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Eurodollar futures pricing in log-normal interest rate models in discrete time

Anxhelo Vasili

betreut von

Associate Prof. Dipl.-Ing. Dr.techn.

Stefan GERHOLD

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Contents

1 Abstract

Accordind to the article we are going to demonstrate the appearance of explosions in three quantities in interest rate models with log-normally distributed rates in discrete time. (1) The expectation of the money market account in the Black, Derman, Toy model, (2) the prices of Eurodollar futures contracts in a model with log-normally distributed rates in the terminal measure and (3) the prices of Eurodollar futures contracts in the one-factor log-normal Libor market model (LMM). We derive exact upper and lower bounds on the prices and on the standard deviation of the Monte Carlo pricing of Eurodollar futures in the one factor log-normal Libor market model. These bounds explode at a non-zero value of volatility, and thus imply a limitation on the applicability of the LMM and on its Monte Carlo simulation to sufficiently low volatilities.

2 Introduction

Interest rate models with log-normally distributed rates in continuous time are known to display singular behaviour. The simplest setting where this phenomenon appears is for log-normal short-rate models such as the Dothan model and the Black–Karasinski model. It was shown by Hogan andWeintraub that the Eurodollar futures prices in these models are divergent. Similar explosions appear in Heath, Jarrow, and Morton model with log-normal volatility specification , where the forward rates explode with unit probability. The case of these models is somewhat special, as the Eurodollar futures prices are well-behaved in other interest rate models of practical interest,such as the CIR , and the Hull-White model. It is widely believed that the same models when considered in discrete time are free of divergences see for an account of the historical devel-opment of the log-normal interest rate models. The discrete time version of the Dothan model is the Black, Derman, Toy model , while the Black–Karasinski model can be simulated both in discrete and continuous time . In this article, we demonstrate the appearance of numerical explosions for several quantities in interest rate models with log-normally distributed rates in discrete time. The explosions appear at a finite critical value of the rate volatility. This phenomenon is shown to appear for accrual quantities such as the money market account and the Eurodollar futures prices. The quantities considered remain finite but their numerical values grow very fast above the critical volatility such that they rapidly exceed machine precision. Thus for all practical purposes, they can be considered as real explosions, and their appearance introduces limitations on the use of the models for the particular application considered. In Section 2, we consider the expectation of the money market account in a discrete time short rate model with rates following a geometric Brownian motion. Using an exact solution one can show the appearance of a numerical explosion for this quantity, for sufficiently large number of time steps or volatility. This phenomenon and the conditions under which it appears have been studied in detail elsewhere . Here, we review the main conclusions of this study and point out its implications for the simulation of interest rate models with log-normally distributed rates in discrete time. Sections 3 and 4 consider the calculation of the Eurodollar futures prices in two interest rate models: a one-factor model with log-normally distributed rates in the terminal measure, and the one-factor log-normal Libor market model, respectively. The Eurodollar futures convexity adjustment is computed exactly in the former model, while for the latter we derive exact upper and lower bounds. Both the exact result and the bounds display numerical explosive behaviour for sufficiently large volatilities, which are of the order of typical market volatilities. These explosions limit the applicability of these models for the pricing of Eurodollar futures to sufficiently small values of the volatility.

3 The expectation of the money market account in the BDT model

Lets consider the Black–Derman–Toy model.The following model is defined on tenor dates t_i with $i = 0, 1, .., n$ which are assumed to be uniformly spaced with time step $\tau = t_{i+1} - t_i$.The model is defined in the Risk-neutral measure Q and Numeraire the money market account with discrete time compounding:

$$B_i = \prod_{k=0}^{i-1} (1 + L_k \tau). \tag{1}$$

where $L_k = L_{k,k+1}$ the Libor Rate for the period (t_i, t_{i+1})
The BDT model is defined by the following distributional assumption for the Libors L_i in the Risk-Neutral measure:

$$L_i = \tilde{L}_i e^{\sigma_i W_i - \frac{1}{2}\sigma_i^2 t_i} \tag{2}$$

where

- W_i is a standard Brownian motion in the Risk-Neutral measure sampled at the discrete times t_i
- $\tilde{L}_i$ are constants, which are determined by calibration to the initial yield curve
- σ_i are the rate volatilities, which are calibrated such that the model reproduces a given set of volatility instruments such as caplets or swaptions.

Now for given initial yield curve $P_{0,i}$ and rate volatilities σ_i the calibration problem consists in finding $\tilde{L}_i$ such that $P_{0,i} = \mathbb{E}_{\mathbb{Q}}[B_i^{-1}]$ for all $1 < i \leq n$.The solution for $\tilde{L}_i$ exists provided that the following condition is satisfied $P_{0,i} > P_{0,i+1} > 0$.The solution to the calibration problem satisfies the inequality $\tilde{L}_i > L_i^{fwd}$ where $L_i^{fwd} = 1/\tau(P_{0,i}/P_{0,i+1-1})$ are the forward rates for the period (t_i, t_{i+1}).

- In the zero volatility limit $\sigma = 0$, the money market account B_n is given by

$$B_n = \prod_{k=0}^{n-1} (1 + L_k^{fwd} \tau)^n \tag{3}$$

- for $\sigma > 0$ the money market account B_n becomes a random variable.

Proposition 2.1. Now the the money market account in the BDT model is

$$B_n = \prod_{k=0}^{n-1} \left(1 + \tilde{L}_i \tau e^{\sigma W_k - \frac{1}{2}\sigma^2 t_k}\right) \tag{4}$$

The p-th moment of B_n is given by

$$\mathbb{E}_{\mathbb{Q}}[B_n^p] = (1 + \rho_0)^p \sum_{k=0}^{p(n-1)} b_k^{(0,p)}, \tag{5}$$

where the coefficients $b_k^{(0,p)}$ are found by solving the backwards recursion:

$$b_k^{(i,p)} = b_k^{(i+1,p)} + \sum_{m=1}^{p} \binom{p}{m} b_{k-m}^{(i+1,p)} \rho_{i+1}^m e^{m(k-\frac{1}{2}m-\frac{1}{2})\sigma^2 t_{i+1}} \tag{6}$$

with $p_k = \tilde{L}_k \tau$ and initial conditions

$$b_0^{(n-1,p)} = 1, \quad b_k^{(n-1,p)} = 0, \quad k > 1 \tag{7}$$

The coefficients $b_k^{(i,p)}$ with negative indices $k < 0$ are zero.

3.1 Discrete time moment explosion of the money market account

In this part we are going to use Proposition 2.1 so we can study the dependence of the moments of the money market account $\mathbb{E}_{\mathbb{Q}}[B_n^p]$ on n, σ, τ.
We assume uniform Parameters $\tilde{L}_k = L_0, \sigma_k = \sigma$. The first two results following form Proposition 2.1 are given by

$$\mathbb{E}_{\mathbb{Q}}[B_n] = (1 + \tilde{L}_0 \tau) \sum_{j=0}^{n-1} c_j^{(0)} \tag{8}$$

first two Results following form Proposition 2.1 are given by

$$\mathbb{E}_{\mathbb{Q}}[B_n^2] = (1 + \tilde{L}_0 \tau)^2 \sum_{j=0}^{2(n-1)} d_j^{(0)} \tag{9}$$

where $c_j^{(i)}$ and $d_j^{(i)}$ are the solutions to the backwards recursions:

$$c_j^{(i)} = c_j^{(j+1)} + \tilde{L}_{i+1} \tau c_{j-1}^{(j+1)} e^{\sigma^2 (j-1) t_{i+1}} \tag{10}$$

$$d_j^{(i)} = d_j^{(j+1)} + 2\tilde{L}_{i+1} \tau d_{j-1}^{(j+1)} e^{\sigma^2 (j-1) t_{i+1}} + (\tilde{L}_{i+1} \tau)^2 d_{j-2}^{(j+1)} e^{\sigma^2 (2j-3) t_{i+1}} \tag{11}$$

with initial conditions $c_0^{(n-1)} = 1$, $d_0^{(n-1)} = 1$ and all other coefficients $c_k^{(n-1)} = d_k^{(n-1)} = 0$. The coefficients $c_j^{(i)}$, $d_j^{(i)}$ for $j < 0$ are zero.

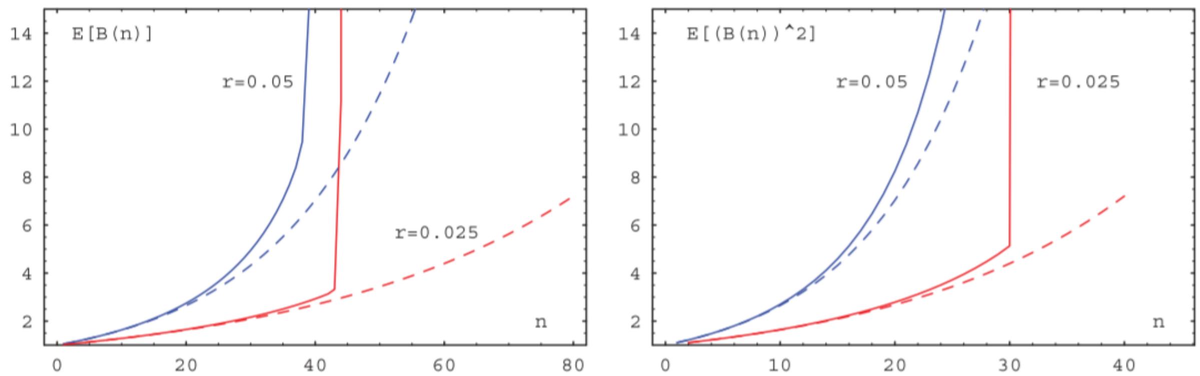

Figure 1. The expectation of the money market account $\mathbb{E}_{\mathbb{Q}}[B_n]$ (left) and of the second moment $\mathbb{E}_{\mathbb{Q}}[B_n^2]$ (right) vs. n assuming yearly compounding $\tau = 1$. $L_0 = 2.5\%$ (red - solid lower curve) and 5.0% (blue - solid upper curve), and the rate volatility is $\sigma = 10\%$. The dashed curves show the result in the zero rate volatility limit $B_n = (1 + L_0 \tau)^n$.

The figure above shows typical plots of the expectation $\mathbb{E}_{\mathbb{Q}}[B_n]$ as function of n at fixed σ ,L_0,τ.The results of this numerical study show that the expectation of the money market account$\mathbb{E}_{\mathbb{Q}}[B_n]$ has an explosive behaviour at a certain time step n.
Although its numerical value remains finite, in the explosive phase this quantity grows very fast and can quickly exceed double precision in a finite time. The same phenomenon is observed by keeping fixed n ,L_0,τ and considering $\mathbb{E}_{\mathbb{Q}}[B_n]$ as function of the volatility σ,and also at fixed σ,$n\tau$,L_0 and making the time step τ sufficiently small.A similar explosion is observed also for the higher integer moments $p > 1$. We show in Table 1. typical results for the average, the second and fourth moments of the money market account in a simulation with time step $\tau = 1$.

Table 1. The expectation $\mathbb{E}_{\mathbb{Q}}[B_n]$, the second and the fourth moments $\mathbb{E}_{\mathbb{Q}}[B_n^2]$, $\mathrm{E}_{\mathrm{Q}}[B_n^4]$ of the money market account with yearly compounding $\tau = 1$ after n years, for $L_0 = 5\%$ and volatility $\sigma = 10\%$.

n	$(1 + L_0\tau)^n$	$\mathbb{E}_{\mathbb{Q}}[B_n]$	$\mathbb{E}_{\mathbb{Q}}[B_n^2]$	$\mathbb{E}_{\mathbb{Q}}[B_n^4]$	$\mathbb{E}_{\mathbb{Q}}[B_n]_{\mathrm{MC}}$	Σ_{MC}
1	1.05	1.05	1.10	1.22	1.05	0
5	1.28	1.28	1.63	2.68	1.28	0
10	1.63	1.64	2.70	7.59	1.63	1.4×10^{-4}
20	2.65	2.75	8.25	5.3×10^{7}	2.73	$7.6 \cdot 10^{-4}$
30	4.32	4.95	2.93×10^{11}	1.8×10^{175}	4.88	0.004
40	7.04	184	3.96×10^{90}	2.1×10^{558}	10.48	0.048
41	7.39	11,871	1.61×10^{102}	1.0×10^{611}	11.59	0.07
42	7.76	1,424,670	3.54×10^{114}	3.6×10^{666}	12.91	0.10
43	8.15	2.89×10^{8}	4.32×10^{127}	1.2×10^{725}	14.52	0.15
44	8.56	9.76×10^{10}	3.05×10^{141}	3.9×10^{786}	16.51	0.23
45	8.99	5.45×10^{13}	1.29×10^{156}	1.6×10^{851}	19.04	0.35

The last two columns show the Monte Carlo estimates of the average $\mathbb{E}_{\mathbb{Q}}[B_n]_{\mathrm{MC}}$ and standard deviation $\Sigma_{B_n,N}$ in a simulation with $N = 10^6$ paths.

Theorem 2.2.The limit

$$\lim_{n\to\infty, \frac{1}{2}\sigma^2 t_n n=\beta} log\mathbb{E}_{\mathbb{Q}}[B_n^q] = \lambda(\rho;\beta;q), \tag{12}$$

with $\rho = L_0\tau$, with $q \in \mathbb{N}$ exists, and depends only on ρ and β.The function $\lambda(\rho;\beta;q)$is the Lyapunov exponent of the positive integer qth momement,and is related to the Lyapunov exponent $\lambda(\rho;\beta;1) = \lambda(\rho\beta)$ of the first momentThe function $\lambda(\rho,\beta)$ is given by

$$\lambda(\rho,\beta) = sup_{d\in(0,1)}\Lambda(d) \tag{13}$$

where

$$\Lambda(d) = \beta d^2 + log(1+\rho) - 2\beta(1+\rho)d^3 \int_0^1 dy \frac{y^2}{1+\rho - e^{\beta d^2(y^2-1)}} \tag{14}$$

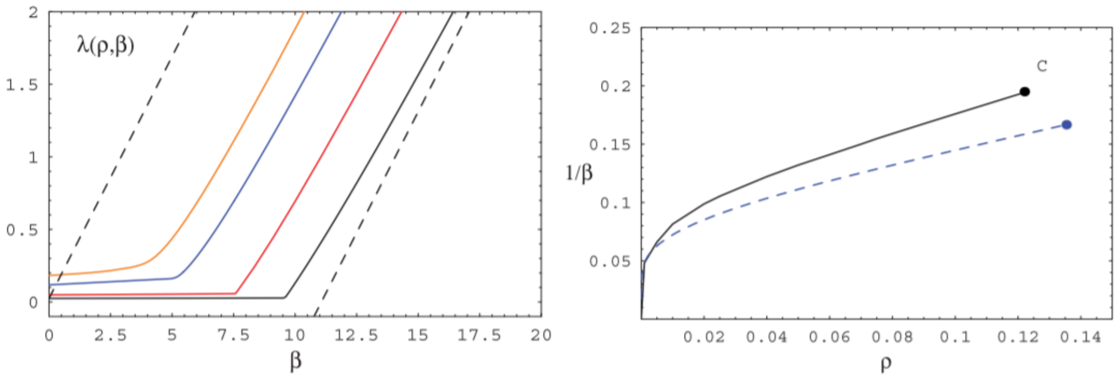

Figure 2. (Left) Plots of the Lyapunov exponent $\lambda(\rho,\beta)$ vs. β for $\rho = 0.025, 0.05, 0.125, 0.2$ (from bottom to top) (Pirjol and Zhu 2015). The dashed curves give upper and lower bounds $\frac{1}{3}\beta + \log(1+\rho) \geq \lambda(\rho,\beta) \geq \frac{1}{3}\beta + \log\rho$, see Proposition 2 in (Pirjol and Zhu 2015). (Right) The transition curve $\beta_{cr}(\rho)$ (solid black), and its simpler approximation $\bar{\beta}_{cr}(\rho) = -3\log\rho$ (dashed blue) ending at $\bar{\beta}_C = 6, \bar{\rho}_C = e^{-2}$ (blue dot).

In figure 2(left panel) we see typical plots of $\lambda(\rho,\beta)$ versus β for several values of ρ. The function $\lambda(\rho,\beta)$ is everywhere continuous in its arguments (ρ,β) but has discontinious derivative $\partial_\beta\lambda(\rho,\beta)$ at a certain point $\beta_{cr}(\rho)$ for ρ below a critical value $\rho < \rho_c = 0.123$.The right panel of the figure shows the critical curve $\beta_{cr}(\rho)$.
The critical curve $\beta_{cr}(\rho)$ is well approximated as

$$\bar{\beta}_{cr}(\rho) = -3log(\rho) \tag{15}$$

This approximation of the critical curve ends at the critical point $(\rho_c,\beta_c) = (e^{-2}, 6)$.

3.2 Implications for Monte Carlo simulations

The moment explosion of the money market account has implications for the Monte Carlo simulation of the process B_t.Lets consider the MC estimate for the expectation $\mathbb{E}_{\mathbb{Q}}[B_n]$ obtained by averaging over N paths.The standard deviation of this estimate is related to the variance of B_n as,

$$\sum\nolimits_{B_n,N} = \frac{1}{\sqrt{N}}\sqrt{varB_n} \tag{16}$$

The explosion of the second moment $\mathbb{E}_{\mathbb{Q}}[B_n^2]$ implies that the variance of B_n grows very fast even as the average value $\mathbb{E}_{\mathbb{Q}}[B_n]$ is well behaved.This is seen in practical MC simulations as a rapid increase in the variance of the sample, but we will show next that a reliable estimate of this variance using the MC sample is problematic.The Monte Carlo simulation methods can not be used to compute precisely the expectation and higher moments of B_n in the explosive phase. The same phenomenon will be seen to appear in several other quantities in models with log-normally distributed rates, and introduces a limitation in the applicability of MC methods for computing these quantities.

3.3 Continuous time limit and relation to the Hogan–Weintraub singularity

In the continuous time limit, the BDT model with constant volatility $\sigma_i = \sigma$ goes over into a short rate model with process for the short rate:

$$dr_t = \sigma r_t dW_t + \mu(t) r_t dt \tag{17}$$

The money market account B_t is given by

$$dB_t = r_t B_t dt \tag{18}$$

with initial condition $B_0 = 1$. The short rate r_t is given by

$$r_t = r_0 e^{\sigma W_t} + \int_0^t ds\mu(s) - \frac{1}{2}\sigma^2 t. \tag{19}$$

The solution of Equation (13) is given by the exponential of the time integral of the geometric Brownian motion

$$B_t = exp\left(r_0 \int_0^t ds e^{\sigma W_s + \int_0^s \mu(u)du - \frac{1}{2}\sigma^2 s} \right) \tag{20}$$

The expectation of B_t is infinite, for any $t > 0$. This follows by noting that the time integral of the geometric Brownian motion is bounded from below by a log-normally distributed random variable, by the arithmetic- geometric means inequality

$$\frac{1}{t}\int_0^t ds e^{\sigma W_s + \int_0^s \mu(u)du - \frac{1}{2}\sigma^2 s} \geq exp\left(\frac{1}{t}\int_0^t ds(\sigma W_s + \int_0^s \mu(u)du - \frac{1}{2}\sigma^2 s) \right) \tag{21}$$

The expectation of the exponential of the quantity on the right-hand side is infinite. This follows from the well-known result that the moment generating function $\mathbb{E}[e^{\theta X}]$ of a log-normally distributed random variable X is infinite for $\theta > 0$.

- Our results show that the approach of the discrete time model to the continuous time limit is not smooth, but proceeds through a discontinuity at some value of the time step size τ where the rate of growth of $\mathbb{E}_{\mathbb{Q}}[B_t]$ has a sudden increase. This is observed in simulations as numerical moment and path explosions.

- The explosion of the expectation of B_t is related to the Hogan–Weintraub singularity.This is shown by comparing the results for the discrete and continuous time settings.

4 Eurodollar futures in a model with log-normal rates in the terminal measure

We consider a one-factor short rate model defined on the tenor of dates $t_0, t_1, .., t_n$. The rate specification is

$$L_{i,i+1} = \tilde{L}_i e^{\sigma_i W_i - \frac{1}{2}\sigma^2 t_i} \tag{22}$$

where

- W_i is a standard Brownian motion in the t_n-th forward measure $\mathbb{P}_n$ with numeraire the zero coupon bond P_{t,t_n}.
- The coefficients $\tilde{L}_i$ are determined by yield curve calibration such that the initial yield curve $P_{0,i}$ is correctly reproduced.

This model is used in financial practice as a log-normal approximation to the log- normal Libor market model or as a parametric representation of the Markov functional model. Lets consider the Eurodollar futures contract on the rate $L_{i,i+1}$. Assuming discrete futures settlement at dates t_i, the pricing of this instrument is related to the expectation of $L_{i,i+1}$ in the spot measure $\mathbb{Q}$.This can be expressed alternatively as an expectation in the terminal measure P_n

$$\mathbb{E}_{\mathbb{Q}}[L_{i,i+1}] = P_{0,n}\mathbb{E}_n[B_i L_{i,i+1} P_{i,n}^{-1}] = P_{0,n}\mathbb{E}_n[B_i L_{i,i+1}\hat{P}_{i,i+1}\left(1 + L_{i,i+1}\tau\right)], \tag{23}$$

where

- $B_i = \prod_{k=0}^{i-1}(1 + L_k\tau)$ is the money market account at time t_i,and we denoted $\hat{P}_{i,j} = P_{i,j}/P_{i,n}$ the numeraire-rebased zero coupon bonds.

The expectation (18) can be computed exactly in the particular case of uniform volatility $\sigma_i = \sigma$. This can be done using a simple modification of the recursion relation in Proposition 2.1, and is given by the following result.
Proposition 3.1. Consider the expectation

$$M_n^{(q)} = \mathbb{E}\left[\prod_{k=1}^{n-1}\left(1 + r_k e^{\sigma W_k - \frac{1}{2}\sigma^2 t_k}\right) e^{q\sigma W_n - \frac{1}{2}(q\sigma)^2 t_n}\right], \tag{24}$$

- where r_k , σ are real positive numbers, and W_i is a standard Brownian motion started at zero $W_0 = 0$ and sampled at times t_k.This expectation is given exactly by

$$M_n^{(q)} = \sum_{p=q}^{n-1+q} c_p^{(0)}, \tag{25}$$

where

- $c_p^{(0)}$ are given by the solution of the recursion relation

$$c_p^{(i)} = c_p^{(i+1)} + r_{i+1} c_{p-1}^{(i+1)} e^{\sigma^2(p-1)t_{i+1}}, \tag{26}$$

- with the initial condition at $i = n - 1$

$$c_q^{(n-1)} = 1, c_p^{(n-1)} = 0 \quad \text{for all} \quad p \neq q. \tag{27}$$

4.1 Results

We consider the Eurodollar futures on the rate $L_{n-1,n}$ spanned by the last time step (t_{n-1}, t_n). The expectation of this rate in the terminal measure $\mathbb{P}_n$ is simply the forward rate $\tilde{L}_{n-1,n} = L^{fwd}_{n-1,n}$,since $\mathbb{P}_n$ coincides with the forward measure for this rate. Also, we have $\hat{P}_{n-1,n} = 1$.The expression (18) simplifies to

$$\mathbb{E}_{\mathbb{Q}}[L_{n-1,n}] = P_{0,n}\mathbb{E}_n[B_{n-1}L_{n-1,n}(1+L_{n-1,n}\tau)] = P_{0,n}L^{fwd}_{n-1,n}(M^{(1)}_{n-1} + L^{fwd}_{n-1,n}\tau M^{(2)}_{n-1}), \quad (28)$$

- where $M^{(1)}_{n-1}$ and $M^{(2)}_{n-1}$ are given by Proposition 3.1 with the substitutions $r_k \to \tilde{L}_k\tau$.
- The multipliers $\tilde{L}$ are obtained from the yield curve calibration of the model to the forward Libors L^{fwd}_k.

The Eurodollar futures convexity adjustment will be parameterized in terms of the ratio

$$\kappa_{ED} = \frac{M^{(1)}_{n-1} + L^{fwd}_{n-1}\tau M^{(2)}_{n-1}}{(1+L_0\tau)^{n-1}(1+L^{fwd}_{n-1}\tau)} \quad (29)$$

This quantity is defined such that it is equal to one in the zero volatility $\sigma \to 0$ limit, and is a multiplicative measure of the convexity adjustment for the Eurodollar futures contract on $L_{n-1,n}$

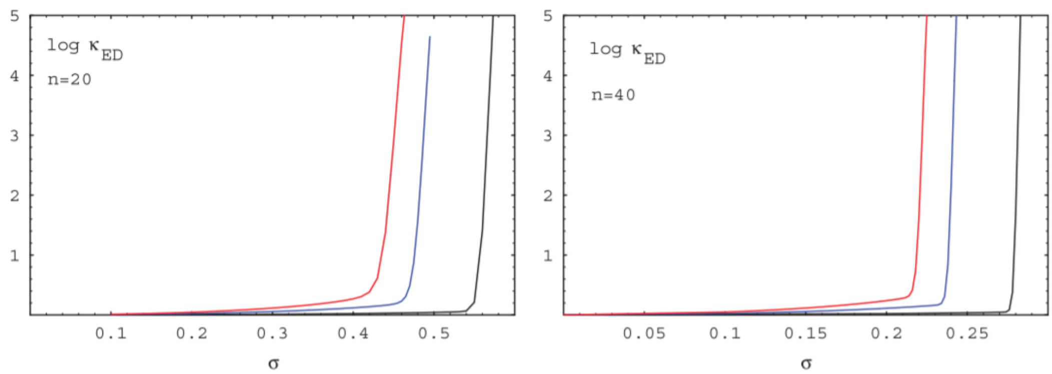

Figure 3. The Eurodollar futures convexity adjustment $\log\kappa_{ED}$ for the rate $L_{n-1,n}$, in the model with log-normal rates in the terminal measure. The rate tenor and time discretization step is $\tau = 0.25$, and the forward rate is $L_0 = 1.0\%$ (black), 5.0% (blue), 10.0% (red) (from bottom to top). The total number of time steps is $n = 20$ (left) and $n = 40$ (right).

In the Figure above we show plots of $log\kappa_{ED}$ versus σ for several values of the forward Libors L_{fwd} and total tenor n.For the numerical simulation we assume for simplicity uniform forward Libors $L^{fwd}_i = L_0$ for $i = 0, 1, .., n-1$.The numerical results for $log\kappa_{ED}$ in the Figure show an explosive behaviour at a certain value of the volatility σ.

5 Log-normal Libor market model

In this section we consider the pricing of Eurodollar futures in the one-factor log-normal Libor market model.We assume the same tenor of dates as in the previous section.We consider a market with given forward Libors L_i^{fwd} for the non-overlapping tenors (t_i, t_{i+1}) and log-normal caplet volatilities σ_i.
The log-normal Libor model gives a possible solution for the dynamics of the forward Libor rates $F_k(t) := F(t; t_k, t_{k+1})$, $k = 1, 2, ..., n-1$ which is compatible with this market.Under the t_n-forward measure, with numeraire $P_{t,n}$,the dynamics of the forward Libors $F_k(t)$ are

$$\frac{dF_{n-1}(t)}{F_{n-1}(t)} = \sigma_{n-1} dW_t \tag{30}$$

$$......$$

$$\frac{dF_k(t)}{F_k(t)} = \sigma_k dW_t - \sigma_k \sum_{j=k+1}^{n-1} \frac{\tau \sigma_j F_j(t)}{1 + F_j(t)\tau} dt, \tag{31}$$

with initial conditions $F_i(0) = L_i^{fwd}$.Here $W_t$is a standard Brownian motion in the t_n-forward measure $\mathbb{P}_n$.We assumed here a one-factor version of the log-normal LMM, where all forward Libors are driven by a common Brownian motion W_t The model can be formulated in a more general form, which can accomodate an arbitrary correlation structure between the n Libor rates.Also,we assumed for simplicity timeindependent volatilities σ_k.Model (23) s the simplest dynamics of the forward Libors compatible with the given market of forward Libors and caplet volatilities.The positivity of the forward rates $F_k(t) > 0$ implies the inequalities

$$0 < \frac{\tau F_k(t)}{1 + F_k(t)\tau} < 1, \quad k = 1, 2, \cdots, n-1 \tag{32}$$

which gives corresponding inequalities for the drift terms in Equation (23).
By the comparison theorem the following inequalities hold with probability one

$$F_k^{down}(t) < F_k(t) < F_k^{up}(t), \tag{33}$$

where

$$F_k^{down}(t) = F_k(0) exp(-\sigma_k \sum_{p=k+1}^{n-1} \sigma_p t) \exp(\sigma_k W_t - \frac{1}{2}\sigma_k^2 t), \tag{34}$$

$$F_k^{up}(t) = F_k(0) \exp(\sigma_k W_t - \frac{1}{2}\sigma_k^2 t) \tag{35}$$

These bounds imply that the probability distributions of the forward Libors $F_k(t)$ in the terminal measure $\mathbb{P}_n$ have log-normal tails.

The pricing of Eurodollar futures on the Libor rate $L_{i,i+1} = F_i(t_i)$ reduces to the evaluation of the expectation:

$$\mathbb{E}_{\mathbb{Q}}[L_{i,i+1}] = P_{0,n}\mathbb{E}_n[B_i L_{i,i+1} P_{i,n}^{-1}] = P_{0,n}\mathbb{E}_n[B_i L_{i,i+1}\hat{P}_{i,i+1}(1 + L_{i,i+1}\tau)], \tag{36}$$

This is identical to the expression (16) in the model considered in the previous section. We will derive upper and lower bounds on this expectation for the last Libor rate $i = n-1$,assuming uniform forward rates and caplet volatilities $L_i^{fwd} = L_0$ and $\sigma_i = \sigma$.The relevant expectations can be evaluated exactly using Proposition 3.1 with the substitutions

$$r_k \to F_k(0)\tau \tag{37}$$

for the upper bound, and

$$r_k \to F_k(0)\tau e^{-(n-k-1)\sigma^2 t_k} \tag{38}$$

for the lower bound.

We start by computing the upper bound on the multiplicative convexity adjustment factor κ_{ED}.This is clearly a finite value, and the finiteness of the Eurodollar futures prices noted in was one of the reasons for the acceptance and widespread use of the Libor market models.

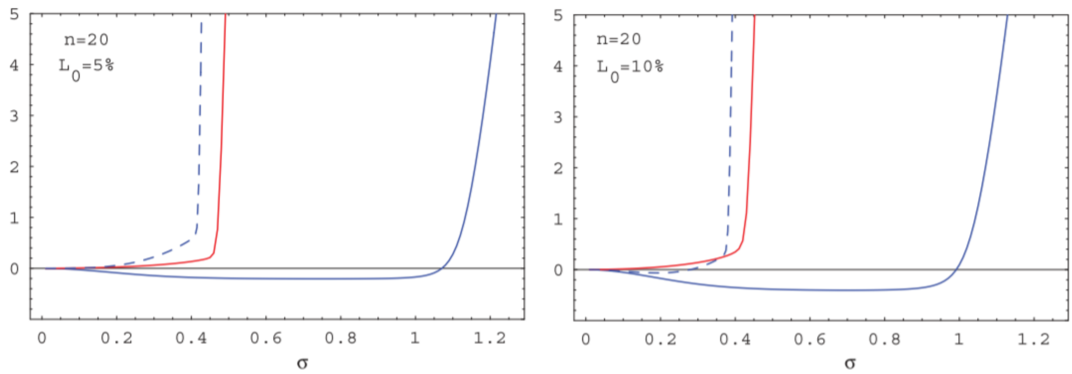

Figure 4. Upper (solid red) and lower (solid blue) bounds on the Eurodollar futures convexity adjustment $\log\kappa_{\rm ED}$ vs. σ for the rate $L_{n-1,n}$, in the one-factor log-normal Libor market model. The dashed blue curves show the lower bound on $\log\kappa_{2,\rm ED}$ defined in (4.13). The rate tenor is $\tau = 0.25$, and the forward Libor rate is flat with $L_0 = 5.0\%$ (left), 10.0% (right). The total number of time steps is $n = 20$ with time step $\tau = 0.25$.

In Figure 4 we show plots of $log\kappa_{ED}^{up}$ with κ_{ED}^{up} the upper bound on the convexity adjustment,
The upper bound has an explosion at a critical value of the volatility, which is relatively small.
The plots show also the lower bound $log\kappa_{ED}^{down}$,which display also an explosion at a higher value of the volatility.These results show that the Eurodollar futures convexity adjustment in the Libor market model explodes to unphysical values for sufficiently large volatilities. For maturity $T = 5$ years and quarterly simulation time step $\tau = 0.25$
e explosion volatility of the lower bound $log\kappa_{ED}^{down}$ is $\sigma_{exp} \simeq 110\%$ for $L_0 = 5\%$ and $\sigma_{exp} \simeq 100\%$ for $L_0 = 10\%$.
This explosion introduces a limitation of the applicability of this model for pricing Eurodollar futures to volatilities below a maximum allowed level, which depends on the rate tenor, maximum maturity and simulation time step. We give next an analytical upper bound on the explosion volatility of the lower bound which makes explicit its dependence on the model parameters.
Proposition 4.1 The explosion volatility of the lower bound on the price of the Eurodollar futures on $L_{n-1,n}$ in the LMM with uniform parameters L_0,σ is bounded from above as

$$\sigma_{exp}^2 t_n \leq -\frac{2n}{n-1} log(L_0\tau) \tag{39}$$

This bound on the explosion volatility σ_{exp} becomes smaller as the rate L_0 increases and as the maturity t_n increases. For the two cases shown in Figure 4.e bound on σ_{exp} is 134% and $121,5\%$ respectively.
These bounds divide the range of the volatility parameter σ into three regions:
(a)The low-volatility region, below the explosion volatility of the upper bound κ_{ED}^{up}.In this region the model is well behaved.
(b) An intermediate volatility region, between the explosion volatilities of the upper and lower bounds. In this region an explosive behaviour of Eurodollar futures prices is possible, but is not required by the bounds.
(c)The large volatility region, above the explosion volatility of the lower bound κ_{ED}^{down}
In this region the Eurodollar futures prices explode to unphysical values.
Although we assumed in this calculation uniform model parameters $L_i^{fwd} = L_0$,$\sigma_i = \sigma$,these bounds can be extended to the general case of arbitrary bounded parameters (L_i^{fwd}, σ_i) by using $\bar{L}^{fwd} = sup_i L_i^{fwd}$, $\bar{\sigma} = sup_i \sigma_i$ for the upper bound, and $\underline{L}^{fwd} = inf_i L_i^{fwd}$,$\underline{\sigma} = inf_i \sigma_i$ for the lower bound.
he Eurodollar futures convexity adjustment has been computed in the log-normal Libor market model in , using an analytical approximation based on the Itô-Taylor expansion, and checked by Monte Carlo simulation. The adjustment was found to be well-behaved and no singularity was observed for maturities up to $T = 5$ years Two scenarios have been considered: (i) normal vols, moderate rates scenario: $\sigma = 40\%$ and $L_0 = 5\%$, (ii) higher vols, low rates scenario:$\sigma = 60\%$ and $L_0 = 1\%$ Both scenarios lie in region b), where an explosion may occur, but is not required by the bounds considered.

6 Summary and discussion

We have shown that certain expectations related to the pricing of financial instruments have explosive behaviour at large volatility in several widely used log-normal interest rate models simulated in discrete time. Although the existence of such explosions has been known for a long time in the continuous-time version of such models, experience with the discrete time version of these models appears to suggest that no divergences are present. While this statement is strictly true mathematically, in the sense that the expectations are finite in the discrete time case, the actual numerical values can become unrealistically large, such that they are clearly unphysical. We discussed the appearance of such numerical explosions in three interest rate models with log-normal rates in discrete time. The first quantity is the expectation of the money market account in the BDT model. The discretely compounded money market account plays a central role in the simulation of interest rate models in the spot measure, where it represents the numeraire (Jamshidian 1997). A good understanding of its distributional properties is clearly of great practical importance. Due to an autocorrelation effect between successive compounding factors, the expectation and the higher positive integer moments of the money market account in discrete time under stochastic interest rates following a geometric Brownian motion have a numerical explosion (Pirjol 2015; Pirjol and Zhu 2015). The criteria for the appearance of this explosion have been derived in (Pirjol and Zhu 2015). The explosion time decreases with the rate volatility and with the time step size, and approaches zero in the continuous time limit, as expected from the continuous time theory (Andersen and Piterbarg 2007). This explosion implies that the distribution of the money market account has heavy tails, and the explosive paths appear when sampling from the tails of this distribution.

We showed in this article that similar explosive phenomena appear in expectations and variances of certain accrual-type payoffs, which have the compounding structure of the money market account, such as the Eurodollar futures prices. We illustrated this phenomenon on the case of two interest rate models: i) a one-factor short rate model with log-normal rates in the terminal measure, and ii) the one-factor log-normal Libor market model. The Eurodollar futures can be priced exactly in the former model, using the exact solution of this model presented in (Pirjol 2013). The result shows explosive behaviour at a critical value of the volatility. While no similar exact result is available in the log-normal Libor market model, we derive exact upper and lower bounds on the Eurodollar futures prices in the log-normal Libor market model with uniform volatility, or more generally with bounded parameters (L_i^{fwd}, σ_i) . Both bounds display the same explosive behaviour at sufficiently large volatility. We also derive an exact lower bound on the error of a Monte Carlo calculation of this quantity, which has a similar explosive behaviour. This introduces a limitation on the applicability of this simulation method to sufficiently low volatilities.

7 References

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