Generalized statistical arbitrage concepts and related gain strategies

Institute of Financial and Actuarial Mathematics at Vienna University of Technology
TU Vienna

Lecturer

Dipl.-Ing. Dr.techn. Stefan Gerhold

Student

Igor Radujko

Vienna, 26.02.2020
Abstract

The following seminar paper is based on the article ”Generalized statistical arbitrage concepts and related gain strategies”, written by C. Rein, L Rüschendorf, and T. Schmidt. The main topic of the article and thus of this seminar paper is the concept of statistical arbitrage, which we here generalize and then introduce related trading strategies. In the article authors constructed several profitable generalized strategies with respect to various choices of the information system, while i focused on the embedded binomial and follow-the-trend strategies as well as their their behaviour on simulated data.
Contents

1 Introduction 1

2 Generalized gain strategies 1

3 On the statistical no-arbitrage notion 2
   3.1 Statistical arbitrage strategies in binomial models 3
   3.2 Risk of statistical arbitrages 6

4 Generalized \( G \)-arbitrage strategies 7

5 Some classes of profitable strategies 8
   5.1 Embedded binomial trading strategies 9
      5.1.1 Simulation results 11
      5.1.2 Varying barrier levels 12
      5.1.3 The role of drift and volatility 13
   5.2 Follow-the-trend strategy 14
   5.3 The embedded binomial follow-the-trend strategy 16
      5.3.1 Simulation results 19
   5.4 Summary on the different strategies 20

Bibliography 21
1 Introduction

Trading strategies which offer profits on average on the one hand, and little remaining risk on the other hand, have been implemented and analyzed since the mid-1980s. The starting point were pairs trading strategies also known as statistical arbitrage strategies (Stat Arb). Many variants of this simple but effective strategy followed which raised interest in a deeper theoretical understanding of these approaches.

Statistical arbitrage (or Stat Arb) refers to a group of trading strategies which utilize mean reversion analyses to invest in diverse portfolios of up to thousands of securities for a very short period of time, often only a few seconds but up to multiple days. Statistical arbitrage strategies are market neutral because they involve opening both a long position and short position simultaneously to take advantage of inefficient pricing in securities, whose prices have a high historic correlation. As mentioned the concept of pairs trading is simple. The idea is to find stocks whose prices have moved together through history and when the spread between them widens, the arbitrageur should short the winner and buy the loser. By doing this he will profit if history repeats itself, which is why the historical correlation plays a big role in this strategy.

A trading strategy with zero initial cost is called statistical arbitrage if

(i) the expected payoff is positive and,

(ii) the conditional expected payoff is non-negative in each final state of the economy.

In contrast to pure arbitrage strategies a statistical arbitrage can have negative payoffs provided the average payoff in each final state is non-negative.

2 Generalized gain strategies

Throughout the whole paper we consider the following finite-horizon economy:

We let $(\Omega, \mathcal{F}, P)$ be a filtered probability space with a filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, where filtration is assumed to satisfy the usual conditions. We suppose $\mathcal{F} = \mathcal{F}_T$.

Following the classical approach to financial markets, we consider a finite time horizon $T \in \mathbb{N}$ as already mentioned.

The market is given by a $\mathbb{R}^{d+1}$-valued locally bounded non-negative semi-martingale $S = (S_0, ..., S_d)$. We also set the numeraire $S_0$ equal to one and can therefore consider the prices as already discounted.

A dynamic trading strategy $\phi$ is an $S$-integrable and predictable process such that the associated value process $V = V(\phi)$ is given by

$$V_t(\phi) = \int_0^t \phi_s \, dS_s, \quad 0 \leq t \leq T.$$ 

The trading strategy $\phi$ is called $a$-admissible if $\phi_0 = 0$ and $V_t(\phi) \geq -a$ for all $t \geq 0$. $\phi$ is called admissible if it is admissible for some $a > 0$. We further assume that the market
is free of arbitrage, which is equivalent to the existence of an equivalent local martingale measure $Q$, where measure $Q$ is equivalent (local) martingale measure (EMM), if $Q$ is equivalent to $P$ i.e. $(Q \sim P)$, such that $S$ is a $\mathcal{F}$-(local) martingale with respect to $Q$. With $\mathcal{M}^e$ we denote the set of all equivalent local martingale measures.

A statistical arbitrage is a dynamic trading strategy which is on average profitable, conditional on the final state of the economy $S_T$, but here we also consider a general information system represented by a $\sigma$-field $\mathcal{G} \subset \mathcal{F}_T$ as well as strategies which are on average profitable conditional on $\mathcal{G}$. Some examples for $\mathcal{G}$ are $\sigma$-field generated by the event $\{S_T > K\}$, the events $S_T \in K_i$, where $(K_i)_{i \in \mathcal{I}}$ is a partition of $\mathbb{R}^d$, or by $\{\max_{0 \leq t \leq T} S_t > K\}$. We call such strategies $\mathcal{G}$-arbitrage strategies. Sometimes we call a statistical $\mathcal{G}$-arbitrage strategy also a $\mathcal{G}$-profitable strategy or $\mathcal{G}$-arbitrage, for short. By $E$ we denote expectation with respect to the reference measure $P$.

**Definition 1.** Let $\mathcal{G} \subseteq \mathcal{F}_T$ be a $\sigma$-algebra. An admissable dynamic trading strategy $\phi$ is called a statistical $\mathcal{G}$-arbitrage strategy, if $V_T(\phi) \in L^1(P)$ and

i) $E[V_T(\phi)|\mathcal{G}] \geq 0$, $P$-a.s.,

ii) $E[V_T(\phi)] > 0$.

Let

$$SA(\mathcal{G}) := \{ \phi : \phi \text{ is a } \mathcal{G} \text{-arbitrage} \}$$

denote the set of all statistical $\mathcal{G}$-arbitrage strategies. The market model satisfies the condition of no statistical $\mathcal{G}$-arbitrage $\text{NSA}(\mathcal{G})$ if

$$SA(\mathcal{G}) = \emptyset$$

**Remark 1.** We note some easy consequences of Definition 1.

i) For $\mathcal{G} = \mathcal{F}_T$, $\text{NSA}(\mathcal{G})$ is equivalent to the classical no-arbitrage condition (NA) since then $E[V_T(\phi)|\mathcal{G}] = V_T(\phi)$

ii) The tower property of conditional expectations immediately yields that larger information systems $\mathcal{G}$ allow for less profitable $\mathcal{G}$-arbitrage strategies i.e. $\mathcal{G}_1 \subset \mathcal{G}_2$ implies that $SA(\mathcal{G}_2) \subset SA(\mathcal{G}_1)$. As a consequence we get that in this case

$$\text{NSA}(\mathcal{G}_1) \Rightarrow \text{NSA}(\mathcal{G}_2).$$

iii) If $\mathcal{G} = \{\emptyset, \Omega\}$, then $\phi \in SA(\mathcal{G})$ iff $E_P[V_T(\phi)] > 0$.

**3 On the statistical no-arbitrage notion**

The notion of no statistical arbitrage is motivated by the question whether it is possible to construct a trading strategy $\phi$ such that in any final state of the price process $S_T$ the
trader gets a gain on average (i. e. conditional on \( \sigma(S_T) \)). Proposition 1 in Bondarenko (2003) implies that (in discrete time), NSA is equivalent to the existence of an equivalent martingale measure \( Q \) with path independent density \( Z \), i. e.

\[
\frac{dQ}{dP} = Z \in \sigma(S_T),
\]

where we use the notation \( Z \in \sigma(S_T) \) for \( Z \) being \( \sigma(S_T) \)-measurable. Even though one can show that this equivalence needs additional assumptions (on a special one-dimensional trinomial model), existence of an equivalent martingale measure with path independent density \( Z \) implies that NSA holds without further assumptions. This also holds true for the generalized notion NSA(\( \mathcal{F} \)), as the following proposition states.

**Proposition 1.** If there exists \( Q \in \mathcal{M}^e \) such that \( \frac{dQ}{dP} \) is \( \mathcal{F} \)-measurable, then NSA(\( \mathcal{F} \)) holds.

**Proof.** The proof follows from the Bayes-formula for conditional expectations. If \( Z = \frac{dQ}{dP} \in \mathcal{F} \), then for any \( X \in L^1(P) \) it hold that

\[
E_P[X|\mathcal{F}] = \frac{E_Q[XZ|\mathcal{F}]}{E_Q[Z|\mathcal{F}]} = E_Q[X|\mathcal{F}].
\]

If there would be a statistical arbitrage strategy \( \phi \) with \( E_P[X|\mathcal{F}] \geq 0 \) and \( E_P[X] > 0 \), where \( X = V_T(\phi) \in L^1(P) \), then by (1.1),

\[
E_P[X|\mathcal{F}] \geq 0, \quad \text{Q-a.s.}
\]

Moreover, since \( \phi \) is admissible, \( V(\phi) \) is a \( Q \)-supermartingale by Fatou’s lemma, and we obtain that

\[
E_Q[X] = E_Q[V_T(\phi)] \leq V_0(\phi) = 0.
\]

Hence,

\[
0 = E_Q[X|\mathcal{F}] = E_P[X|\mathcal{F}]
\]

in contradiction to \( E_P[X] > 0 \). \( \square \)

### 3.1 Statistical arbitrage strategies in binomial models

In this section we propose a method to construct statistical arbitrage strategies in binomial models. We consider the following recombining two-period binomial model: assume that \( \Omega = \{\omega_1, ..., \omega_4\} \), \( T = 2 \) and \( \mathcal{F}_2 = \sigma(\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}) \). Let \( S_0 = s_0 > 0 \) and let \( S_1(\omega_1) = S_1(\omega_2) = s^+ \), and \( S_1(\omega_3) = S_1(\omega_4) = s^- \) as well as \( S_2(\omega_1) = s^{++} \), \( S_2(\omega_2) = s^{+-} \), and \( S_2(\omega_3) = s^- \), and \( S_2(\omega_4) = s^{--} \). This model is illustrated in Figure 1.

Absence of arbitrage is equivalent to \( \Delta S_i := \Delta S_i - \Delta S_{i-1}, i = 1, 2 \) taking positive as well as negative values. We assume without loss of generality that \( s^+ > s_0, s^- < s_0 \) and \( s^{++} > s^+, s^{--} < s^- \) which means we consider binomial models as presented in Figure 1.
3 On the statistical no-arbitrage notion

Figure 1: The considered recombining binomial model with two periods.

Gains from trading are again given by

\[ V_2(\phi) = \phi_1 \Delta S_1 + \phi_2 \Delta S_2. \]

\(\phi_1\) is constant, since for \(t = 0\) we have the trivial \(\sigma\)-Algebra and \(\phi_2\) can take the two values \(\{\phi_2^+, \phi_2^-\}\). According to Definition 1 \(\phi\) is a statistical arbitrage, if and only if

\[ \phi_1 \Delta S_1(\omega_1) P(\omega_1) + \phi_2^+ \Delta S_2(\omega_1) P(\omega_1) \geq 0 \]  
(3.2)

\[ \phi_1 \Delta S_1(\omega_4) P(\omega_4) + \phi_2^- \Delta S_2(\omega_4) P(\omega_4) \geq 0 \]  
(3.3)

\[ \phi_1 \Delta S_1(\omega_2) P(\omega_2) + \phi_2^+ \Delta S_2(\omega_2) P(\omega_2) + \phi_1 \Delta S_1(\omega_3) P(\omega_3) + \phi_2^- \Delta S_2(\omega_3) P(\omega_3) \geq 0 \]  
(3.4)

and at least one of the inequalities is strict. Moreover, the density \(Z\) is path-independent iff \(Z(\omega_2) = Z(\omega_3)\). Since \(P(\omega_i) > 0, i = 1, \ldots, 4\) we can divide (3.2) with \(P(\omega_1)\), (3.3) with \(P(\omega_4)\) as well as (3.4) with \(P(\omega_3)\) and write the equations in an equivalent way as \(A\phi \geq 0\) using \(\phi = (\phi, \phi^+, \phi^-)^T\) and matrix \(A\) given as

\[
A = \begin{pmatrix} \Delta S_1(\omega_1) & \Delta S_2(\omega_1) & 0 \\ \Delta S_1(\omega_4) & 0 & \Delta S_2(\omega_4) \\ q\Delta S_1(\omega_2) + \Delta S_1(\omega_3) & q\Delta S_2(\omega_2) & \Delta S_2(\omega_3) \end{pmatrix},
\]

where \(q = \frac{P(\omega_2)}{P(\omega_3)}\).

**Proposition 2.** In the recombining two-period binomial model NSA holds if and only if \(\det(A) = 0\).

For the proof see Appendix A. Proofs in [1].

**Remark 2.** It turns out that in the binomial model above NSA is equivalent to existence of a path-independent density: indeed, the unique equivalent martingale measure is given
Proposition 2 yields that NSA holds iff
\[
\det(B) = 0
\]
which we can further transform into the following equation:
\[
B = \Delta S_2(\omega_2)\left(\Delta S_1(\omega_3) - \Delta S_1(\omega_1)\right) + \Delta S_1(\omega_1)\left(\Delta S_2(\omega_4) - \Delta S_2(\omega_3)\right).
\]

Proposition 2 yields that NSA holds iff \( \det(A) = 0 \). Calculating \( \det(A) \) we see that \( \det(A) = 0 \) is equivalent to
\[
0 = -\Delta S_2(\omega_1)\left(\Delta S_1(\omega_4)\Delta S_2(\omega_3) - (\Delta S_1(\omega_3) + q\Delta S_1(\omega_2))\Delta S_2(\omega_4)\right)
- q\Delta S_1(\omega_1)\Delta S_2(\omega_2)\Delta S_2(\omega_4)
\]
which we can further transform into the following equation:
\[
q = \frac{P(\omega_2)}{P(\omega_3)} = \frac{\Delta S_2(\omega_1)(\Delta S_1(\omega_3)\Delta S_2(\omega_4) - \Delta S_1(\omega_4)\Delta S_2(\omega_3))}{\Delta S_2(\omega_4)(\Delta S_1(\omega_1)\Delta S_2(\omega_2) - \Delta S_1(\omega_2)\Delta S_2(\omega_1))} =: \tilde{q}.
\]

Using \( q_2 \) and \( q_3 \) we obtain from \( \det(A) = 0 \) that
\[
\frac{dQ(\omega_2)}{dQ(\omega_3)} = \tilde{q} = \frac{P(\omega_2)}{P(\omega_3)},
\]
which means that NSA is equivalent to the existence of a path-independent density.

The question now is what path properties imply absence of statistical arbitrage opportunities.

**Lemma 1.** In the recombining two-period binomial model there exists a statistical arbitrage if and only if
\[
\frac{P(\omega_2)}{P(\omega_3)} \neq \tilde{q}.
\]

**Proof.** From Proposition 1 we know that we need \( \det(A) \neq 0 \) to have the possibility of statistical arbitrage. According to Remark 2, that is equivalent to \( \frac{P(\omega_2)}{P(\omega_3)} \neq \tilde{q} \).

The following Lemma explicitly describes the statistical arbitrage in terms of the vector \( \phi = (\phi, \phi^+, \phi^-)^T \).
Lemma 2. In the recombining two-period binomial model with statistical arbitrage, 
\( \phi = \frac{1}{D}(\xi^1, \xi^2, \xi^3) \) with 
\[ \xi^1 = (q\Delta S_2(\omega_2) - \Delta S_2(\omega_1))\Delta S_2(\omega_1) + \Delta S_2(\omega_1)\Delta S_2(\omega_3), \]
\[ \xi^2 = -(\Delta S_1(\omega_3) + q\Delta S_1(\omega_2) - \Delta S_1(\omega_1))\Delta S_2(\omega_4) - (\Delta S_1(\omega_1) - \Delta S_1(\omega_4))\Delta S_2(\omega_3) \]
\[ \xi^3 = -(q\Delta S_1(\omega_4) - q\Delta S_1(\omega_1))\Delta S_2(\omega_2) - (\Delta S_1(\omega_4) + \Delta S_1(\omega_3) + q\Delta S_1(\omega_2))\Delta S_2(\omega_1), \]
and 
\[ D = (q\Delta S_1(\omega_1)\Delta S_2(\omega_2) + (\Delta S_1(\omega_3) - q\Delta S_1(\omega_2))\Delta S_2(\omega_1))\Delta S_2(\omega_4) \]
\[ + \Delta S_1(\omega_1)\Delta S_2(\omega_1)\Delta S_2(\omega_3) \]
is a statistical arbitrage.

Proof. According to Lemma 1, if \( \frac{P(\omega_2)}{P(\omega_3)} \neq q \) we have statistical arbitrage. Then, according to Proposition 2, it holds that the determinant of the matrix \( A \) (defined as in example above) is not equal to zero, which means that in this case the matrix \( A \) is invertible. Hence, \( \phi = A^{-1}1 \) is a statistical arbitrage and one can easily verify that \( \phi = \frac{1}{D}(\xi^1, \xi^2, \xi^3) \).

3.2 Risk of statistical arbitrages

The word arbitrage might be misleading on the riskiness of statistical arbitrages, because in the classical sense, an arbitrage is a strategy without risk. However statistical arbitrage is not without risk. It depends heavily on the ability of market prices to return to a historical or predicted normal, commonly referred to as mean reversion. Nonetheless, two stocks that operate in the same industry can remain uncorrelated for a significant amount of time due to both micro and macro factors. For this reason, most statistical arbitrage strategies take advantage of high-frequency trading algorithms to exploit tiny inefficiencies that often last for a matter of milliseconds. Large positions in both stocks are needed to generate sufficient profits from such minuscule price movements. This adds additional risk to statistical arbitrage strategies, although options can be used to help mitigate some of the risk.

In mathematical sense, since we consider arbitrage-free markets, all gains come with a certain risk and, higher profits are associated with higher risk. We consider the following example, which points out the riskiness of statistical arbitrage.

Let \( \Delta S_i(\omega_j) \in \{-5, 5\} \) i.e. we assume that the stock either rises by 5 or falls by 5. In addition, assume that \( q = \frac{P(\omega_2)}{P(\omega_3)} = 1.2 \). The using the equations (3.2)-(3.4) it is not difficult to compute \( \phi = (1.6, -1.4, -1.8)^T \). From this strategy we obtain that the gains at time 2, given by

\[ G_2(\omega) = \phi_1(\omega)\Delta S_1(\omega) + \phi_2(\omega)\Delta S_2(\omega), \]
yield \( G_2(\omega_1) = G_2(\omega_4) = 1 \). In addition we get that \( G_2(\omega_2) = 15 \) and \( G_2(\omega_3) = -17 \). If we assume that \( P(\omega_2) = 0.3 \) then \( P(\omega_3) = 0.25 \) and we obtain that the average expected gain on \( \{\omega_2, \omega_3\} \) computes to

\[ P(\omega_2)G(\omega_2) + P(\omega_3)G(\omega_3) = 0.3 \cdot 15 + 0.25 \cdot (-17) = 0.25 \geq 0, \]
such that the strategy is indeed a statistical arbitrage. While the (average) gains in the three relevant scenarios are 1, 0.25, 1, the possible loss in scenario \( \omega_3 \) is equal to -17.
which is attained with probability 0.25. To exploit the averaging property of statistical arbitrage, we repeat this strategy in the following until we first record a positive P&L. These considerations show clearly, that a risk analysis of the implemented strategy is very important.

4 Generalized $\mathcal{G}$-arbitrage strategies

In connection with improvement procedures for payoffs we consider any static or semi-static payoff $X \in L^1(P)$ as a generalized strategy. This leads to the following notion of generalized statistical $\mathcal{G}$-arbitrage strategies and the corresponding notion of generalized statistical $\mathcal{G}$-arbitrage. We denote by $L^1(P, Q) := L^1(P) \cap L^1(Q)$ the set of random variables which are integrable with respect to $P$ and $Q$.

**Definition 2.** Let $\mathcal{G} \subseteq \mathcal{F}$ be a $\sigma$-algebra. The set of generalized statistical $\mathcal{G}$-arbitrage strategies with respect to $Q \in \mathcal{M}^e$ is defined as

$$\overline{SA}(Q, \mathcal{G}) := \{ X \in L^1(P, Q) : E_Q[X] = 0, \ E_P[X | \mathcal{G}] \geq 0 \text{ P-a.s and } E_P[X] > 0 \}$$

The market satisfies $\overline{NSA}(Q, \mathcal{G})$, the condition of no generalized statistical $\mathcal{G}$-arbitrage with respect to $Q$, if

$$\overline{SA}(Q, \mathcal{G}) = \emptyset.$$

**Proposition 3.** Let $Q \in \mathcal{M}^e$. Then $\overline{NSA}(Q, \mathcal{G})$ is equivalent to the existence of a $\mathcal{G}$-measurable version of the Radon-Nikodym derivative $Z = dQ/dP$.

The proof of this result is achieved by Jensen’s inequality and using as candidate of a generalized $\mathcal{G}$-arbitrage

$$X = \frac{E[Z | \mathcal{G}]}{Z} - 1 \geq -1.$$

This equation also shows that the statistical arbitrage, if it exists, may be chosen bounded from below.

One consequence of this characterization result is the characterization of $\overline{NSA}(\mathcal{G})$ for the case of complete market models. Recall that the Radon-Nikodym derivative $Z = dQ/dP$ is path-independent, iff $Z$ is $\sigma(S_T)$-measurable.

A financial market is called complete, if every contingent claim is attainable, i.e. for every $\mathcal{F}$-measurable random variable $X$ bounded from below, we find an admissible self-financing trading strategy $\phi$, such that $x + V_T(\phi) = X$. This is implied by the assumption that $\mathcal{M}^e = \{Q\}$: indeed, under this assumption, Theorem 16 in Delbaen and Schachermayer (1995a) yields that any $X \in L^1(Q)$, bounded from below, is hedgeable and hence attainable.
Proposition 4. Assume that $\mathcal{M}^e = \{Q\}$. Then $\text{NSA}(\mathcal{G})$ holds if and only if $dQ/dP$ is $\mathcal{G}$-measurable.

Proof. We first show that existence of a $\mathcal{G}$-measurable $Q \in \mathcal{M}^e$ implies $\text{NSA}(\mathcal{G})$: we choose $Q \in \mathcal{M}^e$, such that $Z = dQ/dP$ is $\mathcal{G}$-measurable. Then $\text{NSA}(\mathcal{G})$ follows as in the proof of Proposition 1.

For the converse direction assume that $Z$ is not $\mathcal{G}$-measurable. By Proposition 3 it follows that there exists a generalized $\mathcal{G}$-arbitrage, i.e. an $X \in L^1(P,Q)$ with $E_Q[X] = 0$, $E_P[X|\mathcal{G}] \geq 0$ and $E_P[X] > 0$. As remarked above, $X$ can be chosen bounded from below. Hence, Theorem 16 in Delbaen and Schachermayer (1995a) yields existence of an admissible self-financing trading strategy $\phi$, such that $x + V_T(\phi) = X$. Moreover, the superhedging duality, i.e. Theorem 9 in Delbaen and Schachermayer (1995a) implies that $x = E_Q[X] = 0$, and hence $\phi$ is a $\mathcal{G}$-arbitrage. This is a contradiction and the claim follows.

In particular this result implies that Proposition 1 in Bondarenko (2003) gives a correct characterization of NSA for complete markets.

The following definition introduces the generalized $\mathcal{G}$-no-arbitrage condition without dependence on a specific pricing measure $Q$.

Definition 3. Let $\mathcal{G} \subseteq \mathcal{F}$ be a $\sigma$-algebra. The set of generalized statistical $\mathcal{G}$-arbitrage strategies is defined as

$$\text{SA}(\mathcal{G}) := \{X \in L^1(P) : \sup_{Q \in \mathcal{M}^e} E_Q[X] \leq 0, E_P[X|\mathcal{G}] \geq 0 \text{ P-a.s and } E_P[X] > 0\}.$$ 

The market satisfies $\text{NSA}(\mathcal{G})$, i.e. no generalized statistical $\mathcal{G}$-arbitrage, if $\text{SA}(\mathcal{G}) = \emptyset$.

Note that the definition defines a generalized statistical $\mathcal{G}$-arbitrage as a random variable $X \in L^1(P)$, such that $\sup_{Q \in \mathcal{M}^e} E_Q[X] \leq 0$, $E_P[X|\mathcal{G}] \geq 0$ $P$-almost surely, and $E_P[X] > 0$. In this sense, the strategies in $\text{SA}(\mathcal{G})$ are generalized statistical $\mathcal{G}$-arbitrage-strategies under any choice of the pricing measure $Q$.

5 Some classes of profitable strategies

In the following we are considering several classes of simple statistical arbitrage strategies for several classes of information systems $\mathcal{G}$. While these strategies are easy to apply for general stochastic models we investigate them on the Black-Scholes model which will allow for analytic properties of the trading strategies. We will see in the following section that similar results can be expected in more general market models. The Black-Scholes model is, according to [1] Example 4.4, free of statistical arbitrage, and we show in the following how to construct dynamic trading strategies allowing statistical $\mathcal{G}$-arbitrage for different
choices of \( \mathcal{G} \). Accordingly, we assume that \( S \) is a geometric Brownian motion, i.e. the unique strong solution of the stochastic differential equation
\[
dS_t = \mu S_t \, dt + \sigma S_t \, dB_t, \quad 0 \leq t \leq T
\]
where \( B \) is a \( P \)-Brownian motion and \( \sigma > 0 \). In the simulation we will first choose \( \mu = 0.1241, \sigma = 0.0837, S_0 = 2186 \) according to estimated drift and volatility from the S&P 500 (September 2016 to August 2017), and later consider also some other values.

5.1 Embedded binomial trading strategies

We introduce a recombination of several two-step binomial models embedded in the continuous -time model as long as the final time \( T \) is reached. As information system we consider the \( \sigma \)-field \( \mathcal{G} \) generated by the stopping times when the final states of each of the binomial model are reached (or the trivial \( \sigma \)-field otherwise).

Since we repeatedly consider embedded binomial models it makes sense to talk on the outcome of the trading strategy on average conditional on the final states of each binomial model, i.e. we average the outcome over many repeated applications of the trading strategy and can therefore apply the concept of statistical arbitrage here. Let \( i \) denote the current step of our iteration and consider a multiplicative step size \( c > 0 \). We start at time \( t_0^0 = 0 \). Otherwise consider the initial time of our next iteration given by the time where the last repetition finished, denote this time by \( t_0^i \) and the according level by \( s_0^i = S_{t_0^i} \). Then we define the following two stopping times denoting the first and second period of our binomial model by
\[
t_1^i = \inf\{ t \in [t_0^i, T] \mid S_t \in \{ s_0^i(1-c), s_0^i(1+c) \} \}
\]
and
\[
t_2^i = \inf\{ t \in (t_1^i, T] \mid S_t \in \{ s_0^i(1-2c), s_0^i(1+2c) \} \}
\]
with the convention that \( \inf \emptyset = T \). This induces a sequence of \( \sigma \)-fields \( \mathcal{G}^i := \sigma(S_{t_2^i}) \). Since \( S \) is continuous, this scheme allows to embed repeated binomial models \( S_{t_0^i}, S_{t_1^i}, S_{t_2^i}, \ldots \) into continuous time. The considered trading strategy is to execute the statistical arbitrage strategy for binomial models computed in Lemma 2 at the stopping times \( t_0^i, t_1^i, t_2^i \). At \( t_2^i \) we clear the position and start the procedure afresh by letting \( t_0^{i+1} = t_2^i \). Generally, we assume that the time horizon \( T \) is sufficiently large such that the (typically small) levels \( s_0^i(1-2c), \ldots, s_0^i(1+2c) \) are reached at least once.
Some classes of profitable strategies

5 Example 1 Figure 2 illustrates the embedding of the binomial model: the boundary $s_0(1 - c)$ is hit at stopping time $t_1 = t^0_1$ and the boundary $s_0(1 - 2c)$ at stopping time $t_2 = t^0_2$. The trading strategy $\phi$ from Lemma 2 then implies trading buying (selling) $\phi_1$ entities of the underlying at time $t = 0$ and $\phi_2^-$ entities at $t = t_1$. At time $t = t_2$ we will close the position and start this procedure again with $t^1_0 = t_2$ and with the new starting point $s^1_0 = S_{t_2}$. In this way we get a recombination of several 2-period binomial models, as illustrated in Figure 3.

Since we do not want to loose the statistical arbitrage opportunity, the constant $c$ and with it the barriers for the hitting times will be chosen in dependence of $\mu$ and $\sigma$. To be more precise we use

$$c = 0.01 \cdot \frac{\mu}{\sigma}$$

which showed a good performance in our simulations. According to Lemma 1 there is a statistical arbitrage opportunity if $P(\omega_2)/P(\omega_3) \neq \tilde{q}$. In the case considered here it holds that $\tilde{q} = 1$. To guarantee existence of a statistical arbitrage we calculate the path probabilities $P(\omega_2), P(\omega_3)$. We use formula 3.0.4 in Section 9 of Part II from Borodin and Salminen (2012), which yields that in general $q \neq 1$, such that in these cases statistical arbitrage exists. We exploit that in the following.

From Lemma 2 we obtain with $D = 2(q - 2)(cs_0^i)^3$ that the trading strategy $\phi = (\phi_1, \phi_2^+, \phi_2^-)$ is given by

$$\phi_1 = (2 + q)(cs_0^i)^2 D^{-1},$$

$$\phi_2^+ = (q - 4)(cs_0^i)^2 D^{-1},$$

$$\phi_2^- = -3q(cs_0^i)^2 D^{-1}.$$
5 Some classes of profitable strategies

We call the trading strategy which results by repeated application of $\phi$ at the respective hitting times the embedded binomial trading strategy.

5.1.1 Simulation results

As already mentioned, we simulate a geometric Brownian motion according to equation above with $\mu = 0.1241, \sigma = 0.0837, S_0 = 2186, T = 1 \text{ year}$, discretized by 1000 steps and embed the according binomial models repeatedly in this time interval. In this case we have $q = 1.00189$ (rounded to five digits) which is not equal to one and therefore $q \neq \tilde{q}$, i.e. the embedded binomial strategy in this case is a $\mathcal{G}$-arbitrage strategy. We denote by $N$ the (random) number of binomial models that are necessary for each simulated diffusion to gain either a profit from trading or to reach $T$ and by $G^i$ the gain or loss of the $i$-th binomial model. Hence either $\sum_{i=1}^{n} G^i > 0$ or we record a loss at time $N = T$.

For 1 million runs, we obtain the results presented in Table 1. For each run we record either a gain or a loss from trading. The average gain per simulation run (the overall average gain in one year) is shown in column one, its median in column two. Median of 206 in comparison to an average gain of 33 reflects that the distribution of the P&L is skewed to the left with potential large losses with small probability. Column 3 denotes the 95% Value-at-Risk (a statistic that measures and quantifies the level of financial risk within a firm, portfolio or position over a specific time frame) which is of size 5,320. In column 4 we depict the average gain per trade which is obtained by dividing the average gain by the average number of trades (i.e. repeated binomial models). In column 5 we show the (fraction of) losses, i.e. the fraction of simulated processes where the outcome of the trading strategy was negative, followed by their mean in column 6. The average number
of trading repeats $\emptyset N$ is followed by the maximal number of embedded binomial models over all runs (max $N$). As we can see from Table 5.1 we can indeed record an overall profit for many cases. We have a negative outcome in 13.3 percent in average of all simulations with an average size of -628. The median of the profits is about 200, average about 30.

The risk measured by the Value-at-Risk at 95% is 5,320 which pointing to the fact that the average gain by the statistical arbitrage is not without risk as previously mentioned. The associated histogram of the P&L is plotted in Figure 4.

![Histogram of the profits and losses from the embedded binomial trading strategy used in Table 5.1.](image)

This confirms the possibility of statistical arbitrage, even though the actual amount of the profit depends on many parameters. Besides, we see that on average our multi-period binomial model has a small number of periods and the number of periods does not explode, which is important with a view on trading costs.

\[
\begin{array}{cccccccc}
\text{gain p.a.} & \text{median} & \text{VaR(0.95)} & \text{gain/trade} & \text{losses} & \text{(mean)} & \emptyset N & \text{max } N \\
33.4 & 206 & 5.32 & 8.74 & 0.133 & -628 & 3.82 & 24
\end{array}
\]

Table 5.1: Simulation results for the embedded binomial trading strategy for 1 mio runs.

5.1.2 Varying barrier levels

The most interesting parameter turns out to be the parameter $c$. It decodes the varying the barrier level and the results for different values of $c$ may be found in Table 5.2. Table
5.2 implies that the parameter \( c \) allows us to balance gains and risk very well. We observe that the smaller the parameter \( c \) is chosen, the higher are the gains in general. As expected the additional gain implies an increase of risk: most prominently, the mean of the losses decreases with \( c \). On the other side, we observe a decrease in the probability for losses to occur. The Value-at-Risk confirms the increase of risk with decreasing \( c \), except for the lowest \( c = 0.0025 \). In this case the risk is of course still present, but the probability of having large losses is below 5\%, such that the Value-at-Risk at level 0.95\% does no longer see it. A high value of \( c \) corresponds to a larger step sizes, which leads to less trades on average. The largest value of \( c \) gives a statistical arbitrage with small gain and smallest risk.

<table>
<thead>
<tr>
<th>( c )</th>
<th>gain p.a.</th>
<th>median</th>
<th>VaR(0.95)</th>
<th>gain p.t.</th>
<th>losses (mean)</th>
<th>( \mathcal{O}N )</th>
<th>max ( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0025</td>
<td>8,890</td>
<td>48,700</td>
<td>-373</td>
<td>743</td>
<td>0.045</td>
<td>-57,900</td>
<td>12</td>
</tr>
<tr>
<td>0.005</td>
<td>465</td>
<td>3,810</td>
<td>58,400</td>
<td>66</td>
<td>0.077</td>
<td>-6,210</td>
<td>7</td>
</tr>
<tr>
<td>0.01</td>
<td>41</td>
<td>206</td>
<td>5,250</td>
<td>11</td>
<td>0.132</td>
<td>-621</td>
<td>4</td>
</tr>
<tr>
<td>0.02</td>
<td>9</td>
<td>10</td>
<td>371</td>
<td>5</td>
<td>0.185</td>
<td>-50</td>
<td>2</td>
</tr>
<tr>
<td>0.04</td>
<td>3</td>
<td>2</td>
<td>24</td>
<td>3</td>
<td>0.109</td>
<td>-2</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 5.2: Table to test captions and labels

5.1.3 The role of drift and volatility

For the investor it is of interest which drift and which volatility of an asset promises a good profit. To investigate this question we define the fraction

\[
\eta := \frac{\mu}{\sigma}
\]

and show simulation results for different values of \( \eta \). In Table 5.3 we fix the volatility \( \sigma \) and consider varying drift, while in Table 5.4 the drift \( \mu \) is fixed and we consider varying volatility. Larger values of \( \eta \) point to a high drift relative to volatility situations which we would expect to be very well exploitable. In fact, in the simulations we can see quite the contrary: actually we observe large gains when \( \eta \) is small, while for larger \( \eta \) we observe only small gains. More precisely, for fixed \( \sigma \) we obtain decreasing gains for increasing drift, while for fixed \( \mu \) we observe increasing gains for increasing volatility. This effect is even more clear for the latter case (increasing \( \sigma \)). Already from the results with varying step sizes in Table 5.2 such an effect was to be expected, as higher values of \( \eta \) lead to larger step sizes here and to lower gains. Intuitively, larger volatility implies more repetitions and therefore a higher likelihood for the statistical arbitrage to end up with gains, which is also reflected by increasing values of \( N \) in Table 5.4.


5 Some classes of profitable strategies

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>gain p.a.</th>
<th>median</th>
<th>VaR(0.95)</th>
<th>gain p.t.</th>
<th>losses (mean)</th>
<th>$\varnothing N$</th>
<th>max N</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.33</td>
<td>211</td>
<td>11,600</td>
<td>252,000</td>
<td>45</td>
<td>0.13</td>
<td>-29,400</td>
<td>5</td>
</tr>
<tr>
<td>0.50</td>
<td>170</td>
<td>4,360</td>
<td>94,500</td>
<td>36</td>
<td>0.13</td>
<td>-11,000</td>
<td>5</td>
</tr>
<tr>
<td>0.75</td>
<td>109</td>
<td>1,730</td>
<td>38,100</td>
<td>23</td>
<td>0.13</td>
<td>-4,400</td>
<td>5</td>
</tr>
<tr>
<td>1.00</td>
<td>64</td>
<td>913</td>
<td>20,400</td>
<td>14</td>
<td>0.12</td>
<td>-2,340</td>
<td>5</td>
</tr>
<tr>
<td>1.25</td>
<td>77</td>
<td>561</td>
<td>12,400</td>
<td>17</td>
<td>0.12</td>
<td>-1,400</td>
<td>5</td>
</tr>
<tr>
<td>2.00</td>
<td>42</td>
<td>197</td>
<td>4,430</td>
<td>9</td>
<td>0.11</td>
<td>-490</td>
<td>4</td>
</tr>
<tr>
<td>3.00</td>
<td>34</td>
<td>81</td>
<td>1,680</td>
<td>8</td>
<td>0.10</td>
<td>-182</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 5.3: Simulations for the embedded binomial trading strategy with different values of the drift $\mu$ (and hence $\eta$), fixed $\sigma = 0.1$ and $n = 250,000$ runs

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>gain p.a.</th>
<th>median</th>
<th>VaR(0.95)</th>
<th>gain p.t.</th>
<th>losses (mean)</th>
<th>$\varnothing N$</th>
<th>max N</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>74,500</td>
<td>222,000</td>
<td>-48,400</td>
<td>4,340</td>
<td>0.036</td>
<td>-2,770,000</td>
<td>17</td>
</tr>
<tr>
<td>0.75</td>
<td>6,020</td>
<td>59,900</td>
<td>480,000</td>
<td>582</td>
<td>0.056</td>
<td>-79,400</td>
<td>10</td>
</tr>
<tr>
<td>1.00</td>
<td>241</td>
<td>4,710</td>
<td>80,500</td>
<td>37</td>
<td>0.090</td>
<td>-8,520</td>
<td>7</td>
</tr>
<tr>
<td>1.25</td>
<td>67</td>
<td>541</td>
<td>12,700</td>
<td>16</td>
<td>0.124</td>
<td>-1,460</td>
<td>4</td>
</tr>
<tr>
<td>2.00</td>
<td>8</td>
<td>6</td>
<td>165</td>
<td>5</td>
<td>0.144</td>
<td>-22</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 5.4: Simulations for the embedded binomial trading strategy with different values of the volatility $\sigma$ (and hence $\eta$), fixed $\mu = 0.1$

5.2 Follow-the-trend strategy

From the previous section it follows that embedding a binomial model into continuous time is not able to exploit a large drift. This motivates the introduction of a further step into the embedded model in order to exploit existing trends in the underlying. Our focus in the following is the upward trend, while the strategy is easily adopted to the case for a downward trend. We consider two-step binomial embedding: first, we specify barriers (up/down) as previously. If we observed up movements twice, we expect an upward trend and exploit this in a further step. Consequently, here we will consider four stopping times (for iteration $i$): initial time $\tau_0^i$, and stopping times $\tau_1^i$, $\tau_2^i$ as previously and, in addition $\tau_3^i$. In particular this modelling implies a different choice of the filtration $\mathcal{F}$.

The associated strategy is to trade in the following way: the first trading occurs as previously at the first time when the barriers $s(1 + c)$ or $s(1 - c)$ are hit. In the first case next trading takes place when $s$ or $s(1 + 2c)$ are hit and in the second case $s$ or $s(1 - c)$. If a trend was detected (i.e. the upper barrier $s(1 + 2c)$ was hit, as we consider the case of a positive drift), trading continues until a suitable stopping time. This leads to the following procedure: let $i$ denote the current step of our iteration. We start at time $\tau_0^0 = 0$. Otherwise we consider the initial time of our next iteration given by the the time where we finished the last repetition and denote this time by $\tau_0^i$ and the according level by $s_0^i = S_{\tau_0^i}$.

Then, using again the property that $S$ is continuous, we define the following successive
5 Some classes of profitable strategies

stopping times:
first, analogously to \( t_1 \) from embedded binomial trading, let

\[
\tau_1 = \inf\{ t \in [\tau_0, T] \mid S_t \geq s_0(1 + c) \text{ or } S_t = s_0(1 - c) \}
\]

In the same manner the second stopping occurs if either the upper level is reached, or the mid-level is crossed, or the bottom level is reached. The levels of course differ depending on whether \( S_{\tau_1} = s_0(1 + c) \) or \( S_{\tau_1} = s_0(1 - c) \). On this subject, we define (for the first case)

\[
\sigma_1 = \inf\{ t \in (\tau_1, T] \mid S_t \geq s_0(1 + 2c) \}
\]

\[
\sigma_2 = \inf\{ t \in (\tau_1, T] \mid S_t \leq s_0 \}
\]

For the second case, we set

\[
\sigma_3 = \inf\{ t \in (\tau_1, T] \mid S_t \leq s_0(1 - 2c) \}
\]

\[
\sigma_4 = \inf\{ t \in (\tau_1, T] \mid S_t \geq s_0 \}
\]

Altogether we obtain that

\[
\tau_2 = \begin{cases} 
\sigma_1 \land \sigma_2 & \text{if } S_{\tau_1} = s_0(1 + c) \\
\sigma_3 \land \sigma_4 & \text{otherwise.}
\end{cases}
\]

Finally, we set

\[
\tau_3 = \begin{cases} 
\inf\{ t \in (\tau_2, T] \mid S_t \leq s_0 \text{ or } S_t \geq s_0(1 + 4c) \} & \text{if } S_{\tau_2} = s_0(1 + 2c) \\
\tau_2 & \text{otherwise.}
\end{cases}
\]

Figure 5: The embedded binomial model for the follow-the-trend strategy with positive drift
Let $\tau^{\text{max}}$ denote the last stopping time of $\tau_3^1, \tau_3^2, ...$ which lies before $T$. Then the statistical arbitrages traded on the partition of $S_{\tau^{\text{max}}}$ generated by the values $s_0(1+2kc), k = 0, 1, 2, ...$ which defines the $\mathcal{G}$ on the path space of the diffusion.

Trading will be executed at times $\tau^i_1$ to $\tau^i_3$ when the process reaches one of the predefined boundaries (or when trading time is over). At time $\tau^i_2$ we check if a positive trend persists and trade on this trend. Recall the trading strategy $\phi = (\phi, \phi^+, \phi^-)$ from Equations (5.1) to (5.3). First, trading at the first two times is executed as previously at times $t^i_0, t^i_1$, see Lemma 2: we hold on $[\tau^i_0; \tau^i_1)$ the fraction $\phi^1$ shares of $S$. After reaching $s^i_0(1+c)$ ($s^i_0(1-c)$, respectively) at time $\tau^i_1$ the trading strategy changes to holding $\phi^+_2$ ($\phi^-_2$) shares of $S$ until $\tau^i_2$. We distinguish between the three following cases for the next trading strategy:

i) $\tau^i_2 = \sigma^i_1$: in this case we reached the upper level $s^i_0(1+2c)$ and follow the (upward) trend by holding $\phi^{++}_3$ shares of $S$. This position will be equalized at $\tau^i_3$ or if the final time is reached.

ii) $\tau^i_2$ equals $\sigma^i_2$ or $\sigma^i_4$: from the state $s^i_0(1+c)$ resp. $s^i_0(1-c)$ we arrived back at $s^i_0$ (or below resp. above). We did not detect any trend and the embedded binomial trading strategy ends by liquidating the position.

iii) $\tau^i_2$ equals $\sigma^i_4$: again, no (upward) trend was detected and the strategy ends by liquidation the position.

Since Lemma 2 treats a related, but slightly different case we explicitly check in the following that the embedded binomial model indeed allows for statistical arbitrage.

### 5.3 The embedded binomial follow-the-trend strategy

We consider $\check{\Omega} = \{\omega_1, ..., \omega_5\}$ as depicted in Figure 6. Let $S_0 = s_0 \in \mathbb{R}_{\geq 0}$ and $S_1$ take the two values $s^+$ and $s^-$ such that

$$S_1(\omega_1) = S_1(\omega_2) = S_1(\omega_5) = s^+, \quad S_1(\omega_3) = S_1(\omega_4) = s^-.$$ 

At time 2 we have the three possibilities $S_2(\omega_1) = S_2(\omega_2) = s^{++}, S_2(\omega_2) = S_2(\omega_3) = s^{+-}$ and $S_2(\omega_4) = s^{--}$. In the cases of $\omega_2, ..., \omega_4$ the model stops. If, however, we saw two up-movements, the model continues and ends up at time 3 in the states $S_3(\omega_1) = s^{+++}$ or $S_3(\omega_5) = s^{++-}$. We assume without loss of generality that $s^+ > s_0$, $s^- < s_0$, and $s^{++} > s^+, s^+ < s^{+-} < s^+$, and $s^{--} < s^-$ as well as $s^{++-} < s^{++} < s^{+++}$ i. e. we consider binomial models as presented in Figure 6.
The dynamic trading strategies can be described by

\[ V_3(\phi) = \phi_1 \Delta S_1 + \phi_2 \Delta S_2 + \phi_3 \Delta S_3 \]

with \( \phi_1, \phi_2^+, \phi_2^- \) and \( \phi_3^{++} \) being the respective values in the states \( \tilde{\Omega}, \{\omega_1, \omega_2, \omega_5\}, \{\omega_3, \omega_4\} \) and \( \{\omega_1, \omega_5\} \) at times 1, 2, and 3 respectively. Moreover, we choose

\[ \tilde{\mathcal{G}} = \sigma(\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}, \{\omega_5\}) \]

i.e. the \( \sigma \)-field generated by the final states of the embedded binomial model. The following Lemma shows that there is always statistical arbitrage in the follow-the-trend strategy if there is statistical arbitrage in the recombining two-period sub-model consisting only of the first two periods.

Denote

\[ \gamma = \frac{1}{D} \left( \Delta S_1(\omega_4) \Delta S_2(\omega_3) - (q \Delta S_1(\omega_2) + \Delta S_1(\omega_3)) \Delta S_2(\omega_1) \right) \]

with \( D \) given in Lemma 2. The following results shows, that in the follow-the-trend model there is statistical arbitrage, if \( P(\omega_2)/P(\omega_3) \neq \tilde{q} \) holds.

**Proposition 5.** If \( \phi \) is the strategy from Lemma 2, then for any \( \alpha \geq 0 \), \( \psi = (\psi_1, \psi_2^+, \psi_2^-, \psi_3^{++}) \) with

\[ \psi_3^{++} = \frac{1 - \alpha}{\Delta S_3(\omega_1) - \Delta S_3(\omega_5)} \]
and

\[
\begin{pmatrix}
\psi_1 \\
\psi_2^+ \\
\psi_2
\end{pmatrix} = \phi - \Delta S_3(\omega_1)\psi_3^{++}
\]

is a \(\tilde{G}\)-arbitrage strategy, if \(\frac{p_{\omega_2}}{p_{\omega_3}} \neq \tilde{q}\) holds.

One possible choice for \(\alpha\) is \(\alpha = 1\). This leads to \(\psi_3^{++} = 0\), such that in this case the statistical arbitrage in the first two periods is exploited and the strategy coincides with that of Lemma 2.

Proof. Following Definition 1 the strategy \(\psi\) is a statistical \(\tilde{G}\)-arbitrage strategy if the following holds

\[
\begin{align*}
\psi_1 \Delta S_1(\omega_1) &+ \psi_3^+ \Delta S_2(\omega_1) + \psi_3^{++} \Delta S_3(\omega_1) \geq 0 \\
\psi_1 \Delta S_1(\omega_4) &+ \psi_2^- \Delta S_2(\omega_4) \geq 0 \\
\psi_1 \Delta S_1(\omega_2) P(\omega_2) &+ \psi_2^+ \Delta S_2(\omega_2) P(\omega_2) \\
+ \psi_1 \Delta S_1(\omega_3) P(\omega_3) &+ \psi_2^- \Delta S_2(\omega_3) P(\omega_3) \geq 0 \\
\psi_1 \Delta S_1(\omega_5) &+ \psi_3^+ \Delta S_2(\omega_5) + \psi_3^{++} \Delta S_3(\omega_5) \geq 0
\end{align*}
\]

and, in addition, at least one of the inequalities is strict. Here (5.5),(5.6), and (5.9) are divided by \(P(\omega_1)\), \(P(\omega_4)\) and \(P(\omega_5)\), respectively.

We extend the setting from Lemma 2. First, we let

\[
\tilde{A} = \begin{pmatrix}
\Delta S_1(\omega_1) & \Delta S_2(\omega_1) & 0 & \Delta S_3(\omega_1) \\
\Delta S_1(\omega_4) & 0 & \Delta S_2(\omega_4) & 0 \\
q\Delta S_1(\omega_2) + \Delta S_1(\omega_3) & q\Delta S_2(\omega_2) & \Delta S_2(\omega_3) & 0 \\
\Delta S_1(\omega_5) & \Delta S_2(\omega_5) & 0 & \Delta S_3(\omega_5)
\end{pmatrix}
\]

Then Equations (5.5)-(5.9) are equivalent to \(\tilde{A}\psi \geq 0\). Note that \(S_i(\omega_1) = S_i(\omega_5)\) for \(i = 1, 2\) such that \(\tilde{A}\psi = \tilde{x}\) with \(\tilde{x} = (x_1, \ldots, x_4)^T\) reveals

\[
\psi_3^{++} = \frac{x_1 - x_4}{\delta S_3(\omega_1) - \Delta S_3(\omega_5)}.
\]

As for Lemma 2, we will consider the case where \(\tilde{A}\) is invertible. Note that the upper left 3x3 submatrix of \(\tilde{A}\) equals the matrix \(A\) from Proposition 2. Then denoting \(x = (x_1, x_2, x_3)^T\),

\[
\begin{pmatrix}
\psi_1 \\
\psi_2^+ \\
\psi_2
\end{pmatrix} = A^{-1}x - A^{-1}\begin{pmatrix}
\Delta S_3(\omega_1)\psi_3^{++} \\
0 \\
0
\end{pmatrix} = A^{-1}x - \Delta S_3(\omega_1)\psi_3^{++}\gamma
\]

with vector \(\gamma\) from equation (5.4). Up to now free to choose any \(\tilde{x} \in \mathbb{R}^4_{>0}\). If we choose, as for Lemma 3.7., \(x = 1_3\), the \(\phi = A^{-1}1_3\) is the strategy computed in Lemma 2 and the result follows. \(\square\)
5.3.1 Simulation results

We study the performance of the follow-the-trend strategy on basis of different simulations and compare it to the results of the embedded binomial strategies. As in the simulation of the embedded binomial strategies, we simulate a geometric Brownian motion with \( \mu = 0.1241, \sigma = 0.0837, S_0 = 2186, T = 1 \) (year), discretized by 1000 steps and embed the according models repeatedly in this time interval. In this case, Proposition 5. is useful, because according to it we have the existence of statistical arbitrage which we will exploit in the following.

We can note from the simulations that the goal of improving the average gain of the follow-the-trend strategy is not achieved. On the other hand the Value-at-Risk which is decreasing in Tables 5.5 to 5.8 points out to the fact that in general, the follow-the-trend strategy leads to a reduction of risk compared to the embedded-binomial trading strategy. The reduction of the average gain and its mean can be explained in following way: follow-the-trend-strategy introduces additional scenarios with smaller gains (see Figure 6). This leads to a reduction of the average gain and, at the same time, to a reduction of risk.

The results from Table 5.6 to 5.8 show a similar dependence on the choice of the parameters and of the barrier of the follow-the-trend strategy compared to the embedded binomial strategy. In general, we record smaller gains together with smaller risk with one exception: the last line of Table 5.8 shows that a small \( \sigma \) allows the follow-the-trend strategy to exploit the existing (although small) positive trend in the data better. Of course, this comes with a higher risk, which is clearly visible. Summarizing, the follow-the-trend strategy shows (in general) smaller gains together with a smaller risk. The follow-the-trend strategy is, however, able to exploit a positive trend when \( \sigma \) is very small.

<table>
<thead>
<tr>
<th>gain p.a.</th>
<th>median</th>
<th>VaR(0.95)</th>
<th>gain p.t.</th>
<th>losses (mean)</th>
<th>( \varnothing N )</th>
<th>max N</th>
</tr>
</thead>
<tbody>
<tr>
<td>27.8</td>
<td>164</td>
<td>4,180</td>
<td>9.17</td>
<td>0.171</td>
<td>-554</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 5.5: Simulations for the follow-the-trend strategy for 1 mio runs. In comparison to Table 1 we find slightly smaller gains together with a smaller risk.

<table>
<thead>
<tr>
<th>c</th>
<th>gain p.a.</th>
<th>median</th>
<th>VaR(0.95)</th>
<th>gain p.t.</th>
<th>losses (mean)</th>
<th>( \varnothing N )</th>
<th>max N</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005( \mu/\sigma )</td>
<td>404</td>
<td>3,300</td>
<td>51,300</td>
<td>71.1</td>
<td>0.098</td>
<td>-5,590</td>
<td>6</td>
</tr>
<tr>
<td>0.01( \mu/\sigma )</td>
<td>32</td>
<td>162</td>
<td>4,130</td>
<td>10.7</td>
<td>0.169</td>
<td>-548</td>
<td>3</td>
</tr>
<tr>
<td>0.02( \mu/\sigma )</td>
<td>6</td>
<td>8</td>
<td>272</td>
<td>3.9</td>
<td>0.238</td>
<td>-45</td>
<td>2</td>
</tr>
<tr>
<td>0.04( \mu/\sigma )</td>
<td>3</td>
<td>1</td>
<td>23</td>
<td>2.6</td>
<td>0.122</td>
<td>-2</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 5.6: Simulations for the follow-the-trend strategy with varying barrier levels c. In the simulations for Table 5.5 we used \( c = 0.01 \mu/\sigma \)
5 Some classes of profitable strategies

<table>
<thead>
<tr>
<th>η</th>
<th>gain p.a.</th>
<th>median</th>
<th>VaR(0.95)</th>
<th>gain p.t.</th>
<th>losses (mean)</th>
<th>∅ N</th>
<th>max N</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.33</td>
<td>282</td>
<td>9,340</td>
<td>203,000</td>
<td>71</td>
<td>0.16</td>
<td>−26,100</td>
<td>4</td>
</tr>
<tr>
<td>0.5</td>
<td>122</td>
<td>3,500</td>
<td>76,200</td>
<td>31</td>
<td>0.16</td>
<td>−9,780</td>
<td>4</td>
</tr>
<tr>
<td>0.75</td>
<td>99</td>
<td>1,390</td>
<td>30,400</td>
<td>26</td>
<td>0.16</td>
<td>−9,890</td>
<td>4</td>
</tr>
<tr>
<td>1.00</td>
<td>78</td>
<td>734</td>
<td>16,200</td>
<td>20</td>
<td>0.15</td>
<td>−2,050</td>
<td>4</td>
</tr>
<tr>
<td>1.25</td>
<td>54</td>
<td>452</td>
<td>9,950</td>
<td>15</td>
<td>0.15</td>
<td>−1,260</td>
<td>4</td>
</tr>
<tr>
<td>2.00</td>
<td>34</td>
<td>162</td>
<td>3,570</td>
<td>10</td>
<td>0.14</td>
<td>−436</td>
<td>3</td>
</tr>
<tr>
<td>3.00</td>
<td>24</td>
<td>66</td>
<td>1,390</td>
<td>7</td>
<td>0.13</td>
<td>−165</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 5.7: Simulations for the follow-the-trend strategy with varying values of the drift (and hence η) with fixed σ = 0.1.

<table>
<thead>
<tr>
<th>η</th>
<th>gain p.a.</th>
<th>median</th>
<th>VaR(0.95)</th>
<th>gain p.t.</th>
<th>losses (mean)</th>
<th>∅ N</th>
<th>max N</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.33</td>
<td>65,600</td>
<td>2,030,000</td>
<td>22,700,000</td>
<td>6,640</td>
<td>0.06</td>
<td>−2,770,000</td>
<td>10</td>
</tr>
<tr>
<td>0.5</td>
<td>2,010</td>
<td>40,700</td>
<td>586,000</td>
<td>284</td>
<td>0.09</td>
<td>−62,500</td>
<td>7</td>
</tr>
<tr>
<td>0.75</td>
<td>292</td>
<td>3,930</td>
<td>69,200</td>
<td>60</td>
<td>0.12</td>
<td>−7,940</td>
<td>5</td>
</tr>
<tr>
<td>1.00</td>
<td>44</td>
<td>732</td>
<td>16,400</td>
<td>11</td>
<td>0.15</td>
<td>−2,080</td>
<td>4</td>
</tr>
<tr>
<td>1.25</td>
<td>27</td>
<td>200</td>
<td>5,330</td>
<td>9</td>
<td>0.18</td>
<td>−729</td>
<td>3</td>
</tr>
<tr>
<td>2.00</td>
<td>10</td>
<td>15</td>
<td>469</td>
<td>5</td>
<td>0.20</td>
<td>−68</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 5.8: Simulations for the follow-the-trend strategy with varying values of the volatility σ and fixed µ = 0.1.

5.4 Summary on the different strategies

The previous results confirm statistical $G$-arbitrage for introduced strategies with respect to corresponding choices of $G$. We also observe that the main difference between the embedded binomial and the follow-the-trend strategy is that for the first one the average profit achieved is better, while the second one comes with smaller risk for the price of smaller gains on average.
Bibliography