n z^n$, as z tends to 1 in some sector $|\arg(1-z)| \leq C$ with $C < \pi/2$ (Theorem 4). The result is derived directly under the conditions that the associated Dirichlet series $D(s) = \sum_{n\geq 1} a_n n^{-s}$ converges absolutely for $\sigma := \Re(s) > 1$, and has an analytic continuation slightly to the left of $\sigma = 1$, where $D(s) = O(\tau^\nu)$ (with τ the imaginary coefficient of s, and $\nu > 0$).

The motivation for this is to do without the classical (but costly) use of an abelian summation step from an estimate on $A(x) := \sum_{n\leq x} a_n$ (itself being obtainable from a direct use of properties of $D(s)$). The author discusses the case where $a_n = \mu(n)$: in this instance the best known estimate for $A(x) = M(x)$, the Mertens function (due to A. Walfisz and exploiting the Vinogradov-Korobov zero-free region for the Riemann zeta function), is the strong form of the prime number theorem (1) $M(x) = O(x\exp(-c(\log x)^{3/5}(\log\log x)^{-1/5}))$. Abel summation on (1) gives

$$\sum_{n\geq 1} \mu(n) z^n = O(u\exp(-c(\log u)^{3/5}(\log\log u)^{-1/5})),$$

as $u = -1/\log z \to +\infty$, whereas Gerhold's direct method yields

$$(2) \qquad \sum_{n\geq 1} \mu(n) z^n = O(u\exp\left(-d\log u(\log\log u)^{-2/3}(\log\log\log u)^{-1/3}\right))$$

Part II

P. FLAJOLET, S. GERHOLD, AND B. SALVY, *Lindelöf representations and (non-)holonomic sequences*, Electronic Journal of Combinatorics, 17 (2010).

CALCUL DES RÉSIDUS

ET SES APPLICATIONS

A LA THÉORIE DES FONCTIONS

PAR

ERNST LINDELÖF,

PROFESSEUR A L'UNIVERSITÉ DE HELSINGFORS.

PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,

Quai des Grands-Augustins, 55.

1905

...

grale $\int \Phi(x, z)\, dz$ prise le long de C_R tendra effectivement vers zéro, comme nous l'avions dit, de sorte que l'égalité (3) deviendra

$$(4) \qquad \sum_{m}^{\infty} \varphi(\nu) x^{\nu} = -\int_{\alpha - i\infty}^{\alpha + i\infty} \Phi(x, z)\, dz = -\int_{\alpha - i\infty}^{\alpha + i\infty} \frac{\varphi(z) x^z}{e^{2\pi i z} - 1}\, dz.$$

- Lindelöf integral representation for power series
- Taylor coefficients must have an analytic lifting φ
- Applications: Analytic continuation, asymptotics

Lindelöf integrals: precise result

- Assume: $\varphi(s)$ analytic for $\Re(s) > 0$
- Growth assumption: For some $C > 0$ and $0 \leq A < \pi$, we have

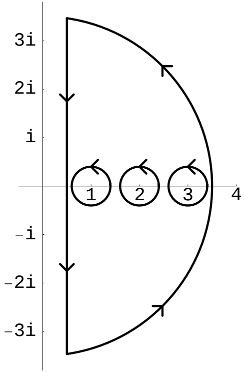
$$\varphi(s) < Ce^{A|s|} \quad \text{as } s \to \infty \text{ in } \Re(s) \geq \tfrac{1}{2}.$$

- **Theorem.** Then $F(z) = \sum_{n\geq 1} \varphi(n)(-z)^n$ is analytic in the sector $-(\pi - A) < \arg(z) < \pi - A$, and has the representation

$$F(z) = -\frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \varphi(s) z^s \frac{\pi}{\sin(\pi s)}\, ds.$$

Proof idea

- Integrate over boundary of a large half-disk
- The contribution from the half-circle vanishes by the growth condition
- Then use the residue theorem
- Residues at integers = power series summands



History and applications Lindelöf integrals

- E. Lindelöf, Le calcul des résidus et ses applications à la théorie des fonctions, Gauthier-Villars, Paris, 1905.
- Analytic continuation of

$$\sum_{n\geq 1} n^{\alpha} z^{n}, \quad \sum_{n\geq 1} (\log n) z^{n}$$

- Asymptotic analysis of these and similar functions (Flajolet 1999)
- Also related to work by Barnes, Mellin, Ramanujan, Ford ...

Our main example

- Define, for real $c \neq 0$ and $0 \neq \theta < 1$, the function

$$E(z; c, \theta) := \sum_{n \geq 1} \exp(cn^{\theta})(-z)^n, \qquad |z| < 1$$

- Lindelöf integral

$$E(z) = -\frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \exp(cs^{\theta}) z^s \frac{\pi}{\sin(\pi s)}\, ds$$

- Analytic continuation to $\mathbb{C} \setminus]-\infty, -1]$
- Singularities at -1 and ∞

Our main example

- Motivation: $\sum_{n\geq 1} \varphi(n)(-z)^n$ is well studied for

$$\varphi(n) = n^\alpha \quad \text{and} \quad \varphi(n) = (\log n)^m.$$

- The coefficient sequence

$$\varphi(n) = e^{cn^\theta}$$

  is a natural next step.
- Applications: asymptotics of finite differences, by singularity analysis; non-holonomicity

Asymptotic results

- Asymptotics of

$$E(z) = \sum_{n \geq 1} \exp(cn^{\theta})(-z)^n$$

for $c = \pm 1$ and $\theta = -1, \frac{1}{2}$ (representative cases)

	$z \to \infty$	$z \to -1$
$e^{1/n}$	$-\frac{e^{2\sqrt{\log z}}}{2\sqrt{\pi}(\log z)^{1/4}}$	$\frac{1}{1+z}$
$e^{-1/n}$	$-\frac{1}{\sqrt{\pi}}(\log z)^{-1/4} \cos\left(2\sqrt{\log z} - \frac{1}{4}\pi\right)$	$\frac{1}{1+z}$
$e^{\sqrt{n}}$	$-1 - \frac{1}{\sqrt{\pi \log z}} + \ldots$	$\frac{\sqrt{\pi} e^{-1/8}}{(1+z)^{3/2}} e^{\frac{1}{4(1+z)}}$
$e^{-\sqrt{n}}$	$-1 + \frac{1}{\sqrt{\pi \log z}} + \ldots$	$E(1) + E'(1)(1+z) + \ldots$

Methods overview

- Asymptotics of

$$E(z) = \sum_{n\geq 1} \exp(cn^{\theta})(-z)^n$$

	$z \to \infty$	$z \to -1$
$c > 0, \theta < 0$	saddle point	series rearrangement
$c < 0, \theta < 0$	2 saddle points	series rearrangement
$c > 0, 0 < \theta < 1$	Hankel contour	Laplace
$c < 0, 0 < \theta < 1$	Hankel contour	Abel's theorem

Saddle point method

- Parameters $c > 0$, $\theta < 0$
- Guiding example:

$$E(z; c = 1, \theta = -1) = \sum_{n \geq 1} e^{1/n} (-z)^n$$

- Lindelöf integral

$$E(z) = -\frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} e^{1/s} z^s \frac{\pi}{\sin \pi s} ds$$

- Goal: asymptotics for $z \to \infty$

Saddle point method

- The function $s \mapsto e^{1/s} z^s$ has an approximate saddle point at

$$s = \frac{1}{\sqrt{\log z}}$$

- New integration contour (depending on z)

$$s = \frac{1}{\sqrt{\log z}} + it, \qquad t \in \mathbb{R}$$

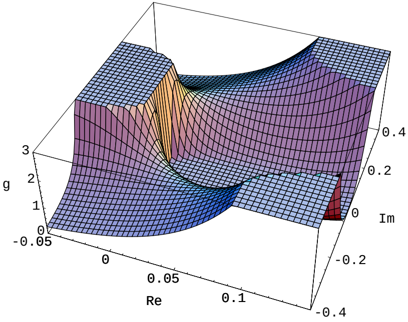
- Main contribution for “small” t
- Integrate local expansion, estimate the tails

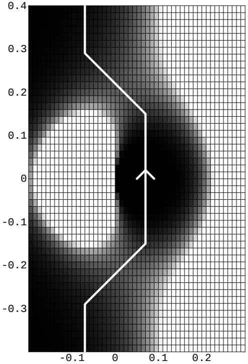
Two saddle points

- The function $s \mapsto e^{-1/s} z^s$ has two approximate saddle points at

$$s = \pm \frac{i}{\sqrt{\log z}}$$

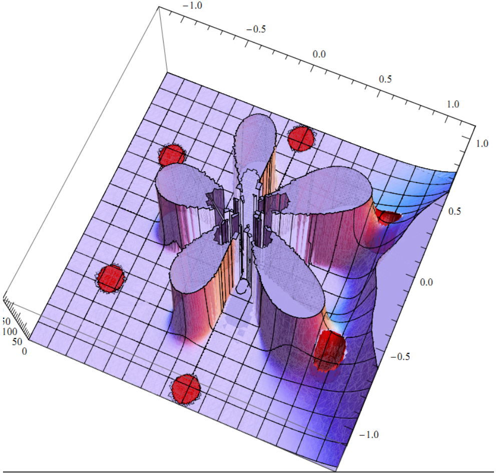
- The new integration contour, passing through the two saddle points





Multiple saddle points

- For $\theta < -1$, there are more than two saddle points
- Have to consider only the rightmost two
- Example: The surface $|z^s \exp(-s^\theta)/\sin \pi s|$ for $\theta = -5$ and $z = 200$ (six saddlepoints).



Asymptotic results

- Asymptotics of

$$E(z) = \sum_{n\geq 1} \exp(cn^{\theta})(-z)^n$$

for $c = \pm 1$ and $\theta = -1, \frac{1}{2}$ (representative cases)

	$z \to \infty$	$z \to -1$
$e^{1/n}$	$-\frac{e^{2\sqrt{\log z}}}{2\sqrt{\pi}(\log z)^{1/4}}$	$\frac{1}{1+z}$
$e^{-1/n}$	$-\frac{1}{\sqrt{\pi}}(\log z)^{-1/4}\cos\left(2\sqrt{\log z} - \frac{1}{4}\pi\right)$	$\frac{1}{1+z}$
$e^{\sqrt{n}}$	$-1 - \frac{1}{\sqrt{\pi \log z}} + \dots$	$\frac{\sqrt{\pi}e^{-1/8}}{(1+z)^{3/2}} e^{\frac{1}{4(1+z)}}$
$e^{-\sqrt{n}}$	$-1 + \frac{1}{\sqrt{\pi \log z}} + \dots$	$E(1) + E'(1)(1+z) + \dots$

Application: e^{cn^θ} is not a holonomic sequence

- A function $f(z)$ is holonomic if it satisfies an LODE

$$p_0(z)f^{(0)}(z) + \cdots + p_d(z)f^{(d)}(z) = 0$$

  with polynomial coefficients.

- A sequence (a_n) is holonomic if it satisfies an LORE

$$p_0(n)a_n + \cdots + p_d(n)a_{n+d} = 0$$

  with polynomial coefficients.

- Our results + classical LODE asymptotics imply: e^{cn^θ} is not holonomic (except for trivial cases).
- "Strong transcendence result" (algebraic power series are holonomic.)

Application: Finite differences

- Singularity analysis (Flajolet-Odlyzko): translates power series asymptotics to coefficient asymptotics
- By our saddle point results, we find the amusing formulas

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k e^{1/k} \sim -\frac{e^{2\sqrt{\log n}}}{2\sqrt{\pi}(\log n)^{1/4}},$$

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k e^{-1/k} = -\frac{\cos\left(2\sqrt{\log n} - \frac{1}{4}\pi\right)}{\sqrt{\pi}(\log n)^{1/4}} + o\left((\log n)^{-1/4}\right).$$

Another result obtained via Lindelöf integrals

- $$\sum_{n=1}^{\infty} \frac{(-z)^n}{n!+1} \sim -\sum_{k\geq 1} \frac{\pi}{\sin \pi s_k} \frac{1}{\Gamma'(s_k+1)} z^{s_k}, \qquad z \to \infty,$$

  where

  $$(s_k)_{k\geq 1} \approx (-3.457, -3.747, -5.039, -5.991, \dots)$$

  are the solutions of $\Gamma(s+1) = -1$.
- Note the difference to

  $$\sum_{n=1}^{\infty} \frac{(-z)^n}{n!} = -1 + \mathrm{e}^{-z}$$

Part III

P. Friz, S. Gerhold, A. Gulisashvili, and S. Sturm, *On refined volatility smile expansion in the Heston model*. To appear in Quantitative Finance, 2011.

- **Volatility** is a measure of the variation of a stock price
- Classic Black-Scholes model: Option prices are essentially functions of the underlying's volatility
- Inferring the **implied volatility** from observed option prices does not give a unique volatility
- Advanced models reproduce this **volatility smile**

Heston Model

- Dynamics

$$dS_t = S_t\sqrt{V_t}dW_t, \qquad S_0 = 1,$$
$$dV_t = (a + bV_t)\,dt + c\sqrt{V_t}dZ_t, \qquad V_0 = v_0 > 0,$$

- Correlated Brownian motions

$$d\langle W, Z\rangle_t = \rho dt, \qquad \rho \in [-1, 1]$$

- Parameters

$$a \geq 0, b \leq 0, c > 0$$

Heston Smile

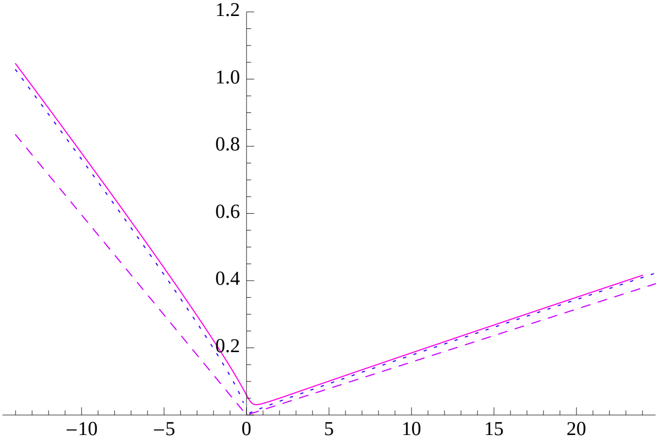


Abbildung: Heston model: Implied variance $\sigma_{BS}(k,1)^2$ in terms of log-strikes compared to the first order (dashed) and third order (dotted) approximations.

Density and smile asymptotics

- Consider a fixed maturity $T > 0$.
- Implied Black-Scholes volatility ($k = \log K$ is the log-strike)

$$\sigma_{BS}^2(k, T) \sim ? \qquad (k \to \pm\infty)$$

- Closely related: How heavy are the tails?

$$D_T(x) \sim ? \qquad (x \to 0, \infty)$$

- $D_T :=$ density of S_T.

Main results (right tail), SG et al. 2011

- Density of spot for $x \to \infty$

$$D_T(x) = A_1 x^{-A_3} e^{A_2\sqrt{\log x}} (\log x)^{-3/4+a/c^2} \left(1+O((\log x)^{-1/2})\right)$$

- Implied volatility for $k = \log K \to \infty$

$$\sigma_{BS}(k, T)\sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{1}{k^{1/2}}\right)$$

Interpretation of smile expansion

- Implied volatility for $k = \log K \to \infty$

$$\sigma_{BS}(k, T)\sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{1}{k^{1/2}}\right)$$

- β_1 does not depend on $\sqrt{v_0}$
- β_2 depends linearly on $\sqrt{v_0}$
- Changes of $\sqrt{v_0}$ have second-order effects
- Increase $\sqrt{v_0}$: parallel shift, slope not affected
- Changes in mean-reversion level $\bar{v} = -a/b$ seen only in β_3

Mellin (Fourier) inversion

- Mellin transform of spot: $M(u) = E[e^{(u-1)X_T}]$
- Analytic in a complex strip
- Density of S_T by Mellin inversion:

$$D_T(x) = \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} x^{-u} M(u) du.$$

- Valid for contour in analyticity strip of the Mellin transform

Heston Model: Mgf of log-spot X_t

- Moment generating function

$$E[e^{sX_t}] = \exp(\phi(s,t) + v_0\psi(s,t))$$

- Riccati equations

$$\partial_t\phi = F(s,\psi),\ \phi(0) = 0,$$
$$\partial_t\psi = R(s,\psi),\ \psi(0) = 0$$

$$F(s,v) = av,$$
$$R(s,v) = \frac{1}{2}(s^2 - s) + \frac{1}{2}c^2v^2 + bv + s\rho cv$$

- Explicit solution possible, but cumbersome expression

Moment explosion

- Critical moment for time T

$$s^* := \sup\{s \geq 1 : E[S_T^s] < \infty\}$$

- Explosion time for moment of order s

$$T^*(s) = \sup\{t \geq 0 : E[S_t^s] < \infty\}$$

- Critical slope, critical curvature:

$$\sigma := -\partial_s T^*|_{s^*} \geq 0 \qquad \text{and} \qquad \kappa := \partial_s^2 T^*|_{s^*}$$

Explicit Explosion time for the Heston model

- Explosion time for moment of order s

$$T^*(s) = \frac{2}{\sqrt{-\,\Delta(s)}}\left(\arctan\frac{\sqrt{-\,\Delta(s)}}{s\rho c + b} + \pi\right),$$

$$\Delta(s) := (s\rho c + b)^2 - c^2\left(s^2 - s\right)$$

- Critical moment s^*: Find numerically from

$$T^*(s^*) = T.$$

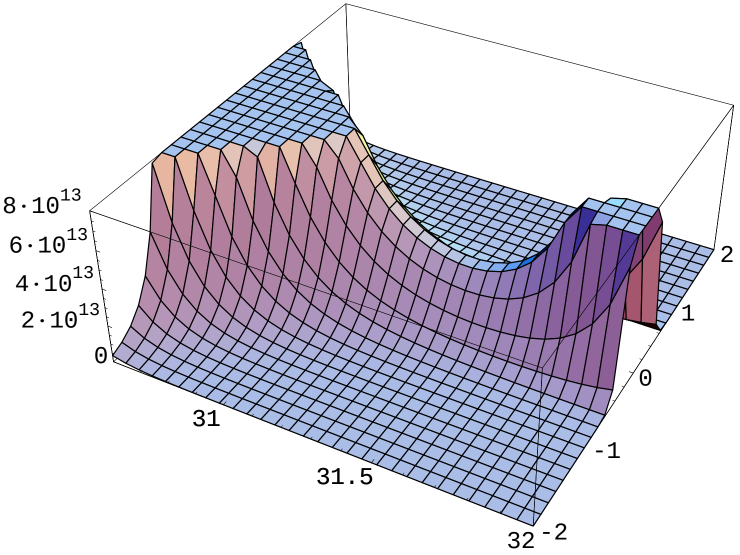
Saddle point method

- Recall:

$$D_T(x) = \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} x^{-u} M(u) du$$

- Shift contour to the right, close to the singularity.
- Let it pass through a saddle point of the integrand.
- For large x, the integral is concentrated around the saddle.
- Local expansion of integrand yields expansion of whole integral.

The surface $|x^{-u}M(u)|$



Asymptotics of ψ and ϕ near critical moment

- Recall $M(u) = \exp(\phi(u-1,t) + v_0\psi(u-1,t))$
- For $u \to u^*$ we have (with $\beta := \sqrt{2v_0}/c\sqrt{\sigma}$)

$$\psi(u-1,T) = \frac{\beta^2}{u^* - u} + const + O(u^* - u),$$
$$\phi(u-1,T) = \frac{2a}{c^2}\log\frac{1}{u^* - u} + const + O(u^* - u)$$

- Found from Riccati equations

Saddle point method

- Finding the saddle point: 0 = derivative of integrand
- Use only first order expansion:

$$0 = \frac{\partial}{\partial u} x^{-u} \exp\left(\frac{\beta^2}{u^* - u}\right)$$

- Approximate saddle point at

$$\hat{u}(x) = u^* - \beta / \sqrt{\log x}$$

- Same singular behavior as in the paper on Lindelöf integrals!

Call prices and Smile asymptotics

- Gulisashvili (2010): Assumes that density of spot varies regularly at infinity

$$D_T(x) = x^{-\gamma} h(x),$$

  h varies slowly at infinity, $\gamma > 2$
- Expansions of call prices and implied volatility
- Similarly for left tail

Conclusion

- Common theme: Asymptotics of functions that have integral representations
- Investigate singularities of the integrand
- Concrete problems may be technically demanding (zero-free regions for ζ; Riccati ODEs)
- Saddle point method: useful in analytic combinatorics and mathematical finance