On a Certain Functional Equation: Oscillations in the Solutions and their Taylor Coefficients

Stefan Gerhold (joint work with Michael Drmota)
Vienna University of Technology

September 8, 2008

The Problem

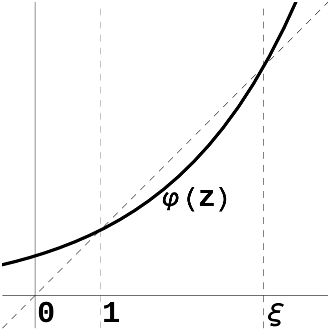
- Consider the functional equation

$$P(z) = a(z)P(\varphi(z))$$

  with given analytic $a(z)$ and $\varphi(z)$.
- Suppose that $\varphi(z)$ has an attracting fixed point at $z = 1$, and a repelling fixed point at $z = \xi > 1$.
- Suppose $a(1) = 1$.
- Asymptotics of $P(z)$ at its dominating singularity?
- Asymptotics of the Taylor coefficients of $P(z)$?

The Function $\varphi(z)$

- Attracting fixed point: $0 < \varphi'(1) < 1$
- Repelling fixed point: $\varphi'(\xi) > 1$



Additive and Multiplicative Formulation

- Multiplicative:
$$P(z) = a(z)P(\varphi(z)).$$
- Solution:
$$P(z) = \prod_{k\geq 0} a(\varphi^{[k]}(z)).$$
- Additive (take logs):
$$Q(z) = u(z) + Q(\varphi(z)).$$
- Solution:
$$Q(z) = \sum_{k\geq 0} u(\varphi^{[k]}(z)).$$

Results

- De Bruijn (1979): Asymptotics of $Q(z)$ in special cases.
- Teufl (2007): Asymptotics of $P(z)$, $Q(z)$ and their Taylor coefficients.
- Our contributions:
  1. New proof, both for the asymptotics of the function and the transfer to the coefficients.
  2. We show that the results are applicable to the case $\varphi(z) = \mathrm{e}^{\lambda(z-1)}$.

Motivation: Galton-Watson Processes with Immigration

- Evolution of a population Z_n:

$$Z_0 = 1, \qquad Z_{n+1} = \sum_{k=1}^{Z_n} X_{n,k} + I_n.$$

- Independent random variables (offspring and immigration)

$$X_{n,k} \sim \text{Poisson}(\lambda), \qquad I_n \sim \text{Poisson}(c).$$

- The law of Z_n tends to an equilibrium distribution if $0 < c, \lambda < 1$.
- Skellam, Haldane (1949)

Galton-Watson Processes with Immigration

- Probability generating function $P(z)$ of the equilibrium distribution satisfies

$$P(z) = \varphi_c(z)P(\varphi_\lambda(z)), \qquad \varphi_\lambda(z) = \mathrm{e}^{\lambda(z-1)}.$$

- Iteration yields

$$P(z) = \prod_{k\geq 0} \varphi_c(\varphi_\lambda^{[k]}(z)).$$

- We will analyze

$$\log P(z) = c \sum_{k\geq 1} (\varphi_\lambda^{[k]}(z) - 1).$$

Analyticity

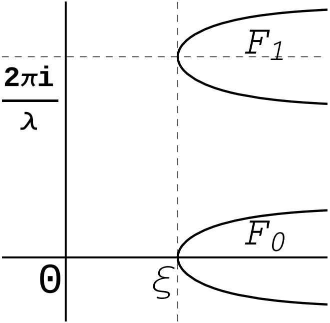
- We are interested in the asymptotics of

$$G(z) := c^{-1} \log P(z) = \sum_{k \geq 0} (\varphi_\lambda^{[k]}(z) - 1).$$

- Analytic in the basin of attraction of 1.
- The radius of convergence is ξ.
- Is $G(z)$ analytic in a Delta domain with indent at ξ?

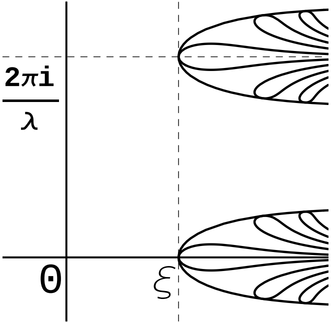
Dynamics of the exponential function

- $\varphi_\lambda(z) = \exp(\lambda(z-1))$ is a contraction for $\Re(z) < \xi$.
- Hence $\{\Re(z) < \xi\}$ is in the basin of attraction of 1.
- Preimage of $\{\Re(z) \geq \xi\}$: Union of "Fingers" F_k, $\quad k \in \mathbb{Z}$.



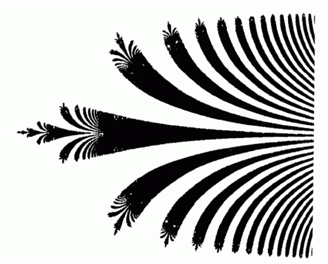
Dynamics of the exponential function

- $\varphi_\lambda(z) = \exp(\lambda(z-1))$ is a contraction for $\Re(z) < \xi$.
- Hence $\{\Re(z) < \xi\}$ is in the basin of attraction of 1.
- Preimage of $\{\Re(z) \geq \xi\}$: Union of "Fingers" $F_k, \quad k \in \mathbb{Z}$.



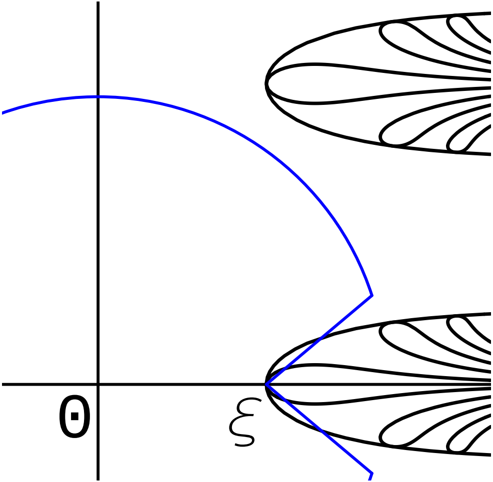
The Julia set of $\varphi_\lambda(z)$

- Closure of the set of points whose orbits tend to infinity.
- A "Cantor bouquet" (Devaney, Krych 1984)



The Delta-Domain

- The "pacman" has to avoid all preimages of ξ.



Linearization

- Poincaré equation

$$f_2(\lambda\xi z) = \varphi_\lambda(f_2(z)), \qquad f_2(0) = \xi.$$

- Unique analytic solution around $z = 0$, up to the transformation

$$f(z) \to f(\text{const} \times z).$$

  We choose $f_2'(0) = -1$.

Linearization

- Poincaré equation

$$f_2(\lambda\xi z) = \varphi_\lambda(f_2(z)), \qquad f_2(0) = \xi.$$

- f_2 is entire, and

$$\varphi_\lambda(z) = f_2(\lambda\xi f_2^{[-1]}(z)).$$

- Schröder equation

$$\frac{1}{\lambda\xi} f_2^{[-1]}(z) = f_2^{[-1]}(\varphi_\lambda^{[-1]}(z)), \qquad f_2^{[-1]}(\xi) = 0.$$

The Linearized Function

- New problems: 1. Asymptotics of

$$\sum_{k\geq 0}(f_2((\lambda\xi)^k u) - 1), \qquad u \to 0;$$

- 2. Asymptotics of

$$f_2^{[-1]}(z), \qquad z \to \xi.$$

- But $f_2^{[-1]}$ is analytic at ξ:

$$f_2^{[-1]}(z) = \xi - z + c_2(\xi - z)^2 + \ldots$$

Auxiliary Functions

- Poincaré equation

$$f_1(\lambda z) = \varphi_\lambda(f_1(z)), \qquad f_1(0) = 1.$$

- Then

$$\varphi_\lambda(z) = f_1(\lambda f_1^{[-1]}(z)).$$

- Schröder equation

$$\lambda f_1^{[-1]}(z) = f_1^{[-1]}(\varphi_\lambda(z)), \qquad f_1^{[-1]}(1) = 0.$$

Auxiliary Functions

- Define
$$g := f_1^{[-1]} \circ f_2$$
- Then
$$g(\lambda\xi z) = \lambda g(z).$$
- Hence
$$g(z) = z^{-1/\eta} E(\log_{\lambda\xi} z),$$
where $\lambda^{-\eta} = \lambda\xi$, and E is analytic in a strip and 1-periodic.

Asymptotics of f_2 at Infinity

- By definition of g
$$f_2 = f_1 \circ g.$$
- f_1 is analytic at zero:
$$f_1(z) = 1 + f_1'(0)z + \dots$$
- Hence there are analytic 1-periodic functions E_k s.t.
$$\begin{aligned} f_2(z) &= f_1(z^{-1/\eta} E(\log_{\lambda\xi} z)) \\ &= 1 + \sum_{k\geq 1} E_k(\log_{\lambda\xi} z) z^{-k/\eta}. \end{aligned}$$

Mellin Transform

- Definition:

$$\mathcal{M}(F(u))(s) = \int_0^\infty u^{s-1} F(u) \mathrm{d}u.$$

- Transform of our sum:

$$\mathcal{M} \sum_{k \geq 0} (f_2((\lambda\xi)^k u) - 1)(s) = \frac{\mathcal{M}(f_2(u) - 1)(s)}{1 - (\lambda\xi)^{-s}}$$

$$=: \frac{M(s)}{1 - (\lambda\xi)^{-s}}.$$

The Fundamental Strip

- $(1-(\lambda\xi)^{-s})^{-1}$ has poles at

$$2k\pi\mathrm{i}/\log\lambda\xi, \qquad k\in\mathbb{Z}.$$

- Expansions of f_2:

$$f_2(u)-1=\xi-1+\mathrm{O}(u), \qquad u\to 0,$$
$$f_2(u)-1=1+\mathrm{O}(u^{-1/\eta}), \qquad u\to\infty.$$

- Hence the transformed function is meromorphic in $-1<\Re(z)<1/\eta$, with a double pole at zero.

Asymptotics

- The inverse Mellin Transform yields the following expansion for $u \to 0$:

$$\sum_{k\geq 0}(f_2((\lambda\xi)^k u) - 1)$$

$$= -\frac{\xi - 1}{\log \lambda\xi} \log u + \sum_{k\neq 0} \frac{M(2k\pi\mathrm{i}/\log \lambda\xi)}{\log \lambda\xi} u^{2k\pi\mathrm{i}/\log \lambda\xi} + \mathrm{o}(1)$$

$$=: -\frac{\xi - 1}{\log \lambda\xi} \log u + H(log_{\lambda\xi} u) + \mathrm{o}(1)$$

with $H(z)$ analytic and 1-periodic.

Asymptotics of the Additive Equation

- Recall: We are interested in

$$G(z) = \sum_{k \geq 0} (f_2((\lambda\xi)^k f_2^{[-1]}(z)) - 1), \qquad z \to \xi.$$

- $$f_2^{[-1]}(z) = \xi - z + c_2(\xi - z)^2 + \dots$$

- Hence the result:

$$G(z) = \frac{\xi - 1}{\log \lambda\xi} \log \frac{1}{\xi - z} + H\left(\frac{1}{\log \lambda\xi} \log \frac{1}{\xi - z}\right) + \mathrm{o}(1).$$

Asymptotics of the Multiplicative Equation

- Recall:
$$P(z) = \varphi_c(z)P(\varphi_\lambda(z)).$$
- We have
$$P(z) = \exp(cG(z)).$$
- Hence the result:

$$P(z) \sim (\xi - z)^{-\alpha} \times \tilde{H}(\frac{1}{\log \lambda\xi} \log \frac{1}{\xi - z}), \qquad z \to \xi,$$

with $\alpha := c(\xi - 1)/\log \lambda\xi$, and $\tilde{H}(z)$ analytic and 1-periodic.

Generalization

- Assumptions:
  1. $\varphi(z)$ is analytic and maps some Delta domain, with indent at ξ, into itself.
  2. $\varphi(z)$ has an attracting fixed point at $z = 1$, and a repelling fixed point at $z = \xi$.
  3. $u(z)$ is analytic in the Delta domain, and $u(1) = 0$.
- Then the solution of the additive equation

$$Q(z) = u(z) + Q(\varphi(z))$$

satisfies

$$Q(z) = \frac{u(\xi)}{\log \lambda\xi} \log \frac{1}{\xi - z} + H\left(\frac{1}{\log \lambda\xi} \log \frac{1}{\xi - z}\right) + o(1).$$

Teufl's Approach (2007), Extending de Bruijn (1979)

- Consider
$$Q(z) = u(z) + Q(\varphi(z)).$$
- Ansatz
$$Q(z) = -\frac{u(\xi)}{\log \lambda\xi} \log(\xi - z) + h^+(z).$$
Define
$$h^-(z) = \sum_{k=-\infty}^{-1} \left(u(\varphi^{[k]}(z)) - \frac{u(\xi)}{\log \lambda\xi} \log \frac{\varphi^{[k+1]}(z) - \xi}{\varphi(z) - \xi} \right).$$
- Then
$$h^-(z) = \mathrm{O}(\xi - z).$$

Teufl's Approach (2007), Extending de Bruijn (1979)

- The function

$$h := h^- + h^+ \qquad \text{satisfies} \qquad h(z) = h(\varphi(z)).$$

- From the latter equation one can deduce

$$h(z) = H\left(\frac{1}{\log \lambda\xi} \log \frac{1}{\xi - z}\right) + \mathrm{O}(\xi - z),$$

where E is 1-periodic and analytic.

Where We Are

- We have determined the asymptotics of the solution $P(z)$ of

$$P(z) = a(z)P(\varphi(z)).$$

- Second step: How do the Taylor coefficients of $P(z)$ behave?
- In the example of branching processes, they constitute the probability mass function of the equilibrium distribution.

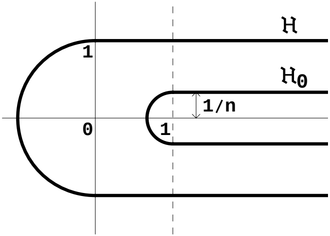
Singularity Analysis

- Express Taylor coefficients by Cauchy's formula

$$[z^n]P(z/\xi) = \frac{1}{2\pi \mathrm{i}} \oint z^{-n-1}(1-z)^{-\alpha} H(\log \frac{1}{1-z}) \mathrm{d}z.$$

- Deform the integration path to $\mathcal{H}_0$, which encircles 1 at the distance

$$c_n/n, \qquad \text{where} \qquad c_n = \exp(\{\log n\}).$$



Cauchy's Coefficient Formula

- Substitute $z = 1 + c_n t/n$:

$$
\begin{aligned}
[z^n]P(z/\xi) &= \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{H}_0} z^{-n-1}(1-z)^{-\alpha} H(\log\frac{1}{1-z})\mathrm{d}z \\
&= \frac{n^{\alpha-1} c_n^{1-\alpha}}{2\pi\mathrm{i}} \int_{\mathcal{H}} \left(1 + \frac{c_n t}{n}\right)^{-n-1} (-t)^{-\alpha} \\
&\qquad\qquad H(\log(-\frac{1}{t}) + \log\frac{c_n}{n})\mathrm{d}t \\
&\sim \frac{n^{\alpha-1} c_n^{1-\alpha}}{2\pi\mathrm{i}} \int_{\mathcal{H}} \mathrm{e}^{c_n t}(-t)^{-\alpha} H(\log(-\frac{1}{t}))\mathrm{d}t \\
&= n^{\alpha-1}\mathrm{func}(\{\log n\}).
\end{aligned}
$$

Further Work

- Are our approaches useful?
- When is the oscillating term constant?
- For the latter question, our representation of the Fourier coefficients could be helpful.

References

- De Bruijn: An asymptotic problem on iterated functions, Nederl. Akad. Wetensch. Indag. Math. 41 (1979), 105–110.
- R.L. Devaney, M. Krych: Dynamics of Exp(z), Ergodic Theory and Dynamical Systems 4 (1984), 35–52.
- E.Teufl: On the asymptotic behaviour of analytic solutions of linear iterative functional equations, Aequationes Math. 73 (2007), 18–55.