Seminar paper

Operational Risk

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1 Introduction

Nowadays the management of financial risk in the banking industry has met with lots of changes over the last years. Financial risk is any of various types of risk associated with financing, including financial transactions that include company loans in risk of default. It is often understood to include only downside risk, meaning the potential for financial loss and uncertainty about its extent. We all know that ability of the company to measure and manage the financial risk is one of the most important tasks. That is why I have chosen one type of the financial risk, namely operational risk, as a central topic of my seminar work. In my introduction I would like to tell about several types of the financial risk.

1.1 Credit risk

Credit risk, also called default risk, is the risk associated with a borrower going into default (not making payments as promised). In the first resort, the risk is that of the lender and includes lost principal and interest, disruption to cash flows, and increased collection costs. The loss may be complete or partial. In an efficient market, higher levels of credit risk will be associated with higher borrowing costs. Because of this, measures of borrowing costs such as yield spreads can be used to infer credit risk levels based on assessments by market participants.

1.2 Operational risk

Operational risk is the risk of a change in value caused by the fact that actual losses, incurred for inadequate or failed internal processes, people and systems, or from external events (including legal risk), differ from the expected losses. Operational risk can be summarized as human risk; it is the risk of business operations failing due to human error. It changes from industry to industry, and is an important consideration to make when looking at potential investment decisions. Industries with lower human interaction are likely to have lower operational risk.

1.3 Market risk

Market risk is the risk of losses in positions arising from movements in market prices.

1.4 Liquidity risk

Liquidity risk is a financial risk that for a certain period of time a given financial asset, security or commodity cannot be traded quickly enough in the market without impacting the market price.

1.5 Other risks

- Reputational risk,
• Settlement risk,
• Profit risk,
• Systematic risk,
• Volatility risk.

1.6 Capital allocation

Under the 1988 Accord, the Basel Committee on Banking Supervision recognizes that the capital charge related to credit risk implicitly covers other risks. Reflecting that risks other than credit and market risks can be substantial, operational risk are now explicitly concerned by the New Basel Capital Accord. A survey issued by the Risk Management Group suggests that economic capital allocation for operational risk ranges between 15 – 25% for the majority of banks.

Figure 1: Illustration of the capital allocation for different types of risks in the majority of banks.
2 Operational Risk. General Information

Banks are required by the regulators to allocate capital against potential losses. It is some sort of the self-insurance. The main risk categories attracting capital charge in financial institutions are credit risk, market risk and operational risk. Operational risk did not require explicit capital allocation until recently; previously, it was implicitly covered by the capital charge for credit risk. The concept of operational risk is generic for organizations of all types.

2.1 Definition of the operational risk

The Basel Committee defines the operational risk as the

"risk of loss resulting from inadequate or failed internal processes, people and systems or from external events".

In most cases legal risk is included in this definition, reputational and strategic risks are not. This definition focuses on causes of loss, called event type, but do not specify their effects (loss type), however event type and loss type should be identified when reporting loss data. In simple words, operational risk is a risk of loss due to the day-to-day operations in the organization.

There are several important instruments, that have to be used in order to manage operational risk:

- developing policies and internal standards;
- developing key risk indicators;
- planning management of major business malfunction;
- supporting a database of risk incidents.

2.2 Top 10 operational risks for 2018

In January and February 2018 the website Risk.net has interviewed chief risk officers, heads of operational risk and senior practitioners at financial services firms, including banks, insurers, asset managers and infrastructure providers, in order to create the list of top 10 operational risks in 2018.

1. IT Disruption:
   a) cyber attack,
   b) human error,
   c) failure of aging hardware,
   d) Distributed Denial of Service (DDoS) attack ... an attempt to make an online service unavailable by overwhelming it with traffic from multiple sources.
For example, in May 2017 WannaCry ransomware attack occurred, worldwide cyberattack by the WannaCry ransomware cryptoworm, which targeted computers running the Microsoft Windows operating system by encrypting data and demanding ransom payments in the Bitcoin cryptocurrency. The attack was stopped within a few days of its discovery due to emergency patches released by Microsoft. It was estimated to have affected more than 200,000 computers across 150 countries, with total damages ranging from hundreds of millions to billions of dollars. Therefore it is not hard to see why operational risk practitioners estimate IT disruption as the most substantial operational danger facing their firms.

2. **Data compromise**: Data compromise is intentional or unintentional release of secure or private/confidential information to an untrusted environment
   - a) unauthorised access,
   - b) accidental disclosure,
   - c) employee negligence.

There are numerous ways in which the extensive quantities of personal information banks and financial services firms hold can fall into the wrong hands. The greatest data breach of 2017 was the cyber attack on credit reporting agency **Equifax**, which threatened personal information including names, social security numbers, driving licence numbers, credit card numbers and personal documents, relating to an estimated 145 million individuals.

3. **Regulatory risk**
   One of the most prominent examples of the consequences of the regulatory risk is the case with **Wells Fargo and Company**, an American international banking and financial services holding company headquartered in San Francisco, California. The Federal Reserve in February 2018 imposed unusually harsh penalties on Wells Fargo, punishing it for years of misconduct and barring it from future growth until the bank fixes its problems. The Fed’s punishment, a forceful intervention by the government into the affairs of a large company, means that one of the country’s largest and most powerful financial institutions will be unable to keep pace with its fast-growing rivals. This singular action caused Wells to slash its profit estimate for the year by up to $400 million.

4. **Theft and fraud**

5. **Outsourcing**
   In business, outsourcing is "an agreement in which one company contracts-out a part of existing internal activity to another company".

6. **Mis-selling**
   Mis-selling means that you were given unsuitable advice, the risks were not explained to you or you were not given the information you needed and ended up with a product that isn’t right for you.

7. **Talent risk**
8. Organisational change
9. Unauthorised trading
10. Model risk

2.3 Regulatory base

As already above mentioned, the official definition of the operational risk is provided by the Basel Committee. The Basel Committee on Banking Supervision (BCBS) is a committee of banking supervisory authorities that was established by the central bank governors of the Group of Ten countries in 1974. It provides a forum for regular cooperation on banking supervisory matters. Its objective is to enhance understanding of key supervisory issues and improve the quality of banking supervision worldwide. The Committee frames guidelines and standards in different areas – some of the better known among them are the international standards on capital adequacy, the Core Principles for Effective Banking Supervision and the Concordat on cross-border banking supervision. The Committee’s Secretariat is located at the Bank for International Settlements (BIS) in Basel, Switzerland.

There are three main generations of standards of BCBS:

1. Basel I: the Basel Capital Accord:
   • was released in 1988,
   • requires a minimum ratio of capital to risk-weighted assets.

2. Basel II: the New Capital Framework:
   • was released in 2004,
   • provides 3 Pillar-model.

3. Basel III:
   • reacts to financial crisis,
   • enhanced capital requirements, liquidity risk management, systemically important banks.

2.4 Three-Pillar Model

Basel II is the second set of recommendations issued by the Basel Committee on Banking Supervision. Published in 2004, the recommendations were intended to improve the international standard for banking regulators set by Basel I. First of all, unlike the original 1988 Basel I recommendations, Basel I does not only focus on credit risk. Other risks such operational risks and market risks are now taken into account. Basel II introduces a three pillars approach:

1. **Pillar I: Minimum capital requirements.**
   Bank have to allocate aside capital for operational risk, that can be calculated using different approaches.
2. Pillar II: Supervisory review process.
This pillar focuses on the supervision of banks’ systems and capital adequacy by regulatory authorities. Basel II second pillar is meant to complement the first pillar. The local regulators are given additional tools to monitor the risk management process of banks.

The goal is to better inform the investors. The idea behind this third pillar is that when market participants have a sufficient understanding of a bank’s activities and the controls it has in place to manage its exposures, they are better able to distinguish between banking organizations so that they can reward those that manage their risks prudently and penalize those that do not.

2.5 OpRisk Capital Estimation Under Basel II
There are three approaches that are suitable to estimate the economic capital for operational risk:

* The Basic Indicator Approach (BIA):

\[
C = \alpha \frac{1}{n} \sum_{j=1}^{3} \max(GI(j), 0), \quad n = \sum_{j=1}^{3} 1_{(GI(j)>0)},
\]

\(GI(j), j = 1, 2, 3 \ldots\) average annual gross income over the previous three years,
\(n \ldots\) the number of years with positive gross income,
\(\alpha = 0.15\).

**BIA** is the most straightforward approach. It is simpler compared to the alternative approaches. Based on the original Basel Accord, banks using the basic indicator approach must hold capital for operational risk equal to the average over the previous three years of a fixed percentage of positive annual gross income. The fixed percentage \(\alpha\) is typically 15% of annual gross income.

* The Standardised Approach (SA):

\[
C = \frac{1}{3} \sum_{j=1}^{3} \max \left( \sum_{i=1}^{8} \beta_i GI_i(j), 0 \right)
\]

\(\beta_i, i = 1, \ldots, 8\) factors for eight business lines (BL), \(\beta_i \in [12\%, 18\%]\)
\(GI_i(j), j = 1, 2, 3\) are the annual gross incomes of the \(i\)-th BL in the previous 3 years.

* The Advanced Measurement Approaches (AMA):

\[
C = \sum_{i=1}^{8} \sum_{k=1}^{7} \gamma_{ik} \epsilon_{ik}
\]

8
In comparison with other approaches, **AMA** is the most sophisticated approach. It allows banks to use their internally generated risk estimates. However, there are some preconditions, which have to be satisfied in order to use AMA:

- Diversification benefits are allowed if dependence modeling is approved by the regulator;
- Bank must meet qualitative and quantitative standards;
- The risk measure used for capital charge should correspond to the 99.9% confidence level for a one-year holding period;
- Capital reduction due to insurance is capped at 20%.

As it was already mentioned, there are 8 business lines and 7 event types, which are considered in the approaches:

**Basel II Business lines (BL):**

<table>
<thead>
<tr>
<th>(i)</th>
<th>Business line, BL((i))</th>
<th>(\beta_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Corporate finance</td>
<td>0.18</td>
</tr>
<tr>
<td>2</td>
<td>Trading and sales</td>
<td>0.18</td>
</tr>
<tr>
<td>3</td>
<td>Retail banking</td>
<td>0.12</td>
</tr>
<tr>
<td>4</td>
<td>Commercial banking</td>
<td>0.15</td>
</tr>
<tr>
<td>5</td>
<td>Payment and Settlement</td>
<td>0.18</td>
</tr>
<tr>
<td>6</td>
<td>Agency Services</td>
<td>0.15</td>
</tr>
<tr>
<td>7</td>
<td>Asset management</td>
<td>0.12</td>
</tr>
<tr>
<td>8</td>
<td>Retail brokerage</td>
<td>0.12</td>
</tr>
</tbody>
</table>

**Basel II event types (ET):**

<table>
<thead>
<tr>
<th>(j)</th>
<th>Event type, ET((j))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Internal fraud</td>
</tr>
<tr>
<td>2</td>
<td>External fraud</td>
</tr>
<tr>
<td>3</td>
<td>Employment practices and workplace safety</td>
</tr>
<tr>
<td>4</td>
<td>Clients, products and business practices</td>
</tr>
<tr>
<td>5</td>
<td>Damage to physical assets</td>
</tr>
<tr>
<td>6</td>
<td>Business disruption and system failures</td>
</tr>
<tr>
<td>7</td>
<td>Execution, delivery and process management</td>
</tr>
</tbody>
</table>
2.6 Operational Risk Data

Basel II specifies the data that should be collected and used for AMA. Briefly, a bank should have internal data, external data and expert opinion data. In addition, internal control indicators and factors affecting the businesses should be used. There are some main features of the required data:

- **Internal data.** Internal data should be collected over a minimum five-year period to be used for capital charge calculations (when the bank starts the AMA, a three-year period is acceptable). Due to a short observation period, typically the internal data for many risk cells contain few low-frequency/high-severity losses or none. A bank must be able to map its historical internal loss data into the relevant Basel II risk cells. The data has to contain all material activities from all sub-systems and geographic locations. A bank can have an appropriate low report boundary for internal loss data collection, typically of the order of EURO 10,000. Besides information on gross loss amounts, a bank should collect information about the date of the event, any recoveries of gross loss amounts, as well as some narrative information about the factors of the loss event.

- **External data.** A bank’s operational risk measurement system must use relevant external data (either public data and/or pooled industry data). These external data should include data on actual loss amounts, information on the scale of business operations where the event occurred, and information on the causes and circumstances of the loss events. Industry data are available through external databases from vendors and consortia of banks. External data are difficult to use directly due to different volumes and other factors. Moreover, the data have a survival bias as typically the data of all collapsed companies are not available.

- **Scenario Analysis/expert opinion.** A bank must use scenario analysis in combination with external data to evaluate its exposure to high-severity events. Scenario analysis is a process conducted by experienced business managers and risk management experts to identify risks, analyze past internal/external events, consider current and planned controls in the banks, etc. It may involve: workshops to identify weaknesses, strengths and other factors; opinions on the severity and frequency of losses; opinions on sample characteristics or distribution parameters of the potential losses. As a result some rough quantitative assessment of the risk frequency and severity distributions can be obtained. Scenario analysis is very subjective and should be combined with the actual loss data. In addition, it should be used for stress testing, for example to assess the impact of potential losses arising from multiple simultaneous loss events.

- **Business environment and internal control factors.** A bank’s methodology must capture key business environment and internal control factors affecting operational risk. These factors should help to make forward-looking estimates, account for the quality of the controls and operating environments, and align capital assessments with risk management objectives.
• *Exposure indicators.* The frequency and severity of operational risk events are influenced by indicators such as gross income, number of transactions, number of staff and asset values. For example, frequency of losses typically increases with increasing number of employees.

• *Near-miss losses.* These are losses that could occur but were prevented. Often these losses are included in internal datasets to estimate severity of losses but excluded in the estimation of frequency.

The last two data types are important for modeling of the risk but often missing in external databases.

### 2.7 Expected and unexpected losses

The Basel II proposal suggested that the capital charge should cover unexpected losses (UL), while expected losses (EL) should be covered by the bank through internal provisions. It is caused by the fact that many activities have regular losses, for example, credit card fraud. The definitive version of the Basel II offered that regulatory capital is calculated as the sum of EL and UL, unless the bank can prove an adequate capture of EL through its internal business practices. For simplification, e consider the capital to be the sum of EL and UL which is the 99.9% Value-at-Risk (VaR). The loss exceeding the 99.9% VaR does not require a capital charge (*catastrophic loss*). On the following illustration we can see the quantities of the EL, UL, VaR.

![Illustration of the expected and unexpected losses at the 99.9% confidence level for a 1-year holding period.](image)

*Figure 2:* Illustration of the expected and unexpected losses at the 99.9% confidence level for a 1-year holding period. \( f(z) \) is the probability density function of the annual loss.
2.8 Difference between regulatory and economical capital

The goal of the regulatory capital is to protect a bank against potential losses. It is the minimum amount imposed by the regulators. Economical capital is the amount that market forces imply for the risk. Regulatory capital is based on the 99.9% confidence level over a 1-year holding period, whereby economic capital is usually higher, for example, 99.95 – 99.97%.

3 Operational Risk Models

There are many models which have been proposed to simulate operational risk under Basel II AMA. There are two main approaches, namely top-down and bottom-up approach.

3.1 Top-down approach

The major feature of this approach is analyzing of data at the macro level, overall bank losses, without specifying of the individual processes or risk types. For example:

- **Multi-factor equity pricing models.** This approach assumes market efficiency, where the current asset price (stock price of the company) reflects all relevant information. Then the stock return process is assumed to be driven by many factors related to the market, credit and other non-operational risks. The residual term of this regression is treated as due to operational risk.

- **Capital asset pricing model.** Here, the asset risk premium is quantified, which is a difference between expected return and risk-free return. Then the contributions from credit and market risks are measured and the operational risk is treated as the residual.

- **Income or expense based models.** These models are based on estimating the historical volatility of income or expense respectively subtracting the contributions from credit and market risks.

- **Risk indicator models.** These models link operational risk and exposure indicators such as gross income, volume of transactions, number of staff, etc. The Basel II BIA and SA are examples of a single indicator and multi-indicator models respectively.

3.2 Bottom-up approach

There are two main bottom-up approaches: process based models and loss distribution approach (LDA) models.

- **Process based models.** This approach includes three important models:
  - Casual networks models. They are usually subjective models. For each bank activity, a tree of events that may lead to operational risk loss is constructed. The probability of each event is specified by an expert.
Multi-factor causal models. They are based on regression of operational risk loss on a number of control factors (explanatory variables) such as number of staff, number of transactions, skill level, etc.

Reliability models. They quantify the probability that a system will operate satisfactorily for a certain period of time. These are the models considered in operational research to study the trustworthiness of system elements. This is relevant to many processes in operational risk, for example, modeling the reliability of transaction processing systems.

- **LDA models.** The LDA model is based on modeling frequency $N$ and severities $X_1, X_2, ...$ of the operational risk events. Afterwards, I would like to concentrate on this model.

### 4 LDA Model

**LDA** is a statistical approach which is commonly used in actuarial sciences to compute total loss distributions. In this section, I will define the mathematical model. We will see different algorithms, which can help us to compute the distributions, and show how to calculate the capital charge based on a $VaR$ measurement of risk.

Under this approach, banks quantify distributions for the frequency and severity of operational losses for each risk cell over a one year time horizon. Banks can use their own risk cell structure but they must be able to map the losses to the relevant Basel II risk cells (eight business lines times seven risk types). Estimation of the capital under the LDA requires evaluation of the compound loss distributions. There are no closed-form solutions for the distributions. That is why numerical evaluation is required. Pavel Shevchenko has written the detailed description of the LDA model in his book *Modelling Operational Risk Using Bayesian Inference*, which I have mostly used to write my seminar work.

#### 4.1 Model Description

The LDA model requires calculation of the distribution for the aggregate (compound) loss $X_1 + ... + X_N$, whereby the frequency $N$ is a discrete random variable. This is the classic risk theory problem. Previously, it was estimated using approximations such as based on the asymptotic central limit theory. Nowadays, the distributions can be calculated using numerical algorithms, such as Monte Carlo method, Panjer recursion or Fourier inversion techniques.

**Model.** The annual loss in a risk cell is modeled by a compound random variable

$$Z = \sum_{i=1}^{N} X_i,$$

where

* $N$ is the number of events(frequency) over one-year modeled as a discrete random variable with probability mass function $p_k = Pr[N = k], k = 0, 1, ...$;
* $X_i, i \geq 1$ are positive severities of the events (loss amounts) modeled as independent and identically distributed function $F(x)$ with $x \geq 0$ and $F(0) = 0$. The corresponding density function is denoted as $f(x)$;

* $N$ and $X_i$ are independent for all $i$, frequencies and severities are independent;

* $H(z)$ is the distribution function of the annual loss $Z$ and $h(z)$ is the density function of $Z$;

* All model parameters (parameter of the frequency and severity distributions) are assumed to be known.

The calculation of the annual loss loss distribution is normally required for the next year, i.e. year $T + 1$. In (4) only one risk cell, namely event type/business line, and one time period is considered. In the following, the model parameters assumed to be known, though, normally they are unknown and can be estimated using past data over the $T$ years. It is also important to note that the finite probability of no loss occurring over the considered period is allowed if $N = 0$, namely, $Pr[Z = 0] = Pr[N = 0]$. Hereafter, only one risk cell and one time period are considered.

### 4.2 Analytic Solution

In general case, there are two main types of the analytic solutions for calculating the compound distribution $H(z)$. These are based on convolutions and method of characteristic functions. These solutions usually cannot be evaluated in a finite number of operations, that is why we need some numerical methods.

#### 4.2.1 Analytic Solution via Convolutions

As we already know from the measure theory, the density and distribution functions of the sum of two independent continuous random variables $Y_1 \sim F_1(.)$ and $Y_2 \sim F_2(.)$ with the densities $f_1(.)$ and $f_2(.)$ respectively, can be calculated via convolution as

$$f_{Y_1+Y_2}(y) = (f_1 * f_2)(y) = \int f_2(y - y_1) f_1(y_1) dy_1$$

as well as

$$F_{Y_1+Y_2}(y) = (F_1 * F_2)(y) = \int F_2(y - y_1) F_1(y_1) dy_1.$$ (6)

Therefore, the distribution of the annual loss (4) can be also calculated via convolutions as

$$H(z) = Pr[Z \leq z] = \sum_{k=0}^{\infty} Pr[Z \leq z|N = k] Pr[N = k] = \sum_{k=0}^{\infty} p_k F^{(k)}(z),$$ (7)

where $F^{(k)}(z) = Pr[X_1 + ... + X_k \leq z]$ is the $k$-th convolution of $F(.)$ that can be recursively calculated as

$$F^{(k)}(z) = \int_0^z F^{(k-1)}(z-x)f(x)dx$$
with the assumption that
\[ F^{(0)*}(z) = \begin{cases} 
1, & z \geq 0, \\
0, & z < 0.
\end{cases} \]

Our integration limits are 0 and \( z \) because the severities have to be non-negative. The direct computation is, in general, not easy as the convolution powers are not available in the closed form. There are several numerical methods, such as Panjer recursion or FFT, which are efficient for the evaluation of these convolution.

### 4.2.2 Analytic Solution via Characteristic Functions

This method is the powerful tool for computing probability distributions, which is commonly used in the insurance, operational risk and credit risk. The frequency-severity compound distributions usually cannot be calculated in a finite number of operations but can be easily expressed through the inverse transform of the characteristic function. The characteristic function of the severity density \( f(x) \) is defined as
\[
\phi(t) = \int_{-\infty}^{\infty} f(x)e^{itx}dx,
\]
where \( i = \sqrt{-1} \) is a unit imaginary number. Also, the probability generating function of a frequency distribution with probability mass function \( p_k = Pr[N = k] \) is
\[
\psi(s) = E[s^k] = \sum_{k=0}^{\infty} s^k p_k.
\]

Then, the characteristic function of the compound loss \( Z \) in (4), denoted by \( \chi(t) \), can be expressed through the probability generating function of the frequency distribution and the characteristic function of the severity distribution as
\[
\chi(t) = \sum_{k=0}^{\infty} (\phi(t))^k p_k = \psi(\phi(t)).
\]

In particular, there are two standard choices of frequency distribution:

1. If frequency \( N \) is distributed from Poisson(\(\lambda\)), then
\[
\chi(t) = \sum_{k=0}^{\infty} (\phi(t))^k \frac{e^{-\lambda} \lambda^k}{k!} = exp(\lambda \phi(t) - \lambda);
\]
The Poisson distribution is popular for modelling the number of times an event occurs in an interval of time or space. The choice of Poisson distribution can be motivated by the fact that the length of time between two events should be exponentially distributed, since the exponential distribution is the unique distribution that is synonymous with "lack of memory". In other words, the fact that no internal fraud was reported last month does not mean that the probability suddenly is greater (or smaller), that an internal fraud will be reported this month. This means that the number of reported events starting from some time \( t_0 \) can be described by a Poisson process, and further that the total number of losses in the interval \([t, t + \tau]\) is Poisson distributed.
2. If \( N \) is from negative binomial distribution \( \text{NegBin}(m, q) \), then

\[
\chi(t) = \sum_{k=0}^{\infty} (\phi(t))^k \binom{k + m - 1}{k} (1 - q)^k q^m = \left( \frac{q}{1 - (1 - q)\phi(t)} \right)^m. \tag{12}
\]

The given characteristic function, the density of the annual loss \( Z \) can be calculated via the inverse Fourier transform as

\[
h(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi(t) \exp(-itz) dt, \quad z \geq 0. \tag{13}
\]

If the severities are nonnegative, for simplicity, we can use the following lemma to calculate the density and distribution functions.

**Lemma.** For a nonnegative random variable \( Z \) with a characteristic function \( \chi(t) \), the density \( h(z) \) and the distribution \( H(z) \) functions, \( z \geq 0 \), are

\[
h(z) = \frac{2}{\pi} \int_0^{\infty} \text{Re}[\chi(t)] \cos(tz) dt, \quad z \geq 0; \tag{14}
\]

\[
H(z) = \frac{2}{\pi} \int_0^{\infty} \text{Re}[\chi(t)] \frac{\sin(tz)}{t} dt, \quad z \geq 0; \tag{15}
\]

### 4.2.3 Value-at-Risk and Expected Shortfall

After we have calculated the compound loss distribution, we can evaluate risk measures, such as VaR and expected shortfall. Analytically, VaR of the compound loss is calculated as the inverse of the compound distribution

\[
\text{VaR}_\alpha[Z] = H^{-1}(\alpha) \tag{16}
\]

and the expected shortfall of the compound loss above the quantile \( q_\alpha = \text{VaR}_\alpha[Z] \), assuming that \( q_\alpha > 0 \), is

\[
\text{ES}_\alpha[Z] = \mathbb{E}[Z|Z \geq q_\alpha] = \frac{1}{1 - H(q_\alpha)} \int_{q_\alpha}^{\infty} z \times h(z) dz = \frac{\mathbb{E}[Z]}{1 - H(q_\alpha)} - \frac{1}{1 - H(q_\alpha)} \int_0^{q_\alpha} z \times h(z) dz,
\]

where \( \mathbb{E}[Z] = \mathbb{E}[N] \mathbb{E}[X_1] \) is the mean of compound loss \( Z \) and \( \text{ES}_\alpha[Z] \) is defined for a given quantile \( q_\alpha \). That is why the quantile \( H^{-1}(\alpha) \) has to be computed first. The above mentioned integral can be estimated via characteristic function

\[
\text{ES}_\alpha[Z] = \frac{1}{1 - H(q_\alpha)} \times \left[ \mathbb{E}[Z] - H(q_\alpha) q_\alpha + \frac{2q_\alpha}{\pi} \int_0^{q_\alpha} \text{Re}[\chi(x/q_\alpha)] \frac{1 - \cos(x)}{x^2} dx \right].
\]
4.3 Numerical Methods for calculation of the compound distribution

As it was already mentioned, it is not easy to evaluate the compound distribution in the closed form because both analytic solutions are not available in the closed form. In this section you may have the overview of the numerical approaches for the calculation of the compound distribution.

4.3.1 Monte Carlo Method

Monte Carlo methods (or Monte Carlo experiments) are a broad class of computational algorithms that rely on repeated random sampling to obtain numerical results. Their essential idea is using randomness to solve problems that might be deterministic in principle. They are often used in mathematical problems and are most useful when it is difficult or impossible to use other approaches. Monte Carlo methods are mainly used in three distinct problem classes: optimization, numerical integration, and generating draws from a probability distribution. That is why it is one of the easiest numerical methods for calculation of the compound loss distribution. This approach has the following logical steps:

1. For \( k = 1, \ldots, K \)
   a) Simulate the annual number of events \( N \) from the frequency distribution.
   b) Simulate independent severities \( X_1, \ldots, X_N \) from the severity distribution.
   c) Calculate \( Z_k = \sum_{i=1}^{N} X_i \).

2. Next \( k \) (i.e. do an increment \( k = k + 1 \) and return to step 1).

4.3.2 Panjer Recursion

The Panjer recursion is an algorithm to compute the probability distribution approximation of a compound random variable \( Z = \sum_{i=1}^{N} X_i \), where both \( N \) and \( X_i \) are random variables. This recursion facilitates the calculation of the compound loss distribution via the convolution, which was already described. It is important to mention, that Panjer recursion is designed for discrete severities, whereby in operational risk the severities are typically continuous. To apply this method we should replace the continuous severities with the discrete. For example, one can round all amounts to the nearest multiple of monetary unit \( \delta \), e.g. to the nearest USD 1000. Define

\[
    f_k = Pr[X_i = k\delta], \quad p_k = Pr[N = k], \quad h_k = Pr[Z = k\delta]
\]

with \( f_0 = 0 \) and \( k = 0,1,\ldots \). Then, the discrete version of (7) is

\[
    h_n = \sum_{k=1}^{n} p_k f_n^{(k)}, \quad n \geq 1,
\]

\[
    h_0 = Pr[Z = 0] = Pr[N = 0] = p_0,
\]
where \( f^{(k)}_n = \sum_{i=0}^{n} f^{(k-1)}_{n-i} f_i \) with \( f^{(0)}_0 = 1 \) and \( f^{(0)}_n = 0 \) if \( n \geq 1 \).

It is important to note that:

- The condition \( f_0 = Pr[X = 0] = 0 \) implies that \( f^{(k)}_n = 0 \) for \( k > n \) and the above summation goes only to \( n \).
- If \( f_0 > 0 \), then \( f^{(k)}_n > 0 \) for all \( n \) and \( k \), and the upper limit in the summation should be replaced by infinity.
- The number of operations to calculate the discrete version is of the order \( n^3 \).

If the frequency \( N \) belongs to the so-called Panjer classes, (18) is reduced to a simple recursion, which is explained in the following theorem, which is called Panjer recursion.

**Theorem 1.** If the frequency probability function \( p_n, n = 0,1,... \) satisfies

\[
p_n = \left( a + \frac{b}{n} \right) p_{n-1}, \quad n \geq 1 \quad \text{and} \quad a, b \in \mathbb{R},
\]

then it is said to be in Panjer class \((a,b,0)\) and the compound distribution satisfies the recursion

\[
h_n = \frac{1}{1 - af_0} \sum_{j=1}^{n} (a + \frac{b j}{n}) f_j h_{n-j}, \quad n \geq 1,
\]

\[
h_0 = \sum_{k=0}^{\infty} (f_0)^k p_k.
\]

(19)

The parameters \((a, b)\) and starting values \( h_0 \) for different types of distributions are listed in the following table:

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( a )</th>
<th>( b )</th>
<th>( h_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poisson(( \lambda ))</td>
<td>0</td>
<td>( \lambda )</td>
<td>( \exp(\lambda(f_0 - 1)) )</td>
</tr>
<tr>
<td>NegBin(( r, q ))</td>
<td>( 1 - q )</td>
<td>( (1 - q)(r - 1) )</td>
<td>( (1 + (1 - f_0)^{\frac{1-q}{q}})^{-r} )</td>
</tr>
<tr>
<td>Bin(( m, q ))</td>
<td>( -\frac{q}{1-q} )</td>
<td>( \frac{m+1}{1-q} )</td>
<td>( (1 + q(f_0 - 1))^{m} )</td>
</tr>
</tbody>
</table>

In the following you can see the implementation on the Panjer recursion algorithm:

1. Initialization: calculate \( f_0 \) and \( h_0 \), using the starting values from the previous table, and set \( H_0 = h_0 \).

2. For \( n = 1, 2, ... \)
   a) Calculate \( f_n \).
   b) Calculate \( h_n = \) with the help of the recursion formula described in (19).
   c) Calculate \( H_n = H_{n-1} + h_n \)
   d) Stop the procedure if \( H_n \) is larger than the required quantile level \( \alpha \), e.g. \( \alpha = 0.999 \). Then the estimate of the quantile \( q_\alpha \) is \( n \times \delta \).

3. Next \( n \) (i.e. do an increment \( n = n + 1 \) and return to step 2).
4.3.3 Fast Fourier Transform

A fast Fourier transform (FFT) is an algorithm that samples a signal over a period of time (or space) and divides it into its frequency components. These components are single sinusoidal oscillations at distinct frequencies each with their own amplitude and phase. An FFT algorithm computes the discrete Fourier transform (DFT) of a sequence, or its inverse (IFFT). Fourier analysis converts a signal from its original domain to a representation in the frequency domain and vice versa. An FFT rapidly computes such transformations by factorizing the DFT matrix into a product of sparse (mostly zero) factors. The basic ideas were popularized in 1965, but some algorithms had been derived as early as 1805. In 1994, Gilbert Strang described the FFT as "the most important numerical algorithm of our lifetime" and it was included in Top 10 Algorithms of 20th Century by the IEEE journal Computing in Science and Engineering.

In our example with the compound distribution the FFT is rather efficient method to estimate the distribution via the inversion of the compound distribution. As with Panjer recursion case, FFT works with discrete severity and based on the discrete Fourier transformation, which is defined as follows:

**Discrete Fourier transformation.** For a sequence \( f_0, f_1, \ldots, f_{M-1} \), the discrete Fourier transformation is defined as

\[
\phi_k = \sum_{m=0}^{M-1} f_m \exp\left(\frac{2\pi i}{M} mk\right), \quad k = 0, 1, \ldots, M - 1
\]  

and the original sequence \( f_k \) can be recovered from \( \phi_k \) by the inverse transformation

\[
f_k = \frac{1}{M} \sum_{m=0}^{M-1} \phi_m \exp\left(-\frac{2\pi i}{M} mk\right), \quad k = 0, 1, \ldots, M - 1.
\]  

Here \( M \) is some truncation point. The number of operations for estimation \( M \) points of \( \phi_m \) is of the order of \( M^2 \). The DFT can be efficiently calculated via FFT algorithms with the number of computations \( O(M \log_2 M) \), if \( M \) is a power of 2.

The logical steps of FFT, where \( M \) is integer power of 2, are as follows:

1. Sort the data in a bit-reversed order. The obtained points are simply one-point transforms.

2. Combine the neighbour points into non-overlapping pairs to get two-point transforms. Then combine two-point transforms into 4-point transforms and continue subsequently until the final \( M \) point transform is obtained. Thus there are \( \log_2 M \) iterations and each iteration involves of the order of \( M \) operations.

What is important for us is the calculation of the compound distribution via FFT, which can be estimated via the following method:

1. Discretise severity to obtain

\[ f_0, f_1, \ldots, f_{M-1}, \]

where \( M = 2^r \) with integer \( r \), and \( M \) is the truncation point in the aggregate distribution.
2. Using FFT, calculate the characteristic function of the severity
\[ \phi_0, \ldots, \phi_{M-1}. \]

3. Calculate the characteristic function of the compound distribution using (10), i.e.
\[ \chi_m = \psi(\phi_m), \quad m = 0, 1, \ldots, M - 1. \]

4. Perform inverse FFT.

4.3.4 Direct Numerical Integration

In the following section you can see another important numerical method for the computation of the compound distribution. It is called the direct numerical integration. The task of the characteristic function inversion is analytically straightforward, but numerically difficult in terms of achieving high accuracy and computational efficiency simultaneously. The computation of compound distribution through the characteristic function involves two steps:

1. Computing the characteristic function (Fourier transform of the density function, referred to as the forward integration). This step requires integration (8), that is, decomposition of the characteristic function for a severity distribution into real and imaginary parts and calculation of both of them:
\[
Re[\phi(t)] = \int_0^\infty f(x) \cos(tx) dx, \quad Im[\phi(t)] = \int_0^\infty f(x) \sin(tx) dx, \tag{22}
\]
Then, the characteristic function of the compound loss can be calculated using (10). The severity density can be normally computed in a finite number of steps, that is why these tasks are relatively simple. This step can be done more or less routinely and many existing methods, including the ones commonly available in many software packages, can be employed. In general, double precision accuracy can be routinely achieved for the forward integrations.

2. Inverting of the characteristic function (referred to as the inverse integration). This task is not as simple as the previous one, but there still exist some algorithms that can help. This step requires integration
\[
H(z) = \frac{2}{\pi} \int_0^\infty Re[\chi(t)] \sin(tz) \frac{dt}{t}, \quad z \geq 0, \tag{23}
\]
which is much more challenging task. If we change the variable \( x = t \times z \), it can be rewritten as
\[
H(z) = \int_0^\infty G(x, z) \sin(x) dx, \quad G(x, z) = \frac{2}{\pi} \frac{Re[\chi(x/z)]}{x},
\]
where \( \chi(t) \) depends on \( Re[\phi(t)] \) and \( Im[\phi(t)] \) calculated from the forward semi-infinite integrations (22) for any required argument \( t \). The total number of forward integrations required by the inversion is usually quite large. This is because in this case the characteristic function could be highly oscillatory due to high frequency and it may decay very slowly due to heavy tails.
5 Conclusion

The purpose of this paper was to give the overview about operational risk, different types of it, main approaches for the calculation of the capital which has to be extra allocated to cover OpRisks. The main goal of this work is to present some algorithms, analytical and numerical, for the estimation of the compound distribution:

1. Analytical solution:
   - via convolutions,
   - via characteristic functions.

2. Numerical solution:
   - Panjer recursion,
   - Monte Carlo Method,
   - Fast Fourier transform (FFT),
   - direct numerical integration.
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