The Financial Mathematics of Market Liquidity

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Chapter 1

Introduction

Quantitative Finance also referred to as (Financial Mathematics) is a young science at the frontier between probability theory, economics and computer science. It has had a major influence on the financial world, far beyond the influence one could expect from a set of quantitative tools.

Since the 2007-2008 crisis, Quantitative Finance has changed a lot. In addition to the classical topics of derivatives pricing, portfolio management, and risk management, a swath of new subfields has emerged, and a new generation of researches is passionate about systemic risk, market impact modelling, high-frequency trading, optimal execution, etc.

1.1 A brief history of Quantitative Finance

1.1.1 From Bachelier to Black, Scholes, and Merton

When exactly did Quantitative Finance emerge as a scientific field? One could indeed go back to Bernoulli, Pascal, Fermat, Fibonacci, or even earlier to Ancient Greece to find some mathematical developments dealing with what would be called today gambling. The real father of (modern) Quantitative Finance is a French mathematician called Bachelier.

His work was inspired by the Paris Bourse. He defended his doctoral thesis in 1900 under the supervision of Henri Poincare. in his thesis, he developed for the first time a theory of option pricing, using processes very close to what was called later Wiener process. However, when Bachelier died in 1946, after a career full of pitfalls, his mathematical research applied to finance was not really famous.

It is only in the mid 1950s that his work started to be read by economists. Savage, a famous statistician, knew the work of Bachelier and thought that Bachelier's mathematical tools could be useful in economics. Samuelson received a postcard from Savage, and read the PhD thesis of Bachelier. Although the assumptions of the Bachelier model were questionable, the mathematical tools were there.

In the 1960s, research was carried out to price warrants. Samuelson participated in this scientific adventure with other economists such as Sprenkle and Boness. They
all proposed formulas for the price of a warrant. All these formulas were close in
their form to the eventual formula of Black and Scholes, but the major methodolog-
ical breakthrough was not there.
At the end of the 60s, Black got interested in the pricing of warrants. By writing
the value of a warrant as a function of time and price, he ended up with a partial
differential equation (PDE), but not with the Black-Scholes formula.
Scholes arrived at MIT at the same period, after finishing his PhD at the University
of Chicago, and he got in touch with Black in Boston. They started working together
on the option pricing problem. After a few months of tinkering, they achieved to
find the Black-Scholes formula.

Merton was a PhD student of Samuelson in 1970 and he discussed a lot with
Black and Scholes, while they were writing a paper to publish their findings. Merton
was interested in option valuation and he noticed that a replication portfolio could
be built in continuous time. Black and Scholes got the right formula, but Merton
introduced replication into the picture.

1.1.2 A new paradigm and its consequences

Black and Scholes' paper was published in 1973. It turned up at the right time,
when financial agents very much needed new financial products and new tools to
manage risk.

Two years before, in 1971, US President Nixon announced the suspension of the
dollar's convertibility into gold, triggering the collapse of the Bretton Woods sys-
tem. In 1973, despite new attempts to go back to fixed exchange rates, all major
currencies were floating. The same year, in 1973, the CBOE (Chicago Board Op-
tions Exchange) was founded, and was the first marketplace for trading options.

Although it took a few years for the new theoretical ideas to be used on trading
floors, the world was ready to use a formula that made it possible to manage option
books in a very simple manner.

Beyond the Black and Scholes formula, which is in fact the tip of a theoretical
iceberg, the theory of derivatives pricing based on replication is the second major
historical breakthrough in risk management. The first one was mutualization, which
led to modern insurance companies. With derivatives pricing, exposure to future
states of the world could be traded on markets, and large books of options could be
managed, through the dynamical hedging of the residual risk of the book.

In short, the findings of Black, Scholes, and Merton have changed the way peo-
ple think about financial risk, and it is one of the reasons why derivatives pricing
has fascinated several generations of students, lectures, and researchers in Financial
Economics and Quantitative Finance.
1.1.3 Quantitative Finance by mathematicians

From the beginning of the 90s, and until the 2007-2008 crisis, Quantitative Finance has been a great and effervescent field, involving the initial participants, other economists, and an increasing number of mathematicians.

The focus, on the equity derivatives side, was on improvements of the Black and Scholes model to account for the volatility surface and its dynamics. The local volatility models of Dupire, and Derman and Kani, constituted a major progress in the industry. Numerous stochastic volatility models have also been proposed in the literature since the early 1990s. Local stochastic volatility models have been developed later in the 2000s and more recently. Other models have been developed for super-replication in incomplete markets, for instance to take account of transaction costs, or to replace hedging by robust super-hedging when the value of parameters is uncertain.

As far as fixed income is concerned, the 1990s have also been a decade of major advances with the new approach proposed by Heath, Jarrow, and Morton, and then through the use of the Libor market model. Later on, the SABR model had a great success because of the asymptotic formula derived within this model.

At the turn of the millennium, Quantitative Finance was used all over the world to manage huge portfolios of derivatives, but also credit derivatives.

However, credit derivatives books could not simply be managed as equity derivatives books or fixed income derivatives books. The 2007-2008 crisis has highlighted the danger of the risk-neutral pricing/hedging models when used in highly incomplete markets. It has also highlighted the importance of model risk: when practitioners believe, without any evidence, that most of the risk is captured by their models, they already have one foot in the grave.

1.1.4 Quantitative Finance today

After the subprime crisis, quantitative analysts and mathematicians involved in Quantitative Finance were often lambasted for having used or built models capturing only part of the risk. They were certainly not at the origin of the crisis, but one cannot say they were only scapegoats. The practitioners often did not take the time to step back and analyze the caveats of the models they intended to use, especially when they were proposed in academic papers bearing the signature of famous academicians, or when a similar modeling approach was used by their competitors.

Clearly, the financial mathematics community as a whole has played its part in the catastrophe. During and just after the crisis, it was deeply shaken, and there was no point in continuing conducting the same kind of research: building new credit derivatives pricing models was simply nonsense, and going on marginally improving the existing models in other areas did not appear as a priority. In fact, new research strands very quickly emerged after the crisis.
Because the turmoil on the subprime market led to the bankruptcy of some of the largest financial institutions, and to a systemic crisis, systemic risk has also become a major concern of the academic research in Quantitative Finance. New models have been built to tackle risk in networks, to model contagions, and to better understand the role of clearing houses. As far as pricing and hedging models are concerned, there is a new interest in transaction costs, in robust (super)-hedging, and more generally in nonlinear approaches.

1.2 Optimal execution and market making in the extended market microstructure literature

1.2.1 The classical literature on market microstructure

In the traditional classification of economists, market microstructure constitutes an area of finance concerned with the price formation process of financial assets, and the influence of the market structure on this process.

The main goal of the economists involved in this strand of research is to understand the mechanisms by which the willingness to buy and sell assets of the different types of market participants translates into actual transactions, and to understand the resulting price process. They do not focus on the macroeconomic supply and demand for stocks or other assets. Instead, they look into the different black boxes that make the actual transactions possible, and analyze the trading process.

Information is often at the heart of their approach. Economists have studied how information is conveyed into prices. They have also studied the impact of asymmetric information on the bid-ask spread and more generally on liquidity. The translation of information into transactions and prices is obviously related to market impact, and market impact modeling is certainly one of the first topics of the market microstructure literature that has also been studied by academics from other fields than economics.

Another very important research topic is the influence of the various possible market structures on the price formation process and on market quality. Today, the economic research on market microstructure is more focused on the importance of pre-trade and post-trade transparency, the role played by dark pools, the optimal tick size, etc.

1.2.2 An extension of the literature on market microstructure

Statisticians, mathematicians, and econophysicists have recently addressed some theoretical and empirical questions belonging to the classical market microstructure literature, or related to it. New approaches have been proposed for the same problems, and some new problems are today considered part of the multidisciplinary
field of market microstructure.

These changes were triggered by the automation of trading and the development of computerized execution algorithms. New technologies have led to new questions for the modelers. The first academic papers dealing with optimal execution were those of Bertsimas and Lo in 1998, and Almgren and Chriss in 1999 and 2001. In spite of this new interest that appeared around 2000, the number of papers on market impact modelling and optimal execution only really skyrocketed at the end of the 2000s. Before the 2007-2008 crisis, Quantitative Finance was more dealing with financial products - and in fact with more and more complex payoffs - than with financial markets. The reappearance of the forgotten word "liquidity" during the crisis, the recent changes in the structure of stock markets, and the rise of high-frequency trading, brought classical market microstructure questions and the new question of optimal execution onto the tables of mathematicians and statisticians involved in Quantitative Finance.

Numerous optimal execution models are based on parameters such as intraday volatility, transaction and execution cost parameters, probabilities to be executed, and other parameters. Therefore, many statisticians and applied probabilists are now involved in the market microstructure literature.

In particular, there is an important literature on high-frequency market making strategies. This literature started in 2008 with the paper of Avellaneda and Stoikov. Since then, some more realistic models have been proposed, to find the optimal strategies of the financial agents who try to make money out of liquidity provision, especially on the stock market - although models from Avellaneda-Stoikov turn out to be more relevant on quote-driven markets, such as many bond markets.

1.3 Conclusion

The story of Quantitative Finance as a scientific field is definitely not linear. It has successively involved a mathematician who defended his PhD in 1900, a great number of brilliant economists and a large community of applied mathematicians.

Until recently, Quantitative Finance was seen as the use of stochastic calculus and other mathematical tools to price and hedge securities and address risk management issues. Since the 2007-2008 crisis, Quantitative Finance has tackled a larger swath of topics, from counterparty risk, to systemic risk, to optimal execution, and market making.
Chapter 2

Organization of markets

2.1 Introduction

Although exchanges have been recurrently criticized and compared to casinos, they have provided over history several fundamental services to the economy. As any marketplace, an exchange allows the matching of buyers and sellers at a given point in time. They guarantee that, at a given time, the prices reflect the balance between demand and supply, and that it will continue in the future.

Exchanges are also strongly linked to the history of technology and telecommunications. The telegraph was utilized to communicate between exchanges in the U.S., between exchanges in Europe, and later on to communicate across the Atlantic Ocean with a latency of a few seconds. Then telephone modified the way traders and brokers worked, and therefore the functioning of exchanges. Computers have also changed the way stocks and bonds are traded: most of the traditional stock exchanges have been replaced by electronic systems, in which agents, can (virtually) meet to buy and sell securities. Nowadays, optic fiber and microwaves have made it possible to achieve ultra-low latency in the communication with exchanges, especially for the trading of stocks, where high-frequency traders represent an important part of trading volumes, and often act as liquidity providers.

2.2 Stock markets

2.2.1 The history of stock exchanges

The history of stock exchanges is linked to the history of business. Although the first stock exchanges date back to the Renaissance - exchanges were created in Bruges (15th century), Lyon (16th century), Amsterdam (17th century), etc. Stock exchanges only started to play a fundamental role in the development of western economies in the 19th century, when the capital needs of companies engaging in large-scale and long-term projects skyrocketed. The need to rely on individual savers to finance long-term projects, and the need to guarantee investors that securities could quickly and cheaply be transferred, led to the emergence of stock exchanges in major advanced countries. These stock exchanges were often club-like/not-for-profit organizations regulated by strict rules, where membership was given to a few
selected agents only.

The success of stock exchanges in the 19th century is not only due to the industrial revolutions, and to the global economic development, but also to the trust in the system. This trust was based on the fact that the members - who were the only people to have access to the floor, and therefore to participate in the price formation process - had a lot to lose if they did not abide by the rules.

By 1914, most advanced countries had set up their own stock exchanges. The New York Stock Exchange (NYSE) had existed for a long time, but it only took its current name in 1865. The London Stock Exchange (LSE) was officially founded in 1801. The current Italian stock exchange (Borsa Italiana) is based upon the one that was established in Milan in 1808. In Nordic countries, the Stockholm Stock Exchange was founded in 1863, and the OMX in Helsinki in 1912. Even in the colonial empires, local exchanges were established: Hong Kong in 1891, Jakarta in 1912, etc.

After the four-year disruption caused by WW1, stock exchanges only partially recovered. Furthermore, there was a series of crises in the 1920s and 1930s, in particular the Wall Street Crash in 1929, followed by the progressive end of the Gold Standard and the Great Depression. The main consequence for stock exchanges was the establishment of new rules, and regulatory bodies to enforce them.

After a new disruption caused by World War 2, stock exchanges recovered but the situation had changed. Regulation was indeed imposed or reinforced almost everywhere, and stock exchanges were no longer only in the hand of their members, but also in those of governments willing to regulate markets. In many countries, stock exchanges were national monopolies. This situation, with regulated monopolies, was characterized by high fees charged to final investors and almost no evolution. In parallel, the development of a powerful banking system made it possible for firms to be financed without going to the market (especially in Europe).

In the 1970s and 1980s, some powerful institutional investors tried to bypass stock exchanges by trading directly with dealers. They also asked the regulator for changes. In particular, they lobbied for more competition between the members of stock exchanges, in order to reduce transaction costs. However, no real change occurred in Europe, because exchanges were protected. The only important change in the 1970s was the creation of the NASDAQ in 1971, that aimed at competing with NYSE. The reaction of the NYSE was the removal of fixed commission charges.

Real changes occurred in the 1990s, because of the deregulation and privatization waves in many advanced countries, because firms were increasingly relying on financial markets to raise capital, and because individuals were increasingly looking for investment opportunities in order to compensate the expected decline of the welfare state, in particular public pension systems. During that decade the trade-off between the current benefits of a statuquo and the opportunities brought by the new economic environment was in favor for the latter. Demutualization of most major stock exchanges occurred in the late 90s, converting the old structures of stock...
exchanges into standard corporate ones - often into self-listed companies. Demutualization made stock exchanges ready to embrace the new economic environment, and recover the central role they used to have a century before.

2.2.2 The influence of technology

In the 90s, the need for change, in a world impacted by a decade of market freedom, was evident. However, in addition to the economic environment, technology played a major role in triggering the changes.

In 1990, except for one, all major stock exchanges were still not fully electronic. Not surprisingly, the electronification of exchanges started with new exchanges, launched directly in an electronic format, to trade derivative products. Electronic trading was then progressively tested on stock exchanges, but the transition was long.

The electronification of trading had a cost for stock exchanges, but some economies of scale were possible. This is why a wave of mergers occurred, leading to the creation of international stock exchange groups. The most famous one is Euronext, created in 2000 following the mergers of Amsterdam Stock Exchange, Brussels Stock Exchange, and Paris Bourse. Later on, in 2007, it merged with NYSE, but it turned back into a stand-alone company in 2014. Another example of this consolidation wave is the acquisition of Borsa Italiana by the LSE in 2007.

2.2.3 Description of the trading environment

The liquidity, foe a given stock, is fragmented over several venues. Among these venues, there is the historical exchange of the stock, often called the main or primary venue. It is organized around a limit order book during the continuous trading phase. In addition to this continuous trading phase, there are periods of auctions that involve a lot of agents at the same time to form a price: opening auctions, closing auctions, and sometimes other auctions during the day, such as volatility interruption auctions. Furthermore, other venues may be available to trade a given stock.

Limit order books

An exchange organized around a visible limit order book is based on a transparent system that matches the buy and sell orders of market participants on a price/time priority basis. It is very often an anonymous system, because no one can see the identity of the market participants.

The limit order book is characterized by a side (buy or sell), a size (the number of shares to be bought or sold), and a price. Buy orders constitute the bid size of the LOB, while sell orders constitute the ask side of the LOB.
**Tick size**

The limit orders cannot be posted at any price. Prices have to be multiples of a given monetary value, called the tick size. The tick size depends on the price of the stock and sometimes on its liquidity. In the U.S., the minimal price increment has been 1/8 of a dollar for a long time, and then became 1/16 of a dollar. The format of prices was not with decimals. But this changed in 2001, and the tick size cannot be below one cent.

**Trading Fees**

Although they are all different, the fee structures of most venues are based on the principle that only executed orders are considered in the pricing. In other words, the insertion of limit orders does not lead to a charge, and the cancellation of limit orders is free. It is clear that new entrants initially applied fee schemes different from those of incumbent exchanges in order to attract part of the trading flows. The optimal make/take fee structure was studied theoretically by a group of economists. They found that it could be optimal to subsidize either liquidity makers or liquidity takers, depending on the market characteristics.

2.2.4 **High-frequency trading**

High-frequency trading is often described as being evil. In fact, high-frequency trading is very diverse. Some high-frequency trading strategies can be assimilated as price manipulation, and they are indeed detrimental to the price formation process. However, some high-frequency traders, play a major role in the current trading environment. They act like liquidity-providing arbitrageurs across the different platforms quoting the same stock.

Although HFT plays an important role in the current competitive structure, it also raises many questions. Some market participants are indeed unwilling to trade with high-frequency traders because of the potential adverse selection, not to say because they fear gaming. Bad practices such as layering or spoofing have been described. To avoid information leakage towards high-frequency traders, utilitarian traders tend to trade more and more on platforms without pre-trade transparency.
Chapter 3
The Almgren-Chriss framework

3.1 Almgren-Chriss model in continuous time

We consider a trader with a single-stock portfolio. At time $t = 0$, his position is denoted by $q_0$. If $q_0$ is positive, then the portfolio contains $q_0$ shares, else the trader has a short position in the stock, of $-q_0$ shares. The trader's position over the time interval $[0, T]$ is modeled by the process $(q_t)_{t\in[0,T]}$.

The dynamics of this process is

$$dq_t = v_t dt,$$

where $(v_t)_{t\in[0,T]}$ is a progressively measurable control process satisfying the unwinding constraint $\int_0^T v_t dt = -q_0$, and the additional technical condition:

$$\int_0^T |v_t| dt \in L^\infty(\Omega).$$

Let be $\mathcal{A}$ the set of admissible controls:

$$\mathcal{A} = \left\{ (v_t)_{t\in[0,T]} \in H^0(\mathbb{R}, (\mathbb{F}_t)_t), \int_0^T v_t dt = -q_0, \int_0^T |v_t| dt \in L^\infty(\Omega) \right\}.$$ 

At time $t$, $v_t$ stands for the trading velocity, that is, the (instantaneous) trading volume.

The mid-price of the stock is modeled by the process $(S_t)_{t\in[0,T]}$. We assume a linear dependence of the form:

$$dS_t = \sigma dW_t + kv_t dt,$$

where $\sigma$ is a positive constant called the arithmetic volatility of the stock, and $k$ is a nonnegative parameter modelling the magnitude of the permanent market impact.

Because of execution costs, the price obtained by the trader at time $t$ is not $S_t$, but instead a price that depends on the volume he trades, and on the market volume at that time.
To model the execution costs, we first introduce the market volume process \((V_t)_{t \in [0,T]}\), which represents the volume traded by other agents. We assume that is deterministic, continuous, positive, and bounded process.

The price obtained by the trader for each share at time \(t\) is of the form \(S_t + g(\frac{v_t}{V_t})\), where \(g\) is an increasing function satisfying \(g(0) = 0\). Instead of working with the function \(g\), we introduce the execution cost function \(L\) that is simply \(L(\rho) = \rho g(\rho)\).

We denote by \((X_t)\), the cash account process modeling the amount of cash on the trader’s account. Given the above assumptions, its dynamics is given by

\[
dX_t = -v_t(S_t + g(\frac{v_t}{V_t}))dt = -v_tS_t dt - V_tL(\frac{v_t}{V_t}).
\]

The assumptions on the function \(L : \mathbb{R} \to \mathbb{R}\) are the following:

- (H1) No fixed cost, i.e., \(L(0) = 0\),
- (H2) \(L\) is strictly convex, increasing on \(\mathbb{R}_+\), and decreasing on \(\mathbb{R}_-\),
- (H3) \(L\) is asymptotically super-linear, i.e., \(\lim_{|\rho| \to \infty} \frac{L(\rho)}{|\rho|} = +\infty\)

Assumption (H1) means that no execution costs are incurred when there is no transaction, hence no fixed cost. Assumption (H2) means that there is always a cost to trade, and that this cost is more than just a linear cost related to the bid-ask spread or a stamp duty. Assumption (H3) is a technical assumption.

### 3.1.1 The optimization problem

The goal is to find an optimal strategy \((v_t)_{t \in A}\) to liquidate the portfolio. For that purpose, we need to decide what optimality means. A good solution corresponds to maximizing \(\mathbb{E}[X_T]\). But Almgren and Chriss considered instead a mean-variance criterion. They proposed to maximize an expression of the form \(\mathbb{E}[X_T] - \frac{\gamma}{2} \mathbb{V}[X_T]\), where \(\gamma\) is a positive constant. To remain into a classical economic framework, we consider an expected utility criterion, and the function we consider is (Constant Absolute Risk Aversion) utility function. In other words, the objective function is of the form

\[
\mathbb{E}[-\exp(-\gamma X_T)],
\]

where \(\gamma\) is a positive constant, called the absolute risk aversion coefficient of the trader.
3.1.2 The case of deterministic strategies

A unique optimal strategy

In this section, we restrict liquidation strategies to deterministic ones. The main consequence of this assumption is that a strategy can be represented by a function $t \rightarrow q(t)$, computed at the beginning of the execution process. In particular, the strategy does not depend on the evolution of the price.

We consider the restricted set of admissible control processes $\mathcal{A}_{\text{det}}$ defined by

$$\mathcal{A}_{\text{det}} = \{(v_t)_{t \in [0,T]} \in \mathcal{A}, \forall t \in [0,T], v_t \text{ is } \mathcal{F}_0 \text{- measurable}\}.$$ 

The computation of the final value of the cash process $X_T$, for a strategy $(v_t)_{t \in [0,T]} \in \mathcal{A}$ leads to:

$$X_T = X_0 - \int_0^T v_t S_t dt - \int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt$$

$$= X_0 + q_0 S_0 + \int_0^T kv_t q_t dt + \sigma \int_0^T q_t dW_t - \int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt$$

$$= X_0 + q_0 S_0 - \frac{k}{2} q_0^2 + \sigma \int_0^T q_t dW_t - \int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt.$$

If $(v_t)_{t \in [0,T]} \in \mathcal{A}_{\text{det}}$, then the final value of the cash process $X_T$ is normally distributed with mean:

$$\mathbb{E}[X_T] = X_0 + q_0 S_0 - \frac{k}{2} q_0^2 - \int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt,$$

and variance,

$$\mathbb{V}[X_T] = \sigma^2 \int_0^T q_t^2 dt.$$

The mean of $X_T$ can be decomposed into three parts:

The first term, $X_0 + q_0 S_0$, is the Mark-to-Market value of the portfolio at time 0. The second term, $-\frac{k}{2} q_0^2$, corresponds to costs coming from the permanent component of market impact. The last term corresponds to the execution costs.

Because $L$ is assumed to be convex, the last term of the mean of $X_T$ is minimal when $v_t$ is proportional to $V_t$. This strategy is not optimal in our framework because we take the variance into account. The variance of $X_T$ is an increasing function of the volatility parameter $\sigma$, and it depends on the strategy through the term $\int_0^T q_t^2 dt$. To minimize this variance term, the trader must quickly liquidate. The trade-off between execution costs and price risk is here a trade-off between minimizing the last term of the $\mathbb{E}[X_T]$, and minimizing the variance of $X_T$.

By using the mean and variance of $X_T$, with the help of Laplace transform of a Gaussian variable, we can compute the value of the objective function:
$$
\mathbb{E}[-\exp(-\gamma X_T)] = -\exp\left(-\gamma \mathbb{E}[X_T] + \frac{1}{2}\gamma^2 \mathbb{V}[X_T]\right)
$$

$$
= -\exp\left(-\gamma \left(X_0 + q_0 S_0 - \frac{k}{2} q_0^2\right)\right)
* \exp\left(\gamma \left(\int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt + \frac{\gamma^2}{2} \sigma^2 \int_0^T q_t^2 dt\right)\right).
$$

This means that the problem boils down to finding a control process \((v_t)_{t\in[0,T]}\) in \(\mathcal{A}_{\text{det}}\) minimizing:

$$
\int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt + \frac{\gamma^2}{2} \sigma^2 \int_0^T q_t^2 dt.
$$

Because \(v_t = \frac{dq_t}{dt}\) is the first derivative of \(q_t\), and \(\int_0^T |v_t| dt < +\infty\), the function \(q : t \to q_t\) is in fact absolutely continuous. Therefore, the problem boils down to a variational problem. We need to find the minimizers of the functional \(J\) defined by

$$
J(q) = \int_0^T \left(V_t L\left(\frac{q'(t)}{V_t}\right) + \frac{1}{2}\gamma \sigma^2 q(t)^2\right) dt,
$$

over the set of absolutely continuous functions \(q \in W^{1,1}(0,T)\) satisfying the constraints \(q(0) = q_0\) and \(q(T) = 0\). Finding a unique minimizer \(q^*\) of the function \(J\) over the set \(\{q \in W^{1,1}(0,T), q(0) = q_0, q(T) = 0\}\) is possible, and \(q^*\) is a monotone function.

**Characterization of the optimal strategy**

To characterize the optimal strategy the optimal strategy \(q^*\), we can use either a Euler-Lagrangian characterization or a Hamiltonian characterization. We better use the Hamiltonian system:

$$
\begin{cases}
    p'(t) = \gamma \sigma^2 q^*(t) \\
    q^*(t) = V_t H'(p(t)) \\
    q^*(0) = q_0 \\
    q^*(T) = 0
\end{cases}
$$

where \(H\) is the Legendre-Fenchel transform of the function \(L\) defined by

$$
H(p) = \sup_{\rho} \rho p - L(\rho).
$$

We can use this transformation because \(L\) is strictly convex, and \(H\) is a function of the class \(C^1\).
The case of quadratic execution costs

Almgren and Chriss initially introduced a model where the additional cost per share due to limited liquidity was a linear function of the number of shares. In our framework, this corresponds to $g$ linear, or equivalently $L$ quadratic. To understand the model and the roles of the different parameters, let us consider a quadratic function $L(\rho) = \eta \rho^2$, and the associated function $H$ is given by $H(\rho) = \frac{\rho^2}{4\eta}$. Therefore, the Hamiltonian system characterizing the optimal liquidation strategy becomes

$$
\begin{align*}
q'(t) &= \gamma \sigma^2 q^*(t) \\
q''(t) &= \frac{V_t}{2\eta} p(t) \\
q^*(0) &= q_0 \\
q^*(T) &= 0.
\end{align*}
$$

Consequently, $q^*$ is the unique solution of the equation

$$q^{**}(t) = \frac{\gamma \sigma^2 V_t}{2\eta} q^*(t)$$

satisfying the boundary conditions $q^*(0) = q_0$ and $q^*(T) = 0$.

If $(V_t)$ is assumed to be constant $(V_t = V, \forall t)$, then we get the classical hyperbolic sine formula of Almgren and Chriss:

$$q^*(t) = q_0 \frac{\sinh \left( \sqrt{\frac{\gamma \sigma^2 V}{2\eta}} (T - t) \right)}{\sinh \left( \sqrt{\frac{\gamma \sigma^2 V}{2\eta}} T \right)}.$$

Associated to this optimal trading curve, the optimal (deterministic) strategy $(v_t^*)_t$ is given by

$$v^*_t = q^*(t) = -q_0 \sqrt{\frac{\gamma \sigma^2 V}{2\eta}} \frac{\cosh \left( \sqrt{\frac{\gamma \sigma^2 V}{2\eta}} (T - t) \right)}{\sinh \left( \sqrt{\frac{\gamma \sigma^2 V}{2\eta}} T \right)}.$$

We see that $q^*$ is a convex function function if $q_0 \geq 0$, and a concave function if $q_0 \leq 0$. This means that the unwinding process is fast at the beginning, and decelerates progressively.

The liquidity parameters $\eta$ and $V$

$\eta$ is a scaling factor for the execution costs paid by the trader. The larger $\eta$, the more the trader pays to buy/sell shares. Similarly, the value of the market volume $V$ is a scaling factor, because the execution costs depend on the participation rate. Hence, a small value of $V$ has the same effect as a large value of $\eta$.

The volatility parameter $\sigma$
The volatility parameter $\sigma$ measures the importance of price risk. Therefore, the larger $\sigma$, the faster the execution to reduce the exposure to price risk.

The risk aversion parameter $\gamma$

The risk aversion parameter sets the balance between execution costs on the one hand, and price risk on the other hand. The larger the parameter $\gamma$, the more the trader is sensitive to price risk. Therefore, high $\gamma$ means fast execution to reduce the exposure to price risk.

When $\gamma$ is really small, we find that the optimal trading curve $q^*$ gets close to a straight-line strategy. This is formally proved by

$$\lim_{\gamma \to 0} q^*(t) = q_0 \left(1 - \frac{t}{T}\right)$$

3.2 Conclusion

In this thesis, we wrote about history of the market exchanges and its structure. We touched different topics such as the influence of the technology on the market exchanges, and the evolution of the exchanges. But the main goal of the thesis was to explain the environment of the market and also trying to offer a better explanation about the changes of the market products’ prices. Almgren-Chriss framework is one of the frameworks which explains that our actions influence the change of price.
Bibliography
