Institute of Financial and Actuarial Mathematics
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Seminar paper

# From Value-at-Risk to Expected Shortfall

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# 1 Introduction

Risk measures serve several different purposes. Generally speaking, a risk measure links the loss $L$ of a financial position with a real number, which in turn, measures the riskiness of $L$. In practice, risk measures determine the capital requirement a financial institution needs as reserve to cover unexpected future losses on its portfolio. In addition, risk measures are used as a risk limiting tool in business units within a company. However, the best measure for risk calculations has not yet been clearly defined in risk management literature. Over the years a variety of risk measures has been proposed. They range from simple measures such as the standard deviation or the quantile measure giving Value-at-Risk (VaR) to complex ones as Expected Shortfall (ES) or Expectiles. The existence of an absolute risk measure is contradicted by the fact that risk management for the insurance industry and the investment banking industry differs. In the insurance industry risk measurement calculations depend on the central limit theorem and large homogeneous portfolios. Meaning that historical performance information is crucial. On top of that, different portfolios can not include the same insurance risk. In investment banking, however, the central limit theorem is not usable, because of investment banking's heterogeneity, which often includes nonlinear portfolio positions. Furthermore, the same entity may be subject to the same risk in several portfolios at the same time.[1] This paper compares and contrasts VaR vs. ES. It analyzes the application of VaR and ES in practice as regulatory risk measures, specifically within the Basel III framework.

[1] See Tunaru [8].

# 2 Loss distribution

Most modern portfolio risk measures are statistical quantities describing the conditional or unconditional loss distribution of the portfolio over some predetermined horizon $\Delta t$. Examples include Value-at-Risk and Expected Shortfall. When working with loss distributions we need to keep in mind two issues. First, any estimate of the loss distribution derives from past data. Any change in the laws governing financial markets curtails the usefulness of these past data to predict future risks. Second, especially when large portfolios are concerned even a stationary environment makes it tricky to evaluate an accurate loss distribution. Therefore, continuous improvements are essential in loss distributions estimations. Furthermore, risk-management models based on estimated loss distributions must be used prudently when applied. This means, that risk measures relaying on the loss distribution need additional information from hypothetical scenarios. Section 2 is based on paragraph 2.2.3 McNeil et al. [7].

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, which is the domain of all random variables we introduce below. We consider a given risk-management time horizon $\Delta t$, which might be one day, ten days or one year. We will make two simplifying assumptions:

- the portfolio composition remains fixed over the time horizon;
- there are no intermediate payments of income during the time period.

Let $V_{t+1}$ be the value of the portfolio at the end of the time period and $\Delta V_{t+1} = V_{t+1} - V_t$ the change of value of the portfolio. The *loss* is defined as $L_{t+1} := -\Delta V_{t+1}$. The distribution of $L_{t+1}$ is termed the *loss distribution*. Whereas, the distribution of $V_{t+1} - V_t$ is called *profit-and-loss* (P&L) *distribution*.
The value $V_t$ is typically modelled as a function of time and a $d$-dimensional random vector $Z_t = (Z_{t,1}, \dots, Z_{t,d})'$ of risk factors, i.e. we have the representation:

$$V_t = f(t, Z_t) \tag{2.1}$$

for some measurable function $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$. Under the assumption that risk factors are observable, the random vector $Z_t$ takes specific realized value $z_t$ at time $t$ and the portfolio value $V_t$ has realized value $f(t, z_t)$. A portfolio value as represented in (2.1) is called a *mapping* of risks.
The *risk-factor changes* over a period (time horizon) is represented as:

$$X_{t+1} := Z_{t+1} - Z_t.$$

In case the current time is given as t and employing the mapping (2.1), the portfolio loss is given by:

$$\begin{aligned} L_{t+1} &= -\left(V_{t+1} - V_t\right) \\ &= -\left(f(t+1, Z_{t+1}) - f(t, Z_t)\right) \\ &= -\left(f(t+1, z_t + X_{t+1}) - f(t, z_t)\right), \end{aligned} \tag{2.2}$$

which shows that the loss distribution is determined by the distribution of the risk-factor change $X_{t+1}$.
If $f$ is differentiable, the first-order approximation $L^{\Delta}_{t+1}$ of the loss in 2.2 is:

$$L^{\Delta}_{t+1} := -\left(f_t(t, z_t) + \sum_{i=1}^{d} f_{z_i}(t, z_t) X_{t+1,i}\right). \tag{2.3}$$

The first-order approximation is convenient as it allows to represent the loss as a linear function of the risk-factor changes. The quality of the approximation 2.3 is obviously best if the risk-factor changes are likely to be small (i.e. measuring risk over a short horizon) and if the portfolio value is almost linear in the risk factors (i.e. if the function $f$ has small second derivatives).

Having mapped the risk of a portfolio, one considers how to derive loss distribution with a view to using them in risk-managemnet applications such as capital setting. Assuming the current time is $t$, the loss over the time period $[t, t+1]$ is:

$$L_{t+1} = -\Delta V_{t+1} = -(f(t+1, z_t + X_{t+1}) - f_(t, z_t)),$$

in order to specify the loss distribution $L_{t+1}$ we need to:

- specify a model for the risk-factor changes $X_{t+1}$;
- determine the distribution of the random value $f(t+1, z_t + X_{t+1})$.

Three possible methods are available for tackling these issues: analytical method, a method based on the idea of historical simulation and a simulation approach (Monte Carlo method). For the further details see Section 2.2.3 McNeil et al. [7].

# 3 Value-at-Risk

In financial and insurance institutions Value-at-Risk is probably the most used risk measure. The Basel regulatory framework uses VaR widely and in Solvency II it has an influential role.
Consider a portfolio of risky assets and a fixed time horizon $\Delta t$, and denote by $F_L(l) = P(L \leq l)$ the distribution function of the corresponding loss distribution. In order to define a statistic based on $F_L$ that measures the severity of the risk of holding the portfolio over the time period $\Delta t$ ,the maximum possible loss, given by $\inf\{l \in \mathbb{R} : F_l = 1\}$, may be used. However, most distributions of interest have an infinite loss. Therefore, the idea in the definition of VaR is to replace "maximum loss" by "maximum loss that is not exceeded with a given probability"[2].

[2]See McNeil et al. [7], p.64.

**Definition 3.1 (Value-at-Risk).** *Given some confidence level $\alpha \in (0,1)$, VaR of a portfolio with loss $L$ at the confidence level $\alpha$ is given by the smallest number $l$ such that the probability that the loss $L$ exceeds $l$ is no larger than $1-\alpha$. In probabilistic terms,*

$$VaR_\alpha = VaR_\alpha(L) = inf\{l \in \mathbb{R} : \mathcal{P}(L > l) \leq 1-\alpha\} = inf\{l \in \mathbb{R} : F_L(l) \leq \alpha\}. \quad (3.1)$$

VaR is therefore a quantile of the loss distribution.

**Example 3.1 (Value-at-Risk for normal loss distribution).** In the case if the underlying loss $L$ is normally distributed $L \sim \mathcal{N}(\mu, \sigma^2)$ the $VaR_\alpha$ for $\alpha \in (0,1)$ can be easily calculated as:

$$VaR_\alpha(L) = \mu + \sigma\Phi^{-1}(\alpha), \quad (3.2)$$

where $\Phi$ is the standard normal distribution function. According to the definition of the generalized inverse and since the loss distribution function $F_L$ is strictly increasing, it suffices to show that:

$$F_L(\mu + \sigma\Phi^{-1}(\alpha)) = \alpha;$$

It holds that,

$$\mathcal{P}(L \leq \mu + \sigma\Phi^{-1}(\alpha)) = \mathcal{P}(\frac{L-\mu}{\sigma} \leq \Phi^{-1}(\alpha)) = \Phi(\Phi^{-1}) = \alpha.$$

## 3.1 Limitations of Value-at-Risk

When using VaR, the level of confidence as well as the holding period are the parameters of choice. These arbitrary parameters, however, depend on context. For instance, when employing VaR to define capital requirements a high confidence interval is preferred. When working with the assumption that a portfolio will not change over the holding period the latter needs to be short. This implies, that VaR estimates can be subject to error and VaR systems can be subject to model risk (i.e., the risk of errors arising from models being based on incorrect assumptions) or implementation risk (i.e., the risk of errors arising from the way in which systems are implemented). However, all risk measurement systems have similar kinds of problems, not only VaR. Nevertheless VaR does have its unique drawbacks. One of these is that VaR only indicates the maximum loss if a tail event does not occur, in other words it tells the most one can lose 95% of the time, but tells nothing about what one can lose on the remaining 5% of occasions. If a tail event does occur, one can expect to lose more than the VaR value, but the VaR figure itself gives no indication of how much that might be.[3] This means, VaR does not alert to any worst case scenarios. This definitely shows, that VaR has limitations as a risk measuring tool.

[3]See Dowd [4] ,p.31.

# 4 Coherent risk measures

With X and Y being the future values of two risky positions, we can call a risk measure $\rho(\cdot)$ to be *coherent* if it satisfies the following axioms[4]:

(i) *Monotonicity*: $Y \geq X \Rightarrow \rho(Y) \leq \rho(X)$.

(ii) *Sub-additivity*: $\rho(X+Y) \leq \rho(X) + \rho(Y)$.

(iii) *Positive homogeneity*: $\rho(hX) = h\rho(X)$ for $h > 0$.

(iv) *Translational invariance*:$\rho(X+n) = \rho(X) - n$ for some certain amount $n$.

Monotonicity (i) implies that for two random values $Y$ and $X$ (e.g. cash flows),the random value that exceeds the other, e.g. $Y \geq X$, has lower risk $\rho(Y) \leq \rho(X)$. Positive homogeneity (iii) requires that the risk of a position is directly proportional to its scale. Translational invariance (iv) means that by adding an amount $n$ to a position the risk of the initial position will decline. According to Dowd [4], the most important property is (ii), sub-additivity. This means that the risk of a portfolio consisting of sub-portfolios will not exceed the sum of the risks of the individual sub-portfolios. Therefore, the sub-additivity of a risk measure ensures the diversification principle.
In practice, regulators who establish capital requirement using non-subadditive risk measures might induce a firm to split into smaller units in order to reduce its regulatory capital requirements, because the total of the capital requirements of these sub-units would be smaller than the capital requirement calculated for the whole firm.

## 4.1 Sub-additivity

The following counter-example proofs that VaR violates the property

$$\rho(X+Y) \leq \rho(X) + \rho(Y).$$

**Example 4.1 (VaR is not sub-additiv).** Let $X$ and $Y$ be two identical independent bonds. Each defaults with probability 4% and results a loss of 100 in this case and a loss of 0 if default does not occur. For the bond X it means:

$$L_X = \begin{cases} 0, & \text{with probability } 0.96; \\ 100, & \text{with probability } 0.04. \end{cases}$$

The same holds for $L_Y$. The 95% VaR of each bond is therefore:

$$VaR_{0.95}(X) = \inf\{l \in \mathbb{R} : \mathcal{P}(X > l) \leq 0.05)\} = 0.$$

Since for $l > 0$ it holds $\mathcal{P}(X > l) = 0.04 \leq 0.05$. The same way $VaR_{0.95}(Y) = 0$. Therefore, $VaR_{0.95}(X) + VaR_{0.95}(Y) = 0$.
For the sum of the bonds $X+Y$ we obtain the following loss:

$$L_{X+Y} = \begin{cases} 0, & \text{with probability } 0.96^2 = 0.9216; \\ 100, & \text{with probability } 2 \cdot 0.04 \cdot 0.96 = 0.0768; \\ 200, & \text{with probability } 0.04^2 = 0.0016. \end{cases}$$

[4]See Dowd [4].

The 95%VaR of $X + Y$ is therefore:

$$VaR_{0.95}(X + Y) = \inf\{l \in \mathbb{R} : \mathcal{P}(X + Y > l) \leq 0.05)\} = 100.$$

Thus, $VaR_{0.95}(X + Y) = 100 > 0 = VaR_{0.95}(X) + VaR_{0.95}(Y)$.

There is ongoing debate about the practical relevance of the non-subadditivity of VaR. McNeil et. al [7] pointed out that the non-subadditivity can be particularly problematic if VaR is used to set risk limits for traders, as this can lead to portfolios with a high degree of concentration risk. Tunaru [8] proved that if the loss distribution of two assets is Pareto(1,1) then VaR is superadditiv.

**Proposition 4.1 (VaR is superadditiv).** *Consider the risk of two assets X and Y that are independent and identically distributed with a Pareto(1,1) distribution.Then for any $\alpha \in (0, 1)$:*

$$VaR_\alpha(x) + VaR_\alpha(Y) < VaR_\alpha(X + Y).$$

**Proof.** For the Pareto distribution $F_X(x) = 1 - \frac{1}{x}$ and therefore:

$$VaR_\alpha(X) = inf\{l \in \mathbb{R} : F_L(l) \leq \alpha\} = inf\{l \in \mathbb{R} : 1 - \frac{1}{l} \leq \alpha\} = \frac{1}{1 - \alpha}; \tag{4.1}$$

Distribution function of $X + Y$ is:

$$F_{X+Y}(u) = \mathbb{P}(X + Y \leq u) = \int_{-\infty}^{\infty} f_Y(y) F_X(u - y) dy; \tag{4.2}$$

Employing (4.2) it holds that:

$$\mathbb{P}(X + Y \leq u) = 1 - \frac{2}{u} - 2\frac{log(u - 1)}{u^2}, \text{ for } u > 2;$$

Therefore, using the fact that $VaR_\alpha(X) = \frac{1}{1-\alpha}$, for any $\alpha$:

$$\mathbb{P}(X + Y \leq 2VaR_\alpha(X)) = \alpha - \frac{(1 - \alpha)^2}{2} log(\frac{1 + \alpha}{1 - \alpha}) < \alpha;$$

Since $X, Y$ are identically distributed this shows that:

$$VaR_{0.95}(X + Y) = 100 > 0 = VaR_{0.95}(X) + VaR_{0.95}(Y);$$

or in other words that VaR is superadditiv at any critical level $\alpha$.

□

# 5 Expected Shortfall

In 2001 Expected Shortfall (ES) was proposed as an alternative risk measure to Value-at-Risk. ES is germane to VaR with regards to worst case scenarios. However, no definite conclusion about which of the two is superior has been arrived at by the risk measurement community. Section 5 is based on paragraph 2.3.4 McNeil et al. [7].

**Definition 5.1 (Expected Shortfall).** Let $L$ be the loss with the density function $F_L$, satisfying the condition $E(|L|) < \infty$. The *Expected Shortfall* at confidence level $\alpha \in (0,1)$ is

$$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 q_u(F_L)du, \tag{5.1}$$

where $q_u(F_L) = F_L^{-1}(u)$ is the quantile function (or the generalized inverse) of $F_L$ which is defined as:

$$F_L^{-1}(\alpha) = inf\{x \in \mathbb{R} : F_L(x) \geq \alpha\}.$$

The condition $E(|L|) < \infty$ ensures that the integral in (5.1) is well defined. By definition, ES is related to VaR by,

$$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(L)du, \tag{5.2}$$

As we can see from the definition of ES, it averages the value of VaR over all levels $u \geq \alpha$ instead of fixing it at a particular one. This allows ES to take into account the events occurred in the "tail", which is not possible when using VaR. An obvious observation is:

$$ES_\alpha \geq VaR_\alpha. \tag{5.3}$$

**Example 5.1 (Expected Shortfall for normal loss distribution).** Here we provide the closed-form expression for ES if the underlying loss distribution is normal $L \sim \mathcal{N}(\mu, \sigma^2)$. $ES_\alpha$ for $\alpha \in (0,1)$ is then

$$ES_\alpha(L) = \mu + \sigma\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}, \tag{5.4}$$

where $\phi$ is the density function of the standard normal distribution and $\Phi$ is the corresponding distribution function. To show the equality (5.4) we recall from the Example 3.1 that

$$VaR_\alpha(L) = \mu + \sigma\Phi^{-1}(\alpha).$$

From the definition of $ES_\alpha$ it follows:

$$\begin{aligned} ES_\alpha(L) &= \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(L)du = \frac{1}{1-\alpha}\int_\alpha^1 \mu + \sigma\Phi^{-1}(u)du \\ &= \mu + \sigma\frac{1}{1-\alpha}\int_\alpha^1 \Phi^{-1}(u)du = \mu + \sigma\frac{ES_\alpha(Z)}{1-\alpha}, \end{aligned}$$

with $Z := (L-\mu)/\sigma \sim \mathcal{N}(0,1)$. Substituting $u = \Phi(z)$ and since $\phi^{'}(z) = -z\phi(z)$ we obtain:

$$ES_\alpha(Z) = \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^\infty z\phi(z)dz = \frac{1}{1-\alpha}[-\phi(z)]_{\Phi^{-1}(\alpha)}^\infty = \frac{1}{1-\alpha}\phi(\Phi^{-1}(\alpha)).$$

The results from Examples 3.1 and 5.1 are illustrated in Table 1, showing ES and VaR values for the standard normal loss distribution at different confidence levels.

Table 1 – VaR and ES for standard normal loss distribution.

| $\alpha$ | pdf | $VaR_\alpha$ | $ES_\alpha$ |
|---|---|---|---|
| 0.50 | 0.3989423 | 0 | 0.7978846 |
| 0.90 | 0.1754983 | 1.281552 | 1.754983 |
| 0.95 | 0.103156 | 1.644854 | 2.062713 |
| 0.99 | 0.02665214 | 2.326348 | 2.665214 |

Table 1 demonstrates that the shorter the right "tail", is the closer the values of VaR and ES are to each other. The Figure 1 illustrates the ES value and its relationship to VaR at 95% confidence level for the standard normal loss distribution.

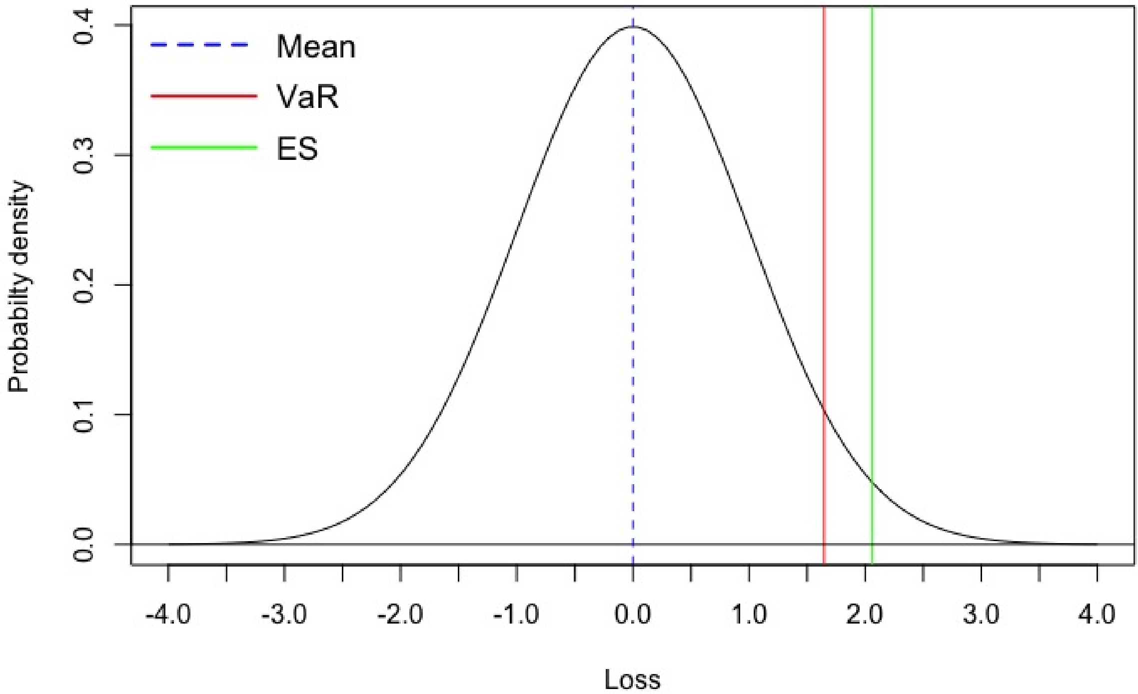


Figure 1 – This is an example of standard normal loss distribution with the 95% VaR marked as red vertical line and 95% ES marked as vertical green line. See Table 1 for the exact values.

The following lemma shows that ES can be interpreted as the expected loss that occurs in the case when the loss exceeds $VaR_\alpha$.

**Lemma 5.1.** *Let $L$ be an integrable loss with continuous distribution function $F_L$, then for any $\alpha \in (0,1)$ we have:*

$$ES_\alpha = \frac{E(L; L \geq q_\alpha(L))}{1-\alpha}, \tag{5.5}$$

*whereas*

$$\frac{E(L; L \geq q_\alpha(L)}{1-\alpha} = E(L|L \geq VaR_\alpha), \tag{5.6}$$

*where we denote $E(X;A) = E(XI_A)$ for a generic integrable random value $X$ and a generic set $A \in \mathcal{F}$.*

*Proof.* Let $U$ be a random variable with uniform distribution on the interval [0,1]. From the measure and probability theory we known that the random variable $F_L^{-1}(U)$ has the distribution function $F_L$. Further, for the generalized inverse $F_L^{-1}$ it holds that:

$$F_L(F_L^{-1}(x)) \geq x \text{ and } F_L(x) \geq y \Leftrightarrow x \geq F_L^{-1}(y).$$

Hence,

$$E(L; L \geq q_\alpha(L)) = E(F_L^{-1}(U); F_L^{-1}(u) \geq F_L^{-1}(\alpha)) = E(F_L^{-1}(U); U \geq \alpha). \tag{5.7}$$

Since it holds that:

$$E(F_L^{-1}(U); U \geq \alpha) = \int_\alpha^1 F_L^{-1}(u)du = (1-\alpha)ES_\alpha;$$

we derive from (5.7) the first representation of ES.
The second representation (5.6) follows since, for a continuous loss distribution $F_L$, we have

$$\mathbb{P}(L \geq q_\alpha(L)) = 1-\alpha.$$

□

## 5.1 Coherence of Expected Shortfall

(i) Taking into account the axioms of a coherent risk measure from Section 4 the positive homogeneity for $\lambda > 0$ is proved as follows:

$$ES_\alpha(\lambda L) = \frac{1}{1-\alpha}\int_\alpha^1 q_u(F_{\lambda L})du.$$

From the measure and probability theory we know that:

$$F_{\lambda L}(x) = \mathcal{P}(\lambda L \leq x) = \mathcal{P}(L \leq x/\lambda) = F_L(x/\lambda).$$

From the definition of the gereralized inverse we obtain:

$$q_u(F_{\lambda L}) = F_{\lambda L}^{-1}(u) = inf\{x \in \mathbb{R} : F_{\lambda L}(x) \geq \alpha\} = inf\{x \in \mathbb{R} : F_L(x/\lambda) \geq \alpha\}$$
$$= inf\{\frac{\lambda}{\lambda} \cdot x \in \mathbb{R} : F_L(x/\lambda) \geq \alpha\} = \lambda \cdot inf\{x/\lambda \in \mathbb{R} : F_L(x/\lambda) \geq \alpha\} = \lambda \cdot q_u(F_L).$$

Therefore, ES satisfies the positive homogeneity property: $ES_\alpha(\lambda L) = \lambda ES_\alpha(L)$.

(ii) The translation invariance and monotonicity can be proved in a similar way.

(iii) The sub-additivity of ES for the case where $L_1, L_2$ and $L_1 + L_2$ have a continuous distribution, which allows the application of the results from Lemma 5.1, is proved as follows:

$$ES_\alpha(L) = \frac{E(LI_{L \geq q_\alpha(L)})}{1-\alpha}.$$

For simplification we denote:

$$I_1 := I_{\{L_1 \geq q_\alpha(L_1)\}}, I_2 := I_{\{L_2 \geq q_\alpha(L_2)\}} \text{ and } I_{12} := I_{\{L_1+L_2 \geq q_\alpha(L_1+L_2)\}}.$$

Then we have,

$$\begin{aligned}(1-\alpha)(ES_\alpha(L_1) + ES_\alpha(L_2) - ES_\alpha(L_1+L_2))&\\ &= E(L_1 I_1) + E(L_2 I_2) - E((L_1+L_2)I_{12})\\ &= E(L_1(I_1 - I_{12})) + E(L_2(I_2 - I_{12})).\end{aligned}$$

When considering the two terms in (7.1) and supposing that $\{L \geq q_\alpha(L_1)\}$, it follows that $I_1 - I_{12} \geq 0$ holds and as a consequence:

$$L_1(I_1 - I_{12}) \geq q_\alpha(L_1)(I_1 - I_{12}).$$

In reverse, when $\{L \leq q_\alpha(L_1)\}$ holds, it follows that $I_1 - I_{12} \leq 0$ and the inequality $L_1(I_1 - I_{12}) \geq q_\alpha(L_1)(I_1 - I_{12})$ remains valid. In the same way we can prove that:

$$L_2(I_2 - I_{12}) \geq q_\alpha(L_2)(I_2 - I_{12}).$$

Therefore, we obtain:

$$\begin{aligned}(1-\alpha)(ES_\alpha(L_1) + ES_\alpha(L_2) - ES_\alpha(L_1+L_2))\\ &\geq E(q_\alpha(L_1)(I_1 - I_{12})) + E(q_\alpha(L_2)(I_2 - I_{12}))\\ &= q_\alpha E((L_1)(I_1 - I_{12})) + q_\alpha E((L_2)(I_2 - I_{12}))\\ &\geq q_\alpha((1-\alpha) - (1-\alpha)) + q_\alpha((1-\alpha) - (1-\alpha))\\ &= 0.\end{aligned}$$

Hence,

$$ES_\alpha(L_1) + ES_\alpha(L_2) \geq ES_\alpha(L_1 + L_2).$$

A general proof of subadditivity of ES is given in Theorem 8.14 McNeil et. al [7]. Subadditivity of ES is obviously an advantage over VaR. Tunaru [8] pointed out that even if VaR calculated under some model is subadditiv it is still possible that the estimated VaR values invalidate the subadditivity inequality condition. In contrast, it is easy to see that even if the risk measure is violating the subadditivity condition theoretically, it is still possible that, due to the in sample estimation error, the estimated risk values do actually obey the subadditivity condition.

## 5.2 Backtesting of Expected Shortfall

VaR has practical advantages over ES: it is easier to estimate and its estimate is easier to backtest. Backtesting of Expected Shortfall is not straightforward. One intuitive way is to work only with the subsample consisting of those periods when VaR is breached and calculate the ratio[5]:

$$ESR_t = \frac{X_t}{ES_\alpha(t)};$$

Then, from Lemma 5.1 it follows:

$$\frac{\mathbb{E}[X_t | X_t < -VaR_\alpha(t)]}{ES_\alpha} = 1;$$

we can say that the average $\overline{ESR}$ will be equal to one when the model forecasts the ES exactly as observed *ex post* on the financial market.

[5]See Tunaru [8], p.199.

# 6 Regulatory risk mesures

Solvency II defines the capital requirement for an insurance or reinsurance company, using Value-at-Risk. However, the Swiss Solvency Test (SST), which serves to supervise the adequate capitalization of insurance companies, uses the Expected Shortfall as risk measure. The Basel II framework uses stress scenarios calibrated on Value-at-Risk, but the Basel III framework will be calibrated using Expected Shortfall. In Section 6 we will discuss the practical usage of Value-at-Risk and Expected Shortfall as regulatory risk measures.

## Solvecy II

Solvency II has its primary purpose to guarantee that insurance companies have sufficient capital to reduce risk and loss, to protect policy holders and to provide financial stability to the insurance sector. Solvency Capital Requirement (SCR) is the capital requirement defined in Solvency II. It must cover all the risks that an insurance company encounters. According to EIOPA [5] the SCR standard formula follows a modular approach, where the overall risk which the insurance or reinsurance company undertakes, is divided into sub-risks and in some risk modules also into sub-sub risks. For each sub-risk (or sub-sub risk) a capital requirement is determined. The capital requirement on sub-risk or sub-sub risk level is aggregated with the use of correlation matrices in order to derive the capital requirement for the overall risk. EIOPA [5] decides that:

> The SCR is calibrated using the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99.5% over a one-year period. This calibration objective is applied to each individual risk module and submodule.

The standard formula of Basic SCR is defined as:

$$Basic\ SCR = \sqrt{\sum_i \sum_j Corr \times SCR_i \times SCR_j},$$

where $SCR_i$ and $SCR_j$ denotes the risk modules $i$ and $j$.
For illustration purposes, here are some examples of risk sub-modules:

- Interest rate risk: arises from interest rate sensitive assets and liabilities in times of high or low interest rate.
- Equity risk: related to the volatility or level of market prices for equities, stems from assets and liabilities that are sensitive to equity price fluctuations.
- Currency risk: stems from fluctuations in the volatility or level of currency exchange rates.

However, not all risk modules are being represented in the stadard formula for Basic SCR. Here are some of the risks that are not being taken into account:

- Inflation risk: arises from the sensitivity of the assets, liabilities and financial instruments to changes of inflation rates. Inflation risk is not selected to a separate risk (sub) module.
- Reputation risk: risk associated with the trustworthiness of strategic decisions is not explicitly covered in the standard formula.
- Liquidity risk: risk that insurance and reinsurance are unable to realize investments and other assets in order to meet their financial obligations, e.g. to convert an asset into cash without capital expenditure.

The correlations in the formula for Basic SCR are based on two assumptions:

(i) Risk distributions are linearly dependant

(ii) The correlation parameters are chosen in such a way as to achieve the best approximation of the 99.5 % VaR for the overall capital requirement in case of skewed non normal distributions.

Unfortunately both assumptions are rarely met in practice. Linear correlations are not sufficient to describe the dependence between distributions adequately and the relevant risks that an insurance or reinsurance company undertakes are typically not normal distributed. They are often skewed. In which case the use of Value-at-Risk might cause the discrepancy of the SCR and the underlying riskiness.

## Basel III

The need to seriously strengthen the Basel II framework had become quite obvious even before Lehman Brothers collapsed in September 2008. The banking sector entered the financial crisis with too much leverage and inadequate liquidity buffers. These weaknesses were accompanied by poor governance and risk management, as well as inappropriate incentive structures. The dangerous combination of these factors was demonstrated by the misprizing of credit and liquidity risks, and excess credit growth. The Basel III framework is a central element of the Basel Committee's response to the global financial crisis of 2007/08. This response states that bank capital requirements must be strengthened. One proposition is to increase a bank's liquidity, while simultaneously decreasing its leverage. Furthermore, it addresses shortcomings of the pre-crisis regulatory framework and provides a regulatory foundation for a more resilient banking system that supports the real economy.[6] Basel III and Solvency II are similar in that each deploys stress scenarios to predict the impact of shocks on certain given drivers. They are different in that each is a regulatory framework that targets a different part of the financial industry. Both regulations use also stress scenarios to see the impact of shocks on certain risk drivers.

[6]https://www.bis.org/publ/bcbs164.htm

# 7 Moving from Value-at-Risk to Expected Shortfall

Section 7 stems from the Basel Committee [2] and [3]. The Basel II framework uses stress scenarios calibrated on VaR with quantile $\alpha = 99\%$. However, the Basel Committee [2],p.3 observed that a number of weaknesses have been identified in using VaR for determining regulatory capital requirements, including its inability to capture tail risk. The Committee has therefore decided to use an Expected Shortfall with parameter $\theta = 97.5\%$ measure for the internal model-based approach and will determine the risk weights for the revised standardized approach using an ES methodology. ES accounts for the tail risk in a more comprehensive manner, considering both the size and likelihood of losses above a certain threshold. The Basel Committee is convinced that VaR should be replaced with a risk measure that covers tail risk more accurately, even though it realizes that shifting to ES might present operational challenges. Based on the more complete capture of tail risks using an ES model, the Committee believes that moving to a confidence level of 97.5% (relative to the 99th percentile confidence level for the VaR measure) is appropriate. This confidence level will provide a broadly similar level of risk capture as the existing 99th percentile VaR threshold, while providing a number of benefits, including generally more stable model output and often less sensitivity to extreme outlier observations.

## 7.1 Quantitative standards for internal models-based measurement

Basel Committee [3] provides quantitative standards for internal model approach to the minimum capital requirement for market risk. The Basel Committee has agreed to the definitions of:

> "*liquidity horizon*" as being: the time required to execute transactions that extinguish an exposure to a risk factor, without moving the price of the hedging instruments, in stressed market conditions, meaning without a significant market influence;
>
> "*trading desk*" as being: a group of traders or trading accounts that implements a well defined business strategy, operating within a clear risk management structure, defined by the bank but with the definition approved by supervisors for capital purposes.

Banks will have flexibility in devising the precise nature of their models, but the following minimum standards will apply for the purpose of calculating their capital charge. Individual banks or their supervisory authorities will have discretion to apply stricter standards.

1. Expected Shortfall must be computed on a daily basis for the bank-wide internal model for regulatory capital purposes. Expected Shortfall must also be computed on a daily basis for each trading desk that a bank wishes to include within the scope for the internal model for regulatory capital purposes.

2. In calculating Expected Shortfall, a 97.5th percentile, one-tailed confidence level is to be used.

3. In calculating Expected Shortfall, the liquidity horizons described in paragraph (6) must be reflected by scaling Expected Shortfall calculated on a base horizon. Expected Shortfall for a liquidity horizon must be calculated from Expected Shortfall at a base liquidity horizon of 10 days with scaling applied to this base horizon result as follows:

$$ES = \sqrt{(ES_T(P))^2 + \sum_{j \geq 2} \left( ES_T(P, j) \sqrt{\frac{(LH_j - LH_{j-1})}{T}} \right)^2} \tag{7.1}$$

where

- $ES$ is the regulatory liquidity-adjusted Expected Shortfall;
- $T$ is the length of the base horizon, i.e. 10 days;
- $ES_T(P)$ is the Expected Shortfall at horizon $T$ of a portfolio with positions $P = (p_i)$ with respect to shocks to all risk factors that the positions P are exposed to;
- $ES_T(P, j)$ is the Expected Shortfall at horizon $T$ of a portfolio with positions $P = (p_i)$ with respect to shocks for each position $p_i$ in the subset of risk factors $Q(p_i, j)$, with all other risk factors held constant;
- $ES_T(P, j)$ must be calculated for changes in the relevant subset $Q(p_i, j)$ of risk factors, over the time interval $T$ without scaling from a shorter horizon;
- $Q(p_i, j)_j$is the subset of risk factors whose liquidity horizons, as specified in paragraph further, for the desk where $p_i$ is booked are at least as long as $LH_j$ according to the table below. For example, $Q(p_i, 4)$ is the set of risk factors with a 60-day horizon and a 120-day liquidity horizon. Note that $Q(p_i, j)$ is a subset of $Q(p_i, j - 1)$;
- the time series of changes in risk factors over the base time interval T may be determined by overlapping observations; $LH_j$ is the liquidity horizon $j$, with lengths in the following table:

| $j$ | $LH_j$ |
|---|---|
| 1 | 10 |
| 2 | 20 |
| 3 | 40 |
| 4 | 60 |
| 5 | 120 |

4. The Expected Shortfall measure must be calibrated to a period of stress. Specifically, the measure must replicate an Expected Shortfall charge that would be generated on the bank's current portfolio if the relevant risk factors were experiencing a period of stress. The identified reduced set of risk factors must be able to explain a minimum of 75% of the variation of the full ES model (i.e. the ES of the reduced set of risk factors should be at least equal to 75% of the fully specified ES model on average measured over the preceding 12 week period). The Expected Shortfall for

the portfolio using this set of risk factors, calibrated to the most severe 12-month period of stress available over the observation horizon, is calculated. That value is then scaled up by the ratio of the current Expected Shortfall using the full set of risk factors to the current Expected Shortfall measure using the reduced set of factors. Expected Shortfall for risk capital purposes is therefore:

$$ES = ES_{R,S} \cdot \frac{ES_{F,C}}{ES_{R,C}}$$

where the Expected Shortfall for capital purposes ($ES$) is equal to Expected Shortfall based on a stressed observation period using a reduced set of risk factors ($ES_{R,S}$) multiplied by the ratio of the Expected Shortfall measure based on the current (most recent) 12-month observation period with a full set of risk factors ($ES_{F,C}$) and the Expected Shortfall measure based on the current period with a reduced set of risk factors ($ES_{R,C}$). For the purpose of this calculation, the ratio is floored at 1.

5. For measures based on current observations ($ES_{F,C}$), banks must update their data sets no less frequently than once every month and must also reassess them whenever market prices are subject to material changes. This updating process must be flexible enough to allow for more frequent updates. The supervisory authority may also require a bank to calculate its Expected Shortfall using a shorter observation period if, in the supervisor's judgement; this is justified by a significant upsurge in price volatility. In this case, however, the period should be no shorter than 6 months.

6. For measures based on stressed observations ($ES_{R,S}$), banks must identify the 12-month period of stress over the observation horizon in which the portfolio experiences the largest loss. The observation horizon for determining the most stressful 12 months must, at a minimum, span back to and including 2007. Observations within this period must be equally weighted. Banks must update their 12-month stressed periods no less than monthly, or whenever there are material changes in the risk factors in the portfolio.

7. No particular type of Expected Shortfall model is prescribed. So long as each model used captures all the material risks run by the bank, as confirmed through P&L attribution and backtesting, and conforms to each of the requirements set out above and below, supervisors may permit banks to use models based on either historical simulation, Monte Carlo simulation, or other appropriate analytical methods.

8. As set out in paragraph (3), a scaled Expected Shortfall must be calculated based on the liquidity horizon n defined below. n is calculated using the following conditions:

   - banks must map each risk factor on to one of the risk factor categories shown below in Table 2 using consistent and clearly documented procedures;
   - the mapping must be (i) set out in writing; (ii) validated by the bank?s risk management; (iii) made available to supervisors; and (iv) subject to internal audit; and;

- $n$ is determined for each broad category of risk factor as set out in the following table. However, on a desk-by-desk basis n can be increased relative to the values in the table below (i.e. the liquidity horizon specified below can be treated as a floor). Where $n$ is increased, the increased horizon must be 20, 40, 60 or 120 days and the rationale must be documented and be subject to supervisory approval. Furthermore, liquidity horizons should be capped at the maturity of the related instrument:

| Risk factor category | $n$ | Risk factor category | $n$ |
|---|---|---|---|
| Interest rate: specified currencies - EUR, USD, GBP, AUD, JPY, SEK, CAD and domestic currency of a bank | 10 | Equity price (small cap): volatility | 60 |
| Interest rate: – unspecified currencies | 20 | Equity: other types | 60 |
| Interest rate: volatility | 60 | FX rate: specified currency pairs[37] | 10 |
| Interest rate: other types | 60 | FX rate: currency pairs | 20 |
| Credit spread: sovereign (IG) | 20 | FX: volatility | 40 |
| Credit spread: sovereign (HY) | 40 | FX: other types | 40 |
| Credit spread: corporate (IG) | 40 | Energy and carbon emissions trading price | 20 |
| Credit spread: corporate (HY) | 60 | Precious metals and non-ferrous metals price | 20 |
| Credit spread: volatility | 120 | Other commodities price | 60 |
| Credit spread: other types | 120 | Energy and carbon emissions trading price: volatility | 60 |
| | | Precious metals and non-ferrous metals price: volatility | 60 |
| Equity price (large cap) | 10 | Other commodities price: volatility | 120 |
| Equity price (small cap) | 20 | Commodity: other types | 120 |
| Equity price (large cap): volatility | 20 | | |

Table 2 – Risk factors with liquidity horizons.

**Example: FX risk**

When effectuating a financial transaction in a currency which is not the base currency of an enterprise *foreign exchange* (FX) risk needs to be considered. Using Table 2 we obtain:

- Risk factor category: FX;
- $n = 10{,}20{,}40$;
- $j = 3$;

$$\begin{aligned} ES_{FX} &= \sqrt{\left(ES_{10}^{10,20,40}\right)^2 + \left(ES_{10}^{20,40}\sqrt{\frac{20-10}{10}}\right)^2 + \left(ES_{10}^{40}\sqrt{\frac{40-20}{10}}\right)^2} \\ &= \sqrt{\left(ES_{10}^{10,20,40}\right)^2 + \left(ES_{10}^{20,40}\right)^2 + 2\cdot\left(ES_{10}^{40}\right)^2}, \end{aligned}$$

where

- $ES_{10}^{10,20,40}$ is Expected Shortfall at base liquidity horizon of 10 days, with respect to all the risk factors which have a liquidity horizon of 10,20 or 40 days;
- $ES_{10}^{20,40}$ is Expected Shortfall at base liquidity horizon of 10 days, with respect to all the risk factors which have a liquidity horizon of 20 or 40 days;
- $ES_{10}^{40}$ is Expected Shortfall at base liquidity horizon of 10 days, with respect to all the risk factors which a have liquidity horizon of 40 days;

## Backtesting

According to Basel Committee [3], backtesting requirements are based on comparing each desk's 1-day static Value-at-Risk measure (calibrated to the most recent 12 months' data, equally weighted) at both the 97.5th percentile and the 99th percentile, using at least one year of current observations of the desk's one-day P&L. If any given desk experiences either more than 12 exceptions at the 99th percentile or 30 exceptions at the 97.5th percentile in the most recent 12-month period, all of its positions must be capitalised using the standardised approach. Positions must continue to be capitalised using the standardised method until the desk no longer exceeds the above thresholds over the prior 12 months.

# 8 Conclusion

There are several issues in dealing with Value-at-Risk: it neglects the loss beyond the VaR level and is not subadditive. This can cause serious practical problems, since information provided by Value-at-Risk may lead investors to wrong conclusions as demonstrated in Section 4. By adopting Expected Shortfall investors can balance this this weakness, because this considers loss beyond the Value-at-Risk level.
Expected Shortfall was designed as a conceptually better alternative. It is a coherent risk measure and it covers the main properties deemed necessary to reach reliable conclusions with regards to risk management. As such it is a reasonable alternative to Value-at-Risk. However, Value-at-Risk is easier to calculate and therefore easier to backtest when determining the accuracy of Value-at-Risk models. Backtesting of VaR is relatively straightforward compared to backtesting of Expected Shortfall. Recent years research has indicated that backtesting of Expected Shortfall is possible and is not as complicated as it was assumed. Different approaches have been proposed, i.e. see Acerbi and Szekeley [1].
However, the effectiveness of Expected Shortfall depends on the stability of estimation and the choice of efficient backtesting methods. Kellner and Roesch [6] showed that Expected Shortfall is more sensitive towards regulatory arbitrage and parameter misspecification than Value-at-Risk. Risk managers should weight the strength and weakness of Expected Shortfall before adopting it as part of the practice of risk management. The summary below illustrates actual known facts[7] about VaR and ES:

| | **VaR** | **ES** |
|---|---|---|
| Strength | -Easier to calculate;<br>-Easily applied to backtesting;<br>-Established as the standard risk measure and equipped with sufficient infrastructure;<br>-Related to the firm's own default probability; | -Sub-additiv;<br>-Is able to consider loss beyond the VaR level<br>-Less likely to give perverse incentives to investors;<br>- Easily applied to portfolio optimizations; |
| Weakness | -Unable to consider tail risk;<br>-Not sub-additive;<br>-Likely to give perverse incentives to investors;<br>-Difficult to apply to portfolio; optimizations; | -Not as easily applied to backtesting;<br>-Needs of more data;<br>-Insufficient infrastructure;<br>-Not ensured with stable estimation; |

[7] see Yamai and Yoshiba [9], p.81

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