Lindelöf integral representations and asymptotic analysis of a certain power series with closed-form coefficients

Stefan Gerhold
(joint work with P. Flajolet and B. Salvy)

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Overview

- Contour integral representations of power series

$$\sum_{n=1}^{\infty} \phi(n)(-z)^n$$

- Assumption: Taylor coefficients $\phi(n)$ are values of an analytic function $\phi(s)$
- Analytic continuation
- Asymptotic analysis
- Focus on special case: $\phi(n) = e^{cn^\theta}$
- Applications: Finite differences, EXBERT distributions, non-holonomicity

Lindelöf integrals

- Power series

$$F(z) := \sum_{n\geq 1} \phi(n)(-z)^n$$

- $\phi(s)$ analytic in a domain containing $]0, \infty[$
- Lindelöf integral

$$\Lambda(z;\mathcal{C}) = -\frac{1}{2i\pi}\int_{\mathcal{C}} \phi(s) z^s \frac{\pi}{\sin(\pi s)}\, ds$$

- contour $\mathcal{C}$ inside domain of analyticity of ϕ, and contains $1, 2, \ldots$

Lindelöf integrals

- Lindelöf integral

$$\Lambda(z;\mathcal{C}) = -\frac{1}{2i\pi}\int_{\mathcal{C}} \phi(s) z^s \frac{\pi}{\sin(\pi s)}\, ds$$

- Note: the residue of $\pi/\sin \pi s$ at $s = n$ equals $(-1)^n$.
- Hence formally, by the residue theorem:

$$\sum_{n\geq 1} \phi(n)(-z)^n = \Lambda(z;\mathcal{C}).$$

Lindelöf integrals: precise result

- Assume: $\phi(s)$ analytic for $\Re(s) > 0$
- Growth assumption: For some $C > 0$ and $0 \leq A < \pi$, we have

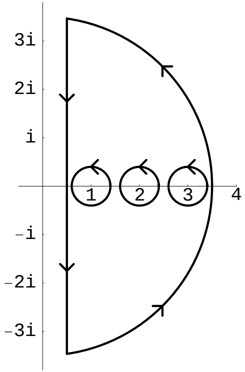
$$\phi(s) < Ce^{A|s|} \quad \text{as } s \to \infty \text{ in } \Re(s) \geq \tfrac{1}{2}.$$

- **Theorem.** Then $F(z) = \sum_{n \geq 1} \phi(n)(-z)^n$ is analytic in the sector $-(\pi - A) < \arg(z) < \pi - A$, and has the representation

$$F(z) = -\frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \phi(s) z^s \frac{\pi}{\sin(\pi s)} \, ds.$$

Proof idea

- Close the contour by a large half-circle on the right
- The contribution from the half-circle vanishes by the growth condition
- Then use the residue theorem



History and applications Lindelöf integrals

- E. Lindelöf, Le calcul des résidus et ses applications à la théorie des fonctions, Gauthier- Villars, Paris, 1905.
- Analytic continuation of

$$\sum_{n\geq 1} n^{\alpha} z^{n}, \quad \sum_{n\geq 1} (\log n) z^{n}$$

- Asymptotic analysis of these and similar functions (Flajolet 1999)
- Euler sums $\sum H_n/n^{\alpha}$ (Flajolet, Salvy 1998)
- Also related to work by Barnes, Mellin, Ramanujan, Ford ...

Our main example

- Define, for real $c \neq 0$ and $0 \neq \theta < 1$, the function

$$E(z; c, \theta) := \sum_{n \geq 1} \exp(cn^{\theta})(-z)^n, \qquad |z| < 1$$

- Lindelöf integral

$$E(z) = -\frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \exp(cs^{\theta}) z^s \frac{\pi}{\sin(\pi s)} \, ds$$

- Analytic continuation to $\mathbb{C} \setminus]-\infty, -1]$
- Singularities at -1 and ∞
- Today's topic: asymptotics of $E(z; c, \theta)$

Our main example

- Motivation: $\sum_{n\geq 1} \phi(n)(-z)^n$ is well studied for

$$\phi(n) = n^\alpha \quad \text{and} \quad \phi(n) = (\log n)^m.$$

- The coefficient sequence

$$\phi(n) = e^{cn^\theta}$$

  is a natural next step.
- Small application: asymptotics of finite differences, by singularity analysis

Asymptotic results

- Asymptotics of

$$E(z) = \sum_{n\geq 1} \exp(cn^{\theta})(-z)^n$$

for $c = \pm 1$ and $\theta = -1, \frac{1}{2}$ (representative cases)

	$z \to \infty$	$z \to -1$
$e^{1/n}$	$-\dfrac{e^{2\sqrt{\log z}}}{2\sqrt{\pi}(\log z)^{1/4}}$	$\dfrac{1}{1+z}$
$e^{-1/n}$	$-\frac{1}{\sqrt{\pi}}(\log z)^{-1/4}\cos\left(2\sqrt{\log z}-\frac{1}{4}\pi\right)$	$\dfrac{1}{1+z}$
$e^{\sqrt{n}}$	$-1-\dfrac{1}{\sqrt{\pi \log z}}+\dots$	$\dfrac{\sqrt{\pi}e^{-1/8}}{(1+z)^{3/2}}e^{\frac{1}{4(1+z)}}$
$e^{-\sqrt{n}}$	$-1+\dfrac{1}{\sqrt{\pi \log z}}+\dots$	$E(1)+E'(1)(1+z)+\dots$

Methods overview

- Asymptotics of

$$E(z) = \sum_{n \geq 1} \exp(cn^{\theta})(-z)^n$$

	$z \to \infty$	$z \to -1$
$c > 0, \theta < 0$	saddle point	series rearrangement
$c < 0, \theta < 0$	2 saddle points	series rearrangement
$c > 0, 0 < \theta < 1$	Hankel contour	Laplace
$c < 0, 0 < \theta < 1$	Hankel contour	Abel's theorem

Saddle point method

- Parameters $c > 0$, $\theta < 0$
- Guiding example:

$$E(z; c = 1, \theta = -1) = \sum_{n \geq 1} e^{1/n} (-z)^n$$

- Lindelöf integral

$$E(z) = -\frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} e^{1/s} z^s \frac{\pi}{\sin \pi s} ds$$

- Goal: asymptotics for $z \to \infty$

Saddle point method

- The function $s \mapsto e^{1/s} z^s$ has an approximate saddle point at

$$s = \frac{1}{\sqrt{\log z}}$$

- New integration contour

$$s = \frac{1}{\sqrt{\log z}} + it, \qquad t \in \mathbb{R}$$

- Main contribution for

$$|t| < (\log z)^{-\alpha}, \qquad \tfrac{2}{3} < \alpha < \tfrac{3}{4}$$

Saddle point method

- The condition $\alpha > \frac{2}{3}$ ensures a uniform expansion

$$\frac{e^{1/s} z^s}{\sin \pi s} = \frac{1}{\pi} L^{1/2} \exp(2L^{1/2} - L^{3/2} t^2) \cdot (1 + o(1)),$$

$$s = L^{-1/2} + it, \qquad |t| < L^{-\alpha},$$

  where $L = \log z$.

- Thence the asymptotics of the central part of the integral:

$$-\frac{L^{1/2} e^{2L^{1/2}}}{2\pi} \int_{-L^{-\alpha}}^{L^{-\alpha}} e^{-L^{3/2} t^2} dt \sim -\frac{e^{2L^{1/2}}}{2\sqrt{\pi} L^{1/4}}$$

Saddle point method

- The condition $\alpha < \frac{3}{4}$ allows to discard the tails $|t| > (\log z)^{-\alpha}$.
- Hence our result:

$$\sum_{n\geq 1} e^{1/n}(-z)^n \sim -\frac{e^{2\sqrt{\log z}}}{2\sqrt{\pi}(\log z)^{1/4}}, \quad z \to \infty$$

  (in the sense of analytic continuation of the left side.)
- Holds in any sector with vertex at zero that does not contain the negative real axis.
- Straightforward generalization to arbitrary parameters $c > 0$, $\theta < 0$.

Methods overview

- Asymptotics of

$$E(z) = \sum_{n \geq 1} \exp(cn^{\theta})(-z)^n$$

	$z \to \infty$	$z \to -1$
$c > 0, \theta < 0$	saddle point	series rearrangement
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Two saddle points

- Parameters $c < 0$, $\theta < 0$
- Guiding example:

$$E(z; c = 1, \theta = -1) = \sum_{n \geq 1} e^{-1/n} (-z)^n$$

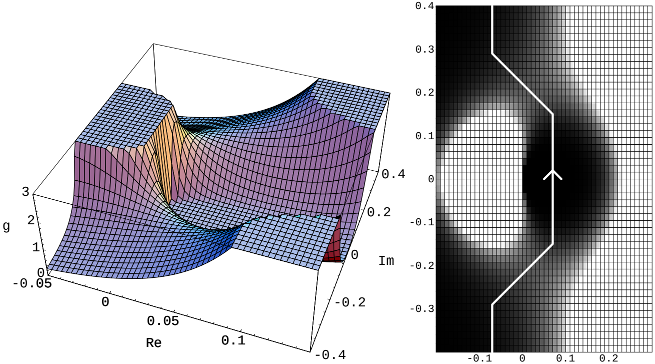
- Lindelöf integral

$$E(z) = -\frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} e^{-1/s} z^s \frac{\pi}{\sin \pi s} ds$$

- Goal: asymptotics for $z \to \infty$

Two saddle points

- The function $s \mapsto e^{-1/s} z^s$ has two approximate saddle points at
$$s = \pm \frac{i}{\sqrt{\log z}}$$
- The new integration contour, passing through the two saddle points



Two saddle points

- Again, we take the central region $|t| < (\log z)^{-\alpha}$ with $\frac{2}{3} < \alpha < \frac{3}{4}$.
- The dominating factor of the integrand becomes

$$e^{\pm 2i\sqrt{\log z}},$$

producing oscillating behavior
- At each saddle point, we proceed as before
- Add the dominating parts of both saddle points
- Tail estimates: very tedious!

Two saddle points

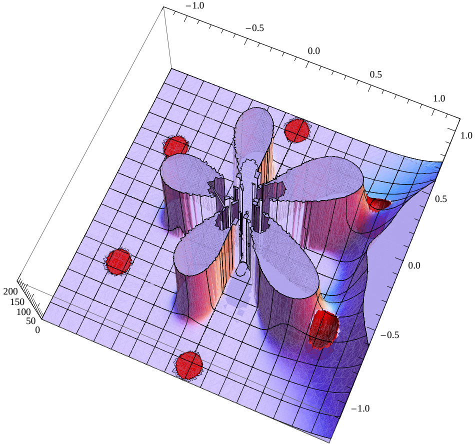
- Result:

$$\sum_{n\geq 1} e^{-1/n}(-z)^n = -\frac{1}{\sqrt{\pi}}(\log z)^{-1/4}\cos\Big(2\sqrt{\log z}-\tfrac{1}{4}\pi\Big) + O((\log z)^{-1/2+\varepsilon}), \qquad z\to\infty.$$

- Again, this holds in any sector with vertex at zero that does not contain the negative real axis.
- Generalization to arbitrary parameters $c<0$, $\theta<0$.
- An exponential factor $\exp\big(C(\log z)^{\frac{\theta}{\theta-1}}\big)$ appears for $\theta\neq -1$.

Two saddle points

- For $\theta < -1$, there are more than two saddle points
- Have to consider only the rightmost two
- Example: The surface $|z^s \exp(-s^\theta)/\sin \pi s|$ for $\theta = -5$ and $z = 200$ (six saddlepoints).



Methods overview

- Asymptotics of

$$E(z) = \sum_{n\geq 1} \exp(cn^{\theta})(-z)^n$$

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Hankel contour

- Parameters c, $0 < \theta < 1$
- Guiding example:

$$E(z; c = 1, \theta = \tfrac{1}{2}) = \sum_{n \geq 1} e^{\sqrt{n}} (-z)^n$$

- Lindelöf integral

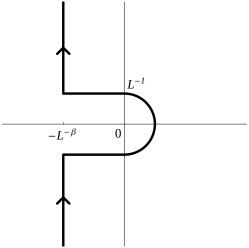
$$E(z; 1, \tfrac{1}{2}) = -\frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} e^{\sqrt{s}} z^s \frac{\pi}{\sin \pi s} ds$$

- Near $s = 0$, we have

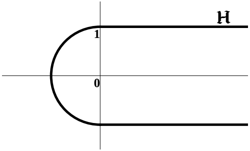
$$e^{\sqrt{s}} \frac{\pi}{\sin \pi s} = s^{-1} + s^{-1/2} + O(1).$$

Hankel contour

- New integration contour (with $L = \log z$ and $0 < \beta < 1$):



- The contributions of the two vertical lines are negligible.
- The U-shaped part is transformed into a Hankel contour $\mathcal{H}$.



Hankel contour

- We want to evaluate the contour integrals

$$\int \frac{z^s}{s^\lambda}, \qquad \lambda = 1, \tfrac{1}{2}.$$

- Substitute

$$s \log z = -w$$

- Use Hankel's formula

$$\frac{1}{\Gamma(\lambda)} = -\frac{1}{2i\pi} \int_{\mathcal{H}} \frac{e^{-w}}{(-w)^\lambda} dw.$$

- Hence

$$\int \frac{z^s}{s^\lambda} \sim \frac{2i\pi}{\Gamma(\lambda)} (\log z)^{\lambda-1}.$$

Hankel contour

- Result:

$$\sum_{n\geq 1} e^{\sqrt{n}}(-z)^n = -1 - \frac{1}{\sqrt{\pi \log z}} + O\left(\frac{1}{\log z}\right), \qquad z \to \infty.$$

- Again, this holds in any sector with vertex at zero that does not contain the negative real axis.
- Straightforward generalization to arbitrary parameters c, $0 < \theta < 1$.

Methods overview

- Asymptotics of

$$E(z) = \sum_{n\geq 1} \exp(cn^{\theta})(-z)^n$$

	$z \to \infty$	$z \to -1$
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Behavior at the singularity $z = -1$

- **Case 1:** $\theta < 0$. We can rewrite

$$E(z; c, \theta) = \sum_{n\geq 1}\sum_{k\geq 0}\frac{c^k}{k!}n^{k\theta}(-z)^n = \sum_{k\geq 0}\frac{c^k}{k!}\mathrm{Li}_{-k\theta}(-z), \qquad |z| < 1,$$

as a sum of polylogarithms

$$\mathrm{Li}_\alpha(z) = \sum_{n\geq 1}\frac{z^n}{n^\alpha}.$$

- Asymptotics of $\mathrm{Li}_\alpha(z)$ are well known.
- For instance,

$$\sum_{n\geq 1} e^{1/n}(-z)^n = \sum_{k\geq 0}\frac{1}{k!}\mathrm{Li}_k(-z) = \frac{1}{1+z} + \log\frac{1}{1+z} + O(1).$$

Behavior at the singularity $z = -1$

- **Case 2:** $c > 0$, $0 < \theta < 1$.
- Guiding example:

$$E(z; c = 1, \theta = \tfrac{1}{2}) = \sum_{n \geq 1} e^{\sqrt{n}} (-z)^n$$

- Laplace method
- Summands have a peak near $n = \frac{1}{4}(\log z)^{-2}$
- Second order expansion around this peak
- Evaluate resulting sum asymptotically
- Discard the tails

Behavior at the singularity $z = -1$

- Result of the Laplace method:

$$\sum_{n \geq 1} e^{\sqrt{n}} (-z)^n \sim \frac{\sqrt{\pi} e^{-1/8}}{(1+z)^{3/2}} \exp\left(\frac{1}{4(1+z)}\right), \qquad z \to -1.$$

- Straightforward generalization to arbitrary parameters $c > 0$, $0 < \theta < 1$.

Behavior at the singularity $z = -1$

- **Case 3:** $c < 0$, $0 < \theta < 1$.
- Series defining $E(z)$ and its derivatives converge at $z = -1$.

$$u_k := \frac{1}{k!} \lim_{z \to -1^+} \frac{d^k}{dz^k} E(z) = (-1)^k \sum_{n \geq 1} \binom{n}{k} \exp(cn^\theta).$$

- Formal Taylor series at $z = -1$ is an asymptotic series for the function (extension of Abel's theorem, see Ford 1916).

$$E(z; c, \theta) \underset{z \to -1^+}{\sim} u_0 + u_1(1+z) + u_2(1+z)^2 + \dots$$

- This formal series does not converge in any neighborhood of $z = -1$.

Application: e^{cn^θ} is not a holonomic sequence

- A function $f(z)$ is holonomic if it satisfies an LODE

$$p_0(z)f^{(0)}(z) + \cdots + p_d(z)f^{(d)}(z) = 0$$

  with polynomial coefficients.

- A sequence (a_n) is holonomic if it satisfies an LORE

$$p_0(n)a_n + \cdots + p_d(n)a_{n+d} = 0$$

  with polynomial coefficients.

- Our results + classical LODE asymptotics imply: e^{cn^θ} is not holonomic (except for trivial cases).
- "Strong transcendence result" (algebraic power series are holonomic.)

Application: Finite differences

- Given a sequence f_n, define

$$g_n := D_n[f] := \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_k$$

- Generating function:

$$g(z) = \frac{1}{1-z} \left(f_0 + F\left(\frac{z}{1-z} \right) \right),$$

where

$$F(z) = \sum_{n \geq 1} f_n (-z)^n.$$

Application: Finite differences

- Asymptotics of $F(z) \Rightarrow$ asymptotics of $g(z)$
- If $F(z)$ has "tame" asymptotics, we get the asymptotics of $g_n = D_n[f]$ by singularity analysis
- Example: For $f_n = e^{\pm\sqrt{n}}$ we have

$$g(z) \sim -\frac{\pm 1}{\sqrt{\pi}} \frac{1}{1-z} \left(\log \frac{1}{1-z} \right)^{-1/2},$$

hence by singularity analysis

$$g_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k e^{\pm\sqrt{k}} \sim -\frac{\pm 1}{\sqrt{\pi \log n}}.$$

Application: Asymptotics of a certain EXBERT distribution

- EXBERT distributions (EXchangeable BERnoulli trials, Madsen 1993)

$$P[X = x] = \binom{n}{x} \sum_{j=0}^{n-x} \binom{n-x}{j} (-1)^j \pi_{x+j}, \qquad x \in \{0, \dots, n\},$$

  where π_k, $0 \leq k \leq n$, is a probability distribution with all differences non-negative.

- "power model": $\pi_k = \exp((\log p) k^a)$ with $0 < a < 1$ and $0 < p < 1$.
- Asymptotics for large n:

$$P[X = x] \underset{n \to \infty}{\sim} \begin{cases} \frac{-\log p}{\Gamma(1-a)(\log n)^a} & x = 0 \\ \frac{-a \log p}{x \Gamma(1-a)(\log n)^{a+1}} & x \geq 1 \end{cases}.$$

Application: Finite differences (continued)

- Slowly varying functions and oscillating functions can be subjected to singularity analysis
- By our saddle point results, we find the amusing formulas

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k e^{1/k} \sim -\frac{e^{2\sqrt{\log n}}}{2\sqrt{\pi}(\log n)^{1/4}},$$

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k e^{-1/k} = -\frac{\cos\left(2\sqrt{\log n} - \frac{1}{4}\pi\right)}{\sqrt{\pi}(\log n)^{1/4}} + o\left((\log n)^{-1/4}\right).$$

Heuristic partition asymptotics (Hardy)

- The function $E(z)$ occurs in a heuristic argument to identify the constant $\pi\sqrt{2/3}$ in $p(n) \approx \exp(\pi\sqrt{\frac{2n}{3}})$.
- Generating function of the partition sequence

$$P(z) = \sum_{n \geq 1} p(n) z^n = \prod_{k \geq 1} (1 - z^k)^{-1}$$

- By an easy estimate

$$P(z) \approx \exp \frac{\pi^2}{6(1-z)} =: \sum a_n z^n.$$

- What is the growth order of a_n?
- Polynomial growth $\Rightarrow \sum a_n z^n$ too small
- Exponential growth $\Rightarrow \sum a_n z^n$ too large

Heuristic partition asymptotics (Hardy)

- Guess: $a_n \approx e^{Bn^b}$ for some constants $B > 0$ and $0 < b < 1$.
- The growth order of

$$\sum e^{Bn^b} z^n, \qquad z \to 1,$$

is given by its maximum term

$$\exp(C(1-z)^{-b/(1-b)}), \quad \text{where } C = B^{1/(1-b)} b^{b/(1-b)} (1-b).$$

- Matching this to $P(z) \approx \exp \frac{\pi^2}{6(1-z)}$ yields

$$p(n) \approx \exp(\pi \sqrt{\frac{2n}{3}}).$$

Heuristic partition asymptotics (Hardy)

- Relation to our work:
- Hardy's argument needs the rough growth order of

$$\sum e^{Bn^b} z^n, \qquad z \to 1,$$

- We determine the asymptotics of

$$\sum e^{Bn^b} z^n, \qquad z \to 1,$$

exactly by Laplace's method.

Conclusion

- Lindelöf integrals provide analytic continuation of power series and allow to do asymptotics
- Asymptotics of $E(z; c, \theta)$ depends strongly on the parameter region (six different expansions)
- Extensions to related functions like

$$\sum_{n\geq 1} \exp(cn^{\theta}(\log n)^{\alpha})(-z)^n$$

should be relatively easy