Ethics in Quantitative Finance

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1 Introduction

In November 2008, when the French newspaper Le Monde reported on the possible causes of the crisis, the idea that ‘mathematicians are guilty of crimes against humanity’ was born. From the Financial Crisis Inquiry Commission (FCIC) of the US Congress came another assessment regarding the topic, as, in 2011, they argued that a ‘systemic breakdown in accountability and ethics’ in the time leading up to the crisis was to blame for the chaos that ensued. These comments, among many others, raise questions such as: ‘How are the culturally defined ethics of a time period influencing financial and mathematical developments?’, and more generally, ‘Is there a distinct relationship between finance, mathematics, and ethics all together?’ This seminar paper is going to explore these questions and other concepts related to it.

The paper starts with a historical overview showing the evolution of the concept of ‘money’. It then continues to illustrate how the formulation and solution to mathematical problems throughout history were not found by mere mathematical pondering, but rather by considering ideas of ethics and society. This is shown in the form of a few key examples such as the ‘Problem of Points’, the ‘Petersburg Game’ and lastly the development of the ‘Fundamental Theorem of Asset pricing’.

The foundation and inspiration for this seminar paper is Timothy Johnson’s deeply philosophical work Ethics in Quantitative Finance: A pragmatic financial Market Theory [1], which he released in 2017 and apart from a few external sources, which are highlighted in the text, the contents of this paper are based on this book in particular.

Though the book is full of fascinating topics and interesting anecdotes, the development of probability theory was the part that especially piqued my interest and therefore the sections of the book related to this topic were the most significant in my research.
2 History

To properly examine the relationship between mathematics, finance and ethics, we must first consider how these concepts came to be. Looking into the past, we can see in Figure 1 how the concept of money developed over time in earlier societies.

![Figure 1: Historical Timeline](image)

2.1 The Beginning of 'Money'

The abstract concept of ‘money’ was first introduced in Archaic Greece. The society that emerged valued individual autonomy over totalitarian views which contrasted greatly with the prevalent authoritarianism of competing societies at that time. As a result of this individualistic approach, inequality, as well as instability, arose and to counteract this development, Greek society started formalizing democracy. Meanwhile, the development of abstract mathematics was driven by the abstract nature of money. When combining these developments, a new society was born, which managed to outlive the hierarchical societies of the time. This new society was based on rational thought and its ability to solve existential problems was regarded as the reason for its great success.

This shows that money, mathematics and especially ethics have always been used to highlight and understand relationships. Repeating this process, the monetization of society in medieval and western Europe led to similar democratic and scientific advances between 1100 and 1350 as well as between 1648 and 1789.
2.2 Demolition of Uncertainty

In the nineteenth century, people started to believe that humans could more easily predict the future than earlier generations thought. As a result, the faith in the accountability of the future by experts led to the degradation of free will and consequently, the need to employ individual judgment. For this reason, scarcity became the main focus of both politics and economics while the general public believed the concept of uncertainty to be completely under control of science. Consequently, practices which featured uncertainty, for instance gambling or financial speculation, became illicit. The government took control and as long as the economy was static, this mirroring of hierarchical society seemed to work.

2.3 Post-War Problems

In the decades following the Second World War, the relative decline of Britain, the US and France in comparison to Germany and Japan led to economic and political tensions. What followed was an unpredictable economy, which was worsened by globalization and the rapid technological change of that time. However, the ideology of control that was prevalent at that time kept a tight grip on modern finance as the people had forgotten how to deal with problems in the face of radical uncertainty.

3 The Problem of Points

After illustrating the historical development of the meaning of money and the concept of uncertainty, demonstrating how different ethical aspects of the time influenced the development of solutions to mathematical problems is imperative in understanding the importance which ethics have in this regard. We are starting with the Problem of Points, which relates to the following Situation:

"Two players, F and P, are playing a game based on a sequence of rounds, and each round consists of, for example, the tossing of a fair coin. The winner of the game is the player who is the first to win 7 rounds, and they will win 80 francs. The question posed is: How should the 80 francs be split if the game is forced to end after P had won 5 rounds while F had won 4?" (Johnson, 2017 pp. 149, 150)
3.1 Pacioli’s statistical Approach

In a text by Pacioli, where he employed the abaco tradition of using stories to solve problems in commercial arithmetic, there was a first attempt of solving a version of the Problem of Points. Here, the problem stands for the tied-up capital used in a business partnership and how it should be divided if the venture finished prematurely. Paciolis approach was of statistical nature as he argued that the pot should be split 5:4, mirroring the history of the game.

3.2 Cordano’s Paths

Cardano recognized the absurdity of Pacioli’s approach since it would lead to an unfair result if the game happened to end after only one round out of a hundred or when F, for example, had 99 wins out of a hundred to P’s 90. Cardano came to the conclusion that in order to find the correct solution, not the past should be considered, but the future events that would follow it. In other words, what is important is not the number of rounds each player has won already, but the number of rounds each player still needs to win in order to achieve overall victory. More precisely, the different paths the game would follow should be closely examined.

Yet, despite his insight, Cardano’s solution was still incorrect. The correct solution was provided a century later by Pascal and Fermat in a series of letters to each other.

3.3 Fermat and Pascal’s Solution

Fermat and Pascal realized together, that Cardano’s calculation of P winning, which he stated would be the case if the game followed the path PP, meaning P wins and P wins again, was not sufficient. P winning actually represented four different paths: PPPP, PPPF, PPFP, PPFF. It wasn’t a feature of the game itself that the game ended after PP, but merely the player’s choice. This realization demonstrates the ability mathematicians have of abstracting the problem to get to its fundamental structure, which is independent from any behavioral disentanglements. Driven by this attitude, Pascal and Fermat were the first to develop the mathematical theory of probability. Calculating the winning paths in a universe of all possible outcomes would eventually come down to employing Pascal’s Triangle. Nowadays, mathemati-
cians use the Binomial model to solve problems concerning probability, which is, in itself, formally equivalent to the modern Cox–Ross–Rubenstein formula for pricing options.

In addition, it is important to mention that Pascal’s view of the Problem of Points was partly motivated by his religious background. He wanted the people to live virtuously because that was God’s will. He did not want them to persuade God to give them salvation. This is mirrored in the game, as there technically exists a predestined state, but it finishes prematurely before the ending is known. So, the task of the player is, in a way, to conclude the ending based on the information that is available to them at any given point.

3.4 Huygens’ Version

Christiaan Huygens also addressed the Problem of Points. When he visited Paris in the second half of 1655, he was told about the Problem of points, but apparently not of its solution. Yet, after devoting some time to the problem, he independently deduced a solution. After returning to the Netherlands, he wrote Van Rekeningh in Spelen van Geluk (‘On the Reckoning at Games of Chance’), where he addressed many probability-related matters. He started Van Rekening with an axiom, that reads as follows:

"I take as fundamental for such games that the chance to gain something is worth so much that, if one had it, one could get the same in a fair game, that is a game in which nobody stands to lose.” (Johnson, 2017, p. 152)

His deep emphasis on fairness followed the scholastic approach and Huygens deemed this aspect fundamental to understanding the topic. The result he came up with would today be recognized as the formal definition of mathematical expectation. To come to this conclusion, he started by postulating that the game needed to be fair, so there had to be equality between the cost of the game and the possible payoffs. These variables were all unknown and this is especially remarkable as today, the starting point usually consists of knowing the probabilities and payoffs and the game’s price is what is sought after.
3.5 De Moivre’s Adaptation

Pascal and Fermat had begun the study of probability though they only focused on a few coin tosses. But what if there were many consecutive coin tosses? It would be extremely difficult to manually calculate the result. This was De Moivre’s motivation for transforming this discrete problem into a continuous one. He employed Leibnitz’s ideas and assumed that the number of coin tosses became infinite, thereby identifying a function that could give him the probability of winning a specific proportion of the finite coin tosses. This function became later known as the Normal Distribution Function and its derivative, the Bell Curve. Incidentally, De Moivre considered his result the hardest Problem that could be proposed on the subject of chance and he apologized to his readers about the complexity of his findings when he published his work on the matter in 1733.

In a new edition of the doctrine, published in 1738, he also showed ”that the chance that the difference between an ‘experiment’—testing how many heads appeared from tossing a fair coin” (Johnson, 2017, p. 159) and the theoretical result, which would come down to \(\frac{1}{2}\), was represented by the Normal distribution. This became the foundation of the Central Limit Theorem and offered a way of testing if a scientific result is ‘significant’.

As a result, in 1735, it was generally believed that mathematicians had unlocked the concept of ‘random’ events and that through mathematics, every possible outcome could be explained.

4 The Petersburg Game

”The Petersburg Game is based on tossing a fair coin. The pot starts with 1. If the coin comes up heads, the player wins the pot, if it comes up tails, the pot is doubled and the coin tossed again. The minimum the player can win is 1, but, in theory, the coin could come up with 50 tails before a head, yielding a winning of:

\[
2^{(50-1)} = 562,949,953,421,312
\]

The question Bernoulli and Montmort were interested in was:

What is the ‘fair’ price of the Game, which equated with the expected winnings?”
(Johnson, 2017, p. 161)
4.1 Cramer’s Discreptancy

The problem that Bernoulli and Montmort encountered was that after the ARS, it had been generally agreed upon that the value of a game should be determined by calculating the product of a winning path with its probability and summing up these values for all the winnings possible. This defined the mathematical expectation. So, when closely examining the Petersburg Game, there is a 1 in 2 chance of winning 1 coin, a 1 in 4 chance of winning 2, a 1 in 8 chance of winning 4, and so on. (Bergquist, 2019)

As a result, the mathematical expectation, as you can see in Figure 2, would be an infinite sum of \( \frac{1}{2} \), which is, in itself, infinite. But in reality, nobody would offer more than 20 coins to play the game and most would only stake 4-6 coins. Evidently, the mathematical theory that needed addressing wasn’t what people in the real world were exploring. (Bergquist, 2019)

Yet, in 1728, Gabriel Cramer, who was a Swiss mathematician at the time, offered a solution when he stated that the paradox ‘comes from this; that the mathematicians estimate money in proportion to its quantity, and men of good sense in proportion to the usage that they may make of it’. He was the first one to argue that as the amount of money a person possesses increases, the usefulness of the additional money they earn diminishes. A beggar is happy about every single coin they receive, but a millionaire would probably not even recognize if one coin was missing. This aspect of money is usually associated with Nikolaus’ cousin Daniel, who wrote about...
the utility of money in a paper about insurance which he presented to the Imperial Academy of Sciences of St. Petersburg, thereby giving the problem its famous name.

There were two completely incompatible approaches to the game’s solution, which was a problem for mathematicians at the time, as mathematics claimed to be the ‘art of certain knowledge’. This proved to be a great motivator for resolving the problem, which happened by rational discussions in the salons at that time. Three of its most famous participants ended up greatly impacting the solution: D’Alembert, Buffon and Condorcet.

4.2 D’Alembert’s Scepticism

D’Alembert’s attitude towards science was that it should be objective. Gambling, however, was not and to him, probability in itself was subjective, and therefore he deemed it unscientific. His point was that in reality, perfect symmetry is unattainable, and so even the most basic probability of heads or tails could never be accurately determined. Although he was such a skeptic, D’Alembert did provide some insight into the Petersburg Game during his studies. According to him, a paradox emerged because theoretically, the game is able to continue on forever, resulting in the coin tosses to become infinite. In reality, this would be absurd as at some point in the game, the pot would become so big that there would not be any coins left to add to it. Additionally, if the game continued on for too long, the players involved would eventually die of old age. As a result, he proposed the idea of ending the game after the person doubling the pot was bankrupt, therefore limiting the scale of the winnings.

4.3 Buffon’s practical Approach

The Comte de Buffon further developed D’Alembert’s line of thought in his book Essai d’Arithmetique Moral (‘Essay on Moral Mathematics’), which he released in 1777. Buffon’s approach was quite practical in nature, as he asked a young man, a student of his, to conduct 2048 experiments of the game and to tabulate the results. Surprisingly, the theoretical results he came up with closely matched the experimental findings and the total payout of the game ended up being a little over 10,057. This suggested that the fair price of the game should be around 5, which was close to Nikolaus’ and Montmort’s original observation.
4.4 Condorcet’s Observation

The problem would eventually be resolved by the Marquis de Condorcet, when he published his contribution to the game in 1781. He started by making an important, yet trivial, observation:

"According to Bernoulli, if you play a game where you will win ten francs on the toss of a head and lose ten francs on the toss of a tail, the mathematical expectation is that you will win (lose) nothing. But the reality is that you would win ten, or lose ten, francs." (Johnson, 2017, pp. 163, 164)

After making this observation, Condorcet put more structure to the problem which allowed him to think of probability as a tradeoff between certainty and uncertainty. He deduced that the right value of the game would be arrived at by considering a function on the maximum number of coin-tosses possible.

![Figure 3: New Expected Value](image)

$$E[g(k)] = \sum_{k=1}^{n} 2^{k-1} \left(\frac{1}{2}\right)^{k} = \sum_{k=1}^{n} \frac{1}{2} = \frac{n}{2}$$

So, according to him, if there were a maximum of $2n$ coin tosses, the mathematical expected value of the game would be $\frac{n}{2}$, meaning that if events with a chance of less than 1 in 10,000 were classified as ‘morally impossible’, which would be the case if 14 heads in a row came up, the value of the game would be around 6.5 to 7. (Bergquist, 2019)

His interpretation would explain empirical observations of the game without the use of complicated utility functions.
5 The Law of rare Events

With the publication of his ‘Recherches sur la probabilité des jugements en matière criminelle et en matière civile (‘Research on the Probability of Judgements in Criminal and Civil Matters’), Siméon Denis Poisson, a student of Laplace’s, was one of the most successful people studying the development of ‘probability calculus’ and was regarded as having solved many problems on the matter in 1837.

In his Recherches, he focused on the probability of someone being falsely convicted in a courtroom. The jurors who made the decision each would have their own probability of being wrong after assessing the police’s evidence of the accused’s alleged guilt. This chance of the jurors being wrong has become known as Poisson’s ‘Law of rare Events’.

As criminal judgment only has two possible outcomes, guilt or innocence, Poisson started with Pascal and Fermat’s Binomial model, which was the foundation of the Normal distribution and further developing his ideas he came to the following conclusion:

"Developing these ideas, Poisson worked out that if the rate of a rare event occurring, the number of false judgements, was given by a constant, $\rho$, then the chance of there being $k$ false judgements from $n$ jurors was given by

$$\text{Prob} \left( k \text{ rare events out of } n \right) = \frac{(\rho n)^k}{k!} e^{-\rho n}$$

This defines the Poisson Distriution.

On Application, consider a banker lending a sum of money, $L$. The banker is concerned that the borrower defaults, which should be a rare event. The banker might also assume that if the borrower makes no defaults, they will get all their money back, while if the borrower makes one or more defaults in the loan period, the lender gets nothing back.

Therefore, if the banker assesses that the borrower will default at a rate of $r$ defaults a day, and the loan

$$E[\text{value of loan}] = (\text{Prob} (\text{no defaults}) \times L) + (\text{Prob} (\text{one or more defaults}) \times 0).$$
Using the Law of Rare Events, the probability of 'no defaults', when $k = 0$, is given by

$$\text{Prob}(0 \text{ rare events in } n \text{ days}) = \frac{(rn)^0}{0!}e^{-rT} = e^{-rT}$$

meaning that

$$E[\text{value of loan}] = Le^{-rT}.$$ 

The banker is handing over $L$ with the expectation of only getting $Le^{-rT} < L$ back. To make the initial loan amount equal to the expected repayment, the banker needs to inflate the expected repayment by $e^{rT}$ so that

$$E[Le^{rt}] = (Le^{rt})Le^{-rt} = L$$

This justifies the payment of interest at a rate of $r$ per day. The banker is not charging for the use of money — they are not solving the growth equation — they are equating what they lend with what they expect to get back, just as the Scholastics had argued in the thirteenth century.” (Johnson, 2017, pp. 195, 196)

6 The Black-Scholes-Merton Model

"The Black Scholes Model is one of the most important concepts in modern financial theory. The Black Scholes Model is considered the standard model for valuing options. A model of price variation over time of financial instruments such as stocks that can, among other things, be used to determine the price of a European call option.” (Harvey, 2018)

"The idea of the Black-Scholes Model was first published in 'The Pricing of Options and Corporate Liabilities’ of the Journal of Political Economy by Fischer Black and Myron Scholes and then elaborated in 'Theory of Rational Option Pricing’ by Robert Merton in 1973.” (Harvey, 2018)

The Option Pricing Model didn’t appear overnight. In fact, creating the model was a big process, which involved a few people. It went as follows:
6.1 Black’s Differential Equation

Black and Scholes met at the beginning of the 1970ies when they worked closely together at the University of Chicago. Here, after working with him in a consultancy, Black started applying Treynor’s Capital Asset Pricing Model to different assets, including warrants, which are stock options issued by companies and acted as an enhancement to the bonds the companies already issued. Black tackled the problem as any applied mathematician would, he started by considering the price of the warrant as a function. This function depended on input parameters such as the company’s stock price and the time to its maturity. From there he derived a differential equation that would explain its evolution. Unfortunately, Black didn’t manage to solve this differential equation.

6.2 Scholes’ Hedging Portfolio

In the meantime, Scholes also thought about the pricing of warrants but approached it from a completely different angle to Black. Having studied assets before, Scholes concluded that ‘if two assets behave the same, they should cost the same’. Based on this assumption he came up with the idea of constructing a portfolio by selling a single warrant and additionally buying as many stocks as needed to fulfill the condition of a zero-beta value, meaning that its value did not change as the market changed. If this was the case, the value of the warrant must be the same as the stocks’ value in the portfolio. Scholes’ approach was quite unusual at the time, as he opted for a portfolio which offered no returns, at no risk, while most of his colleagues were attempting to create portfolios that offered the highest returns possible. Nonetheless, Scholes’ far-sighted approach seemed promising as it was founded on the ‘law of one price’, the principle of ‘no arbitrage’. But, similarly to Black, Scholes did not manage to solve his problem as he did not know how he should construct the balancing hedging portfolio.

It was only when Scholes told Black of his problem and Black retorted by telling Scholes about his unsolvable differential equation that the two of them realized that combining their ideas would lead to solutions to both of their problems. They concluded together, that the so-called hedge ratio, which gave the exact ‘number of stocks that would be needed to mirror the change in the value of a warrants’ price’, was described by the slope of the curve that depicted the relationship between the stock price and the warrant price.
6.3 Merton’s Addition

When Merton was introduced to their work, he quickly realized how he could incorporate the Black–Scholes approach of hedging into his own projects, namely his continuous-time models. Based on this revelation he got to work and showed that an option, which is the name for a portfolio made up of only one single warrant, positioned in the asset underlying the warrant as well as money lent or borrowed from the bank could offer the same certain return as merely depositing money in a bank would.

6.4 The Outcome

Black and Scholes showed in their paper, ‘The Pricing of Options and Corporate Liabilities’ that the value of a European-style option could be arrived at by considering a function dependent on five parameters: “the current stock price; the time until maturity; the ‘strike’, the fixed price at which the asset is to be bought (or sold) at maturity; the interest rate for borrowing and ending; and the ‘volatility’, which captured the distribution that the asset price was expected to have at maturity.” (Johnson, 2017, p. 227)

This was a breakthrough because for the first time the function used to determine the price was deterministic. It meant that, apart from volatility, which could be estimated based on historical data, all the other parameters were known. This was a big revelation as the value of the option could now be determined with certainty and thus the notion, that option-trading was gambling, was cleared up for good. As a result, legal objections to derivatives ceased to exist.

“The Black–Scholes–Merton model (BSM) enabled a synthesis of investment and speculation, justifying the belief that financiers had the power to control the markets.” (Johnson, 2017, p. 227)
7 Cox-Ross-Rubenstein-Model

As the continuous-time approach of the BSM required advanced knowledge of mathematics, which for most students and academics in business schools, was beyond their capabilities, John Cox, Stephen Ross, and Mark Rubinstein published a paper in 1979 that would simplify the argument made in the model. In their paper, they employed Pascal and Fermat’s discrete-time model as well as the martingale property of Fama and Mandelbrot and confirmed that their simplified version of the model could give the same results as the BSM when passing from discrete to continuous time.

They named their model the Cox–Ross–Rubenstein model (CRR). This model would give great insight into the methodology of option pricing: “First, the price of an option was governed by the distribution of the future asset prices; when BSM chose the process that resulted in log-normally distributed prices, they chose the option’s price. It was an act of faith that asset prices did have this property. Also, it was recognised that without knowing this distribution, the market was ‘incomplete’.” (Johnson, 2017, p. 228)

Gérard Debreu and Kenneth Arrow were the first ones to introduce the concept of incompleteness in the 1950s, by formalizing economics into advanced mathematics, that was also quite abstract. They started with a simple market model that only consisted of ‘now’ and an uncertain future which was represented by a finite number of states. So, if the number of unique assets would match the number of different states, by Cramer’s rule – often used in mathematics for solving differential equations with as many variables as equations – a unique pricing vector exists that can foresee the price of each additional asset. If, however, the number of assets would be smaller than the number of different states, the market is called ‘incomplete’. As a result, the pricing vector would not be unique, making mechanically pricing assets impossible. So, practical reasoning, as well as subjective judgement would be needed for pricing the assets.

Additionally, an ‘arbitrage-free’ market could be arrived at by postulating that all the elements of the pricing vector should be positive, meaning that each asset costs actual money.
8 The Fundamental Theorem of Asset Pricing

"While Cox, Ross and Rubinstein were developing an interpretation of BSM that was easily understandable, Michael Harrison, David Kreps and Stanley Pliska, between 1979 and 1983, used mathematics that went far beyond Merton’s ‘rocket science’, to distil out the essence of asset pricing from all the different strands of financial research. The result of this mathematical endeavour is two statements, the Fundamental Theorem of Asset Pricing.” (Johnson, 2017, p. 228)

"The Fundamental Theorem of Asset Pricing (FTAP) is a mathematical theorem that integrates modern financial economics into a paradigm but does not make any claims as to the value of assets; it does not deliver a number. The use of the phrase ‘if and only if’ in ‘A market admits no arbitrage if and only if the market has a martingale measure’ establishes a relationship. It does not correspond to a matter of fact in the same way that BSM claims to correspond to the price of an option, on the basis of the material, dynamic hedging process. The FTAP is called the ‘fundamental theorem’ because it analyzed existing asset pricing models used in practice and distilled out the essential axioms. These axioms are then used to synthesize new asset pricing models.” (Johnson, 2017, p. 251)

It says:

1. "A market admits no arbitrage if and only if the market has a martingale measure."

2. "The market is complete if and only if the martingale measure is unique.” (Johnson, 2017, p. 228)

8.1 Statement No. 1

"The first statement of the FTAP synthesises two concepts: the idea of arbitrage that goes back to Fibonacci and the idea of a martingale from post Kolmogorov probability. An explanation for this synthesis comes from considering the simplest application of the FTAP to a single-period market based on a single, risky asset. The initial price of the asset, $X_0$ can take on one of two values, $X_0^D < X_0^U$, in the future at time $T > 0$. If $X_0 \leq X_T^D < X_T^U$, then buying the asset at a price that is less than or equal to either of the future payoffs would lead to a possible profit with the guarantee of no loss: an arbitrage.
Similarly, if \( X_0 \geq X_T^U > X_T^D \), short-selling the asset now and buying it back at time \( T \) would also lead to an arbitrage. For there to be no arbitrage opportunities, the following relationship must hold:

\[
X_T^D < X_0 < X_T^U
\]

This implies that there is a number, \( 0 < q < 1 \), that stands for where \( X_0 \) lies between \( X_T^D \) and \( X_T^U \), or

\[
X_0 = X_T^D + q(X_T^U - X_T^D) = (1 - q)X_T^D + qX_T^U.
\]

Using Kolmogorov’s definition of probability as an abstract measure, \( q \) is the probability of the asset price ending up as \( X_T^U \). This can be written as

\[
X_0 = E_Q[X_T]
\]

and is interpreted as the asset price being its expected future value - a martingale with respect to a special probability measure, the ‘martingale measure’, so named because it makes prices martingales.

This connection between the ratio, \( q \), and a probability only comes about if the initial price of the asset lies between its maximum and minimum possible (real) future price: there is no arbitrage. If \( q = 1 \), (respectively, \( q = 0 \)), it implies that the price \( X_T^U \), (respectively, \( X_T^D \)), will occur almost surely and that \( X_0 = X_T^U \), (respectively, \( X_0 = X_T^D \)).

However, this contradicts the assumption that either price will occur.

If \( X_0 \leq X_T^D < X_T^U \), we have that \( q < 0 \), whereas if \( X_0 \geq X_T^D > X_T^U \), then \( q > 1 \). In either case, \( q \) does not stand for a probability, which must lie between 0 and 1.

Arbitrage opportunities are precluded only if the ratio is a probability, and similarly, if \( q \) is a probability, then there are no arbitrage opportunities. This is the essence of the first statement of the FTAP.” (Johnson, 2017, pp. 252, 253)
8.2 Statement No. 2

"In an idealised market, the values $X_0$, $X_D^T$ and $X_U^T$ are known with certainty and the value of $q$ is unique. In more realistic situations, where the future price distribution is unknown or there are transaction costs, a unique martingale measure cannot be found. This ‘incompleteness’ does not imply that there is no true price for an asset, only that traders do not have a Spinozian viewpoint that makes the world deterministic. In these circumstances, an agent should use their judgement, practica, and choose a martingale measure. Having made the choice, they should be consistent in using the same measure in pricing all assets.

Another trader might choose a different measure - such that the two do not agree on a price - making a market. Kolmogorov’s abstract formulation of probability, disconnected from counting outcomes, historical prices or subjective sentiment, reveals that embedded in the FTAP is the old association of the concept of ‘no arbitrage’ with fairness and reciprocity, which is essential in defining the price.” 

(Johnson, 2017, pp. 254, 255)

In the second statement of the FTAP, the importance of incomplete markets is mentioned. It essentially says, that in our practical world of transaction costs and imperfect knowledge, a precise price cannot be arrived at by employing the FTAP. Only by ignoring these transaction costs, or ‘frictions’, as they are sometimes called, the models presented by Black and Scholes, Merton, and Cox, Ross, and Rubinstein could deliver unique prices and perfect information about the future distribution of the asset prices.

8.3 Importance of the FTAP

What makes the FTAP so important, is its unification of different themes in financial economics. It brought together many concepts of abstract mathematics, such as stochastic calculus, martingales - a mathematical concept employed in the development of the ‘Efficient Markets Hypothesis’, and Arrow and Debreu’s concept of incomplete markets.

Precisely, it merged Black and Scholes financial practice-rooted approach with Merton’s ‘top-down’ approach, which is based on the abstract mathematics of the Wiener process.
“The FTAP synthesized the two approaches by identifying the ‘market price of risk’, or ‘Sharpe ratio’, as featuring in the Radon-Nikodym derivative, which governs how the martingale measure is related to observed asset prices.” (Johnson, 2017, pp. 230, 231)

The FTAP is truly a synthesis of a ‘constellation of beliefs, values, techniques’ and it represented a new direction for financial economics. The plurality of techniques that were merged in this so-called ‘theory of everything’ was a breakthrough for finance and mathematics alike.

8.4 The Concept of Reciprocity

The definition of the term reciprocity is quite hazy, but in the FTAP it says that “the objective truth in a radically uncertain market is that there should be equality between what is given and what is received, of reciprocity.” (Johnson, 2017, p. 261)

What is so significant about reciprocity is that it is needed for social cohesion. Aristotle once said, it ‘keeps the city together’, meaning it enabled cooperation. In the financial world, where competition is usually the ideology people follow, this seems counterintuitive, but the pragmatic explanation is that the market’s purpose is to enable market-makers to find an agreement on the price of an asset, which reciprocity allows them to do.

“The presence of the norm reciprocity in the FTAP shows that market prices should be founded on practical judgement and so need to conform to the subjective criteria of truthfulness and the social criterium of rightness.” (Johnson, 2017, p. 262)
9 Conclusion

Lastly, it is important to mention that every time mathematical models are applied to real-life problems, which are usually ripe with uncertainty, they can merely show the current state of the world, they cannot represent the future. Models act as mere approximations of complex systems to help people understand the bigger problem and to come to a consensus.

The tension that surfaces whenever the concept of money comes into play, depends greatly on the (un)certainty of the financial future, as on one hand, money can be used to enable individuality, but on the other hand, it can also act as an instrument of control. If the future should be predictable, then calculating the best distribution of resources by an institution with foresight would be the biggest priority of finance and economics. Conversely, if we as a society believe that the future is unpredictable, then finance is concerned with establishing a consensus, therefore its statements – the so-called prices – have to be justified in the form of not only objective but also subjective and social criteria.

What I wanted to show is that quantitative finance has always been created as a reflection of the society of people at their point in time. Their religious beliefs, culturally motivated stances on politics and philosophy as well as scientific knowledge that was available at the time greatly influenced their views on financial matters and shaped the way they tackled mathematical problems, as their theses directly reflect the environment they lived in.

This shows, that the concept of ethics, mathematics, and finance are all heavily linked to each other and neither of them could successfully exist without the other.
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2. Expected Value: available at http://httprover2.blogspot.com/2013/12/the-st-petersburg-paradox.html (accessed 25.02.19) . . . . . . . . . . . . . 8

References

