Lindelöf integral representations and asymptotic analysis of a certain power series with closed-form coefficients

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(joint work with P. Flajolet and B. Salvy)

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Overview

- Contour integral representations of power series
  \[ \sum_{n=1}^{\infty} \phi(n)(-z)^n \]
- Assumption: Taylor coefficients \( \phi(n) \) are values of an analytic function \( \phi(s) \)
- Analytic continuation
- Asymptotic analysis
- Focus on special case: \( \phi(n) = e^{cn^\theta} \)
- Applications: Finite differences, EXBERT distributions, non-holonomicity
Lindelöf integrals

- **Power series**

\[ F(z) := \sum_{n \geq 1} \phi(n)(-z)^n \]

- **\( \phi(s) \) analytic in a domain containing \( ]0, \infty[ \)**

- **Lindelöf integral**

\[ \Lambda(z; C) = -\frac{1}{2i\pi} \int_C \phi(s)z^s \frac{\pi}{\sin(\pi s)} \, ds \]

- **contour \( C \) inside domain of analyticity of \( \phi \), and contains \( 1, 2, \ldots \)**
Lindelöf integrals

- Lindelöf integral

\[
\Lambda(z; C) = -\frac{1}{2i\pi} \int_C \phi(s)z^s \frac{\pi}{\sin(\pi s)} \, ds
\]

- Note: the residue of \( \pi/\sin \pi s \) at \( s = n \) equals \((-1)^n\).

- Hence formally, by the residue theorem:

\[
\sum_{n \geq 1} \phi(n)(-z)^n = \Lambda(z; C).
\]
Assume: $\phi(s)$ analytic for $\Re(s) > 0$

Growth assumption: For some $C > 0$ and $0 \leq A < \pi$, we have

$$\phi(s) < Ce^{A|s|} \text{ as } s \to \infty \text{ in } \Re(s) \geq \frac{1}{2}.$$ 

**Theorem.** Then $F(z) = \sum_{n \geq 1} \phi(n)(-z)^n$ is analytic in the sector $-(\pi - A) < \arg(z) < \pi - A$, and has the representation

$$F(z) = -\frac{1}{2i\pi} \int_{1/2+i\infty}^{1/2-i\infty} \phi(s)z^s \frac{\pi}{\sin(\pi s)} \, ds.$$
Proof idea

- Close the contour by a large half-circle on the right
- The contribution from the half-circle vanishes by the growth condition
- Then use the residue theorem
History and applications Lindelöf integrals

- Analytic continuation of
  \[ \sum_{n \geq 1} n^\alpha z^n, \quad \sum_{n \geq 1} (\log n)z^n \]
- Asymptotic analysis of these and similar functions (Flajolet 1999)
- Euler sums \( \sum H_n/n^\alpha \) (Flajolet, Salvy 1998)
- Also related to work by Barnes, Mellin, Ramanujan, Ford . . .
Our main example

- Define, for real $c \neq 0$ and $0 \neq \theta < 1$, the function
  
  $$E(z; c, \theta) := \sum_{n \geq 1} \exp(cn^\theta)(-z)^n, \quad |z| < 1$$

- Lindelöf integral
  
  $$E(z) = -\frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \exp(cs^\theta)z^s \frac{\pi}{\sin(\pi s)} \, ds$$

- Analytic continuation to $\mathbb{C} \setminus [0, -1]$.

- Singularities at $-1$ and $\infty$.

- Today's topic: asymptotics of $E(z; c, \theta)$.
Our main example

- Motivation: \( \sum_{n \geq 1} \phi(n)(-z)^n \) is well studied for 
  \[ \phi(n) = n^\alpha \quad \text{and} \quad \phi(n) = (\log n)^m. \]

- The coefficient sequence 
  \[ \phi(n) = e^{cn^\theta} \]
  is a natural next step.

- Small application: asymptotics of finite differences, by singularity analysis
Asymptotic results

Asymptotics of

\[ E(z) = \sum_{n \geq 1} \exp(cn^\theta)(-z)^n \]

for \( c = \pm 1 \) and \( \theta = -1, \frac{1}{2} \) (representative cases)

<table>
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\( E(1) + E'(1)(1 + z) + \ldots \)
Methods overview

Asymptotics of

\[ E(z) = \sum_{n \geq 1} \exp(cn^\theta)(-z)^n \]

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Saddle point method

- Parameters $c > 0$, $\theta < 0$
- Guiding example:

$$E(z; c = 1, \theta = -1) = \sum_{n \geq 1} e^{1/n}(-z)^n$$

- Lindelöf integral

$$E(z) = -\frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} e^{1/s}z^s \frac{\pi}{\sin \pi s} ds$$

- Goal: asymptotics for $z \to \infty$
The function $s \mapsto e^{1/s} z^s$ has an approximate saddle point at

$$s = \frac{1}{\sqrt{\log z}}$$

New integration contour

$$s = \frac{1}{\sqrt{\log z}} + it, \quad t \in \mathbb{R}$$

Main contribution for

$$|t| < (\log z)^{-\alpha}, \quad \frac{2}{3} < \alpha < \frac{3}{4}$$
The condition $\alpha > \frac{2}{3}$ ensures a uniform expansion

\[
\frac{e^{1/s} z^s}{\sin \pi s} = \frac{1}{\pi} L^{1/2} \exp(2L^{1/2} - L^{3/2} t^2) \cdot (1 + o(1)),
\]

\[
s = L^{-1/2} + it, \quad |t| < L^{-\alpha},
\]

where $L = \log z$.

Thence the asymptotics of the central part of the integral:

\[
- \frac{L^{1/2} e^{2L^{1/2}}}{2\pi} \int_{-L^{-\alpha}}^{L^{-\alpha}} e^{-L^{3/2} t^2} dt \sim - \frac{e^{2L^{1/2}}}{2\sqrt{\pi} L^{1/4}}
\]
The condition $\alpha < \frac{3}{4}$ allows to discard the tails $|t| > (\log z)^{-\alpha}$.

Hence our result:

$$\sum_{n \geq 1} e^{1/n} (-z)^n \sim -\frac{e^{2\sqrt{\log z}}}{2\sqrt{\pi}(\log z)^{1/4}}, \quad z \to \infty$$

(in the sense of analytic continuation of the left side.)

Holds in any sector with vertex at zero that does not contain the negative real axis.

Straightforward generalization to arbitrary parameters $c > 0$, $\theta < 0$. 

Saddle point method
Methods overview

Asymptotics of

\[ E(z) = \sum_{n \geq 1} \exp(cn^{\theta})(-z)^n \]

\( z \to \infty \)

\( z \to -1 \)

\( c > 0, \theta < 0 \) saddle point series rearrangement

\( c < 0, \theta < 0 \) 2 saddle points series rearrangement

\( c > 0, 0 < \theta < 1 \) Hankel contour Laplace

\( c < 0, 0 < \theta < 1 \) Hankel contour Abel’s theorem
Two saddle points

- Parameters $c < 0$, $\theta < 0$
- Guiding example:

$$E(z; c = 1, \theta = -1) = \sum_{n \geq 1} e^{-1/n}(-z)^n$$

- Lindelöf integral

$$E(z) = -\frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} e^{-1/s} z^s \frac{\pi}{\sin \pi s} ds$$

- Goal: asymptotics for $z \to \infty$
Two saddle points

- The function $s \mapsto e^{-1/s}z^s$ has two approximate saddle points at

$$s = \pm \frac{i}{\sqrt{\log z}}$$

- The new integration contour, passing through the two saddle points
Two saddle points

- Again, we take the central region $|t| < (\log z)^{-\alpha}$ with $\frac{2}{3} < \alpha < \frac{3}{4}$.
- The dominating factor of the integrand becomes $e^{\pm 2i\sqrt{\log z}}$, producing oscillating behavior.
- At each saddle point, we proceed as before.
- Add the dominating parts of both saddle points.
- Tail estimates: very tedious!
Two saddle points

Result:

\[
\sum_{n \geq 1} e^{-1/n}(-z)^n = -\frac{1}{\sqrt{\pi}} (\log z)^{-1/4} \cos \left( 2\sqrt{\log z} - \frac{1}{4}\pi \right) + O((\log z)^{-1/2+\varepsilon}), \quad z \to \infty.
\]

Again, this holds in any sector with vertex at zero that does not contain the negative real axis.

Generalization to arbitrary parameters \(c < 0, \theta < 0\).

An exponential factor \(\exp(C(\log z)^{\frac{\theta}{\theta-1}})\) appears for \(\theta \neq -1\).
Two saddle points

- For $\theta < -1$, there are more than two saddle points
- Have to consider only the rightmost two
- Example: The surface $|z^s \exp(-s^\theta) / \sin \pi s|$ for $\theta = -5$ and $z = 200$ (six saddlepoints).
Methods overview

Asymptotics of

\[ E(z) = \sum_{n \geq 1} \exp(cn^\theta)(-z)^n \]

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Hankel contour

- Parameters $c$, $0 < \theta < 1$

- Guiding example:

$$E(z; c = 1, \theta = \frac{1}{2}) = \sum_{n \geq 1} e^{\sqrt{n}(-z)^n}$$

- Lindelöf integral

$$E(z; 1, \frac{1}{2}) = -\frac{1}{2i\pi} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} e^{\sqrt{s}z^s} \frac{\pi}{\sin \pi s} ds$$

- Near $s = 0$, we have

$$e^{\sqrt{s}} \frac{\pi}{\sin \pi s} = s^{-1} + s^{-1/2} + O(1).$$
Hankel contour

- New integration contour (with $L = \log z$ and $0 < \beta < 1$):

- The contributions of the two vertical lines are negligible.
- The U-shaped part is transformed into a Hankel contour $\mathcal{H}$. 
We want to evaluate the contour integrals

\[ \int \frac{z^s}{s^\lambda}, \quad \lambda = 1, \frac{1}{2}. \]

Substitute

\[ s \log z = -w \]

Use Hankel’s formula

\[ \frac{1}{\Gamma(\lambda)} = -\frac{1}{2i\pi} \int_{\mathcal{H}} \frac{e^{-w}}{(-w)^\lambda} dw. \]

Hence

\[ \int \frac{z^s}{s^\lambda} \sim \frac{2i\pi}{\Gamma(\lambda)} (\log z)^{\lambda-1}. \]
Result:

\[
\sum_{n \geq 1} e^{\sqrt{n}} (-z)^n = -1 - \frac{1}{\sqrt{\pi \log z}} + O\left(\frac{1}{\log z}\right), \quad z \to \infty.
\]

Again, this holds in any sector with vertex at zero that does not contain the negative real axis.

Straightforward generalization to arbitrary parameters \(c, 0 < \theta < 1\).
Methods overview

- Asymptotics of

\[ E(z) = \sum_{n \geq 1} \exp(cn^\theta)(-z)^n \]

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Behavior at the singularity \( z = -1 \)

- **Case 1:** \( \theta < 0 \). We can rewrite

\[
E(z; c, \theta) = \sum_{n \geq 1} \sum_{k \geq 0} \frac{c^k}{k!} n^{k\theta} (-z)^n = \sum_{k \geq 0} \frac{c^k}{k!} \text{Li}_{-k\theta}(-z), \quad |z| < 1,
\]

as a sum of polylogarithms

\[
\text{Li}_\alpha(z) = \sum_{n \geq 1} \frac{z^n}{n^\alpha}.
\]

- Asymptotics of \( \text{Li}_\alpha(z) \) are well known.
- For instance,

\[
\sum_{n \geq 1} e^{1/n} (-z)^n = \sum_{k \geq 0} \frac{1}{k!} \text{Li}_k(-z) = \frac{1}{1 + z} + \log \frac{1}{1 + z} + O(1).
\]
Behavior at the singularity $z = -1$

- **Case 2:** $c > 0$, $0 < \theta < 1$.
- Guiding example:
  
  $$E(z; c = 1, \theta = \frac{1}{2}) = \sum_{n \geq 1} e^{\sqrt{n}}(-z)^n$$

- Laplace method
- Summands have a peak near $n = \frac{1}{4}(\log z)^{-2}$
- Second order expansion around this peak
- Evaluate resulting sum asymptotically
- Discard the tails
Behavior at the singularity \( z = -1 \)

- Result of the Laplace method:
  \[
  \sum_{n \geq 1} e^{\sqrt{n}}(-z)^n \sim \frac{\sqrt{\pi}e^{-1/8}}{(1 + z)^{3/2}} \exp\left(\frac{1}{4(1 + z)}\right), \quad z \to -1.
  \]

- Straightforward generalization to arbitrary parameters \( c > 0, \quad 0 < \theta < 1 \).
Behavior at the singularity $z = -1$

- **Case 3:** $c < 0$, $0 < \theta < 1$.
- Series defining $E(z)$ and its derivatives converge at $z = -1$.

\[
\frac{1}{k!} \lim_{z \to -1^+} \frac{d^k}{dz^k} E(z) = (-1)^k \sum_{n \geq 1} \binom{n}{k} \exp(cn^\theta).
\]

- Formal Taylor series at $z = -1$ is an asymptotic series for the function (extension of Abel’s theorem, see Ford 1916).

\[
E(z; c, \theta) \underset{z \to -1^+}{\sim} u_0 + u_1(1 + z) + u_2(1 + z)^2 + \ldots
\]

- This formal series does not converge in any neighborhood of $z = -1$. 

Application: $e^{cn^\theta}$ is not a holonomic sequence

- A function $f(z)$ is holonomic if it satisfies an LODE
  \[ p_0(z)f^{(0)}(z) + \cdots + p_d(z)f^{(d)}(z) = 0 \]
  with polynomial coefficients.
- A sequence $(a_n)$ is holonomic if it satisfies an LORE
  \[ p_0(n)a_n + \cdots + p_d(n)a_{n+d} = 0 \]
  with polynomial coefficients.
- Our results + classical LODE asymptotics imply: $e^{cn^\theta}$ is not holonomic (except for trivial cases).
- “Strong transcendence result” (algebraic power series are holonomic.)
Given a sequence $f_n$, define

$$g_n := D_n[f] := \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_k$$

Generating function:

$$g(z) = \frac{1}{1-z} \left( f_0 + F \left( \frac{z}{1-z} \right) \right),$$

where

$$F(z) = \sum_{n \geq 1} f_n (-z)^n.$$
Asymptotics of $F(z)$ ⇒ asymptotics of $g(z)$

If $F(z)$ has “tame” asymptotics, we get the asymptotics of $g_n = D_n[f]$ by singularity analysis

Example: For $f_n = e^{±\sqrt{n}}$ we have

$$g(z) \sim -\frac{±1}{\sqrt{\pi}} \frac{1}{1 - z} \left(\log \frac{1}{1 - z}\right)^{-1/2},$$

hence by singularity analysis

$$g_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k e^{±\sqrt{k}} \sim -\frac{±1}{\sqrt{\pi} \log n}.$$
Application: Asymptotics of a certain EXBERT distribution

- EXBERT distributions (EXchangeable BERnoulli trials, Madsen 1993)

\[ P[X = x] = \binom{n}{x} \sum_{j=0}^{n-x} \binom{n-x}{j} (-1)^j \pi_{x+j}, \quad x \in \{0, \ldots, n\}, \]

where \( \pi_k, 0 \leq k \leq n \), is a probability distribution with all differences non-negative.

- “power model”: \( \pi_k = \exp((\log p)k^a) \) with \( 0 < a < 1 \) and \( 0 < p < 1 \).

- Asymptotics for large \( n \):

\[
P[X = x] \xrightarrow{n \to \infty} \begin{cases} 
  -\log p / \Gamma(1-a)(\log n)^a & x = 0 \\
  -a \log p / x \Gamma(1-a)(\log n)^{a+1} & x \geq 1 
\end{cases}
\]
Slowly varying functions and oscillating functions can be subjected to singularity analysis.

By our saddle point results, we find the amusing formulas

\[ \sum_{k=1}^{n} \binom{n}{k} (-1)^k e^{1/k} \sim -\frac{e^{2\sqrt{\log n}}}{2\sqrt{\pi}(\log n)^{1/4}}, \]

\[ \sum_{k=1}^{n} \binom{n}{k} (-1)^k e^{-1/k} = -\frac{\cos \left( 2\sqrt{\log n} - \frac{1}{4}\pi \right)}{\sqrt{\pi}(\log n)^{1/4}} + o \left( (\log n)^{-1/4} \right). \]
The function $E(z)$ occurs in a heuristic argument to identify the constant $\pi \sqrt{2/3}$ in $p(n) \approx \exp(\pi \sqrt{2n/3})$.

Generating function of the partition sequence

$$P(z) = \sum_{n \geq 1} p(n)z^n = \prod_{k \geq 1} (1 - z^k)^{-1}$$

By an easy estimate

$$P(z) \approx \exp \frac{\pi^2}{6(1 - z)} =: \sum a_nz^n.$$  

What is the growth order of $a_n$?

- Polynomial growth $\Rightarrow \sum a_nz^n$ too small
- Exponential growth $\Rightarrow \sum a_nz^n$ too large
Heuristic partition asymptotics (Hardy)

- **Guess:** $a_n \approx e^{Bn^b}$ for some constants $B > 0$ and $0 < b < 1$.
- The growth order of
  \[ \sum e^{Bn^b} z^n, \quad z \to 1, \]
  is given by its maximum term
  \[ \exp(C(1 - z)^{-b/(1-b)}), \quad \text{where} \quad C = B^{1/(1-b)} b^{b/(1-b)} (1 - b). \]
- Matching this to $P(z) \approx \exp(\frac{\pi^2}{6(1-z)})$ yields
  \[ p(n) \approx \exp(\pi \sqrt{\frac{2n}{3}}). \]
Heuristic partition asymptotics (Hardy)

- Relation to our work:
  Hardy's argument needs the rough growth order of
  \[ \sum e^{Bn^b} z^n, \quad z \to 1, \]

- We determine the asymptotics of
  \[ \sum e^{Bn^b} z^n, \quad z \to 1, \]
  exactly by Laplace’s method.
Conclusion

- Lindelöf integrals provide analytic continuation of power series and allow to do asymptotics.
- Asymptotics of \( E(z; c, \theta) \) depends strongly on the parameter region (six different expansions).
- Extensions to related functions like

\[
\sum_{n \geq 1} \exp(cn^\theta (\log n)^\alpha)(-z)^n
\]

should be relatively easy.