Seminar work

Financial Networks

Author: Maja Prlja

Supervisor: Dr. Stefan Gerhold

March 2015
Contents

List of Figures ......................................................... 2

1 Introduction ......................................................... 2
  1.1 Brief history of the Network Theory .......................... 3

2 Financial Optimization problems ................................. 5
  2.1 Harry Markowitz Model ........................................ 7

3 General Financial Equilibrium Problems ....................... 10
  3.1 A Multisector, Multi-Instrument Financial Equilibrium Model . 12
    3.1.1 A Sector’s Portfolio Optimization Problem ............ 14
    3.1.2 Optimality Conditions .................................. 15
    3.1.3 Economic System Conditions ............................ 15
  3.2 Model with Utility Functions .................................. 17
  3.3 Computation of Financial Equilibria .......................... 17
Chapter 1

Introduction

Finance is the science which deals with capital flow over space and time in the presence of risk. It is considered that economics is the base of financial science where micro as well as macroeconomics affect decision and results at all levels.

Network Theory is an area of applied mathematics and part of graph theory. It represents asymmetric relation between discrete objects. Network Theory is a medium for abstracting complex problems ranging from portfolio optimization to multisector, multi-instrumental general financial equilibrium problems, dynamic multiagent financial problems with intermediation, as well as the financial engineering of the integration of social networks with financial systems.

Finance uses mathematical and engineering methodologies for modeling, analysis and computation of solutions. One of those methodologies are networks, in addition, the topic of this seminar work. Networks consist of nodes, links and flows. One must represent the underlying behavior and interactions of decision makers and costs, risks and prices. Firstly, we are going to get introduced with the brief history of the network theory, starting with the seven bridges of Konigsberg along financial optimization problem to financial equilibrium problems. In second chapter, we are going to get introduced with the financial optimization problems. Some light will be shed on the Harry Markowitz’s model which is considered to be a milestone in area of financial optimization. In third chapter, we are going to observe financial systems with multiple decision makers. Some more attention will be given to theory of Nagurney and Skios.
1.1 Brief history of the Network Theory

Through the history, different problems led us to the theories and methodologies that are network based. You may ask yourself why? First of all, networks prove connectivity for our societies and economies and their methodology is a powerful medium for abstracting complex problems. Those problems may not seem to be networks but in addition they are.

The seven bridges of Konigsberg is the first proof in the network theory. The Konigsberg, which is today Kaliningrad (Russia), was set on both sides of the Pregel River. It consisted of two islands that were connected to each other by seven bridges. The question was: ‘Does there exist any single path that crosses all seven bridges exactly once each?’

In 1736, the Swiss mathematician Euler has written in his paper that the problem has no solution. In the history of mathematics, Euler’s solution led us to the topology and also is considered to be the first theorem of graph theory.

In 1758, the French economist Quesnay made a concept of the circular flow of the financial funds in an economy as a network. His paper is considered to be the first paper which topic were financial networks.

Monge (1781), a French mathematician, published the first paper on transportation in the minimizing cost.

Cournot (1838), a French philosopher and mathematician, stated that a competitive price is determined by intersection of supply and demand curves in the context of the spatially separate markets with transportation costs included.

Kohl (1941) observed a two node - two route transporation problem.

Pigou (1920) studied a transportation network system of two routes.

Konig (1936) published the book Theorie der endlichen und unendlichen Graphen, which is considered to be the first book on a graph theory.

In 1952 Copeland published a book Studies of Moneyflows where he asked the question ‘Does money flow like water and electricity?’.

Also in 1952, Harry M. Markowitz has revealed portfolio optimization theory.

In 1962 L.R. Ford and D.R. Fulkerson published *Flows in Networks* where they presented algorithm which computes the maximum flow in a flow network (Ford-Fulkerson algorithm – FFA).
Chapter 2

Financial Optimization problems

Financial problems as portfolio optimization, asset allocation, currency translation, risk management and many other are proposed to be solved with network models. Those problems usually contain non-linear objective function mainly due to modeling of the risk function which lies in the domain of non-linear network flow problems. The work by Markowitz prefaced a new epoch in financial economics and it became the basis for many optimization models. Even though the work of Markowitz as well as many other works after him had a network structure, the financial network optimization models were developed later. Exceptions are models by Charnes and Miller (1957) and Charnes and Cooper (1961). The first application of financial networks appeared in the literature during late 60s and early 70s.

Rutenberg (1970) used networks in order to formulate a series of currency-translation problems. He stated that the translation between different currencies could be performed through the use of arc multipliers. In a pure network the sum of flows that enter an arc is equal to the sum of flows that leave it. In generalized networks there are gains and losses and each arc should have a multiplier which represents them. In Rutenberg’s network linear costs were on the arcs. Particular currency was represented with the nodes and the amount of cash moving from one currency to another with the flows. Currency-translation models were also observed in works of Christofides et al. (1979) and Shapiro and Rutenberg (1976).

There were also other applications of financial networks. For example R.S.Barr (1972), V.A.Srinivasan (1974), R.L.Crum (1976), D.J.Nye and R.L.Crum (1981), Crum et al. (1983) and many others described with the networks cash management problems.
Chapter 2 Financial Optimization problems

Crum (1976) introduced a linear network model for the cash management of a multinational firm where the possible cash flow patterns were the links and incorporated costs, fees, liquidity changes and exchange rates were the multipliers. Crum et al. (1979) stated that contemporary financial capital allocation problems could be modeled as an integer generalized network where the flows on the particular arcs were integers.

As already mentioned, many financial network optimization problems should have a non-linear objective function but it was not recognized by all authors. For example Rudd and Rosenberg (1979) and Soenen (1979). On the other hand, John M. Mulvey (1987) has recognized that and also that the mean-variance minimization problem by Markowitz was also a financial network optimization problem with non-linear objective function.
2.1 Harry Markowitz Model

In finance, a portfolio is a collection of assets, stocks, bonds, investments and it is usually held by an investment company, financial institution or individual.

Harry Markowitz is an American economist and a Nobel Memorial Prize winner who invented the modern portfolio theory in 1952. His theories emphasized the importance of portfolios, risk, the correlations between securities and diversification. His work changed the way that people invested.

According to the work of Markowitz, the process of selecting a portfolio may be divided into two stages. He stated \(^1\):

"The process of selecting a portfolio may be divided into two stages. The first stage starts with the observation and experience and ends with beliefs about the future performance of available securities. The second stage starts with the relevant beliefs about future performances and ends with the choice of portfolio."

His paper was mainly about the second stage. Markowitz’s model is also called mean-variance model because it is based on expected returns (mean) and standard deviation (variance) of various portfolios. Firstly, Markowitz considered that the investor should maximize discounted expected or anticipated returns. Secondly, he considered the rule where expected return is observed as a desirable and variance of return as an undesirable. The first rule was rejected and the second had many sound points according to Markowitz. The relations between beliefs and choice of portfolio according to the “expected returns-variance of returns” rule are illustrated geometrically. Markowitz stated that portfolio optimization problem is associated with risk minimization with the objective function:

\[
\text{minimize}(V = X^T Q X)
\] (2.1)

\(Q\) denotes the \(n\times n\) variance-covariance matrix on the return of the portfolio and \(V\) the variance vector. Markowitz has provided the way to the (2.1) in his work from 1952. First of all he defined the following: there are \(n\) securities available in the economy, \(r_i\) is the anticipated return, \(X\) is the relative amount invested in security \(i\). \(R\) denotes the expected rate of return. The discounted anticipated return of the portfolio is:

\[
R = \sum_{i=1}^{n} X_i r_i
\] (2.2)

\(^1\)Markowitz, H.M. Portfolio Selection, 1952
Graphically it is presented in Figure 1 where the links 1, ..., \(n\) represent the securities and the flows represent relative amounts of securities \(X_1, ..., X_n\). In order to maximize \(R\), we let \(\sum_{i=1}^{n} X_i = 1\). Short sales are not included and that is the reason why \(X_i \geq 0\) for all \(i\). According to Markowitz, the expected returns or anticipated returns rule is not very adequate and he considered the expected returns - variance of returns (E-V) rule.

The following elementary concepts of statistics will be needed:
\(Y = (y_1, ..., y_n)\) is a random variable, the value of this variable is unknown, it can take multiple different values. The probability that \(y_1\) occurs is denoted by \(p_1\), \(y_2\) by \(p_2\) etc. The mean (expected value) of \(Y\) is:

\[
E[Y] = p_1y_1 + p_2y_2 + ... + p_ny_n
\]  

(2.3)

And variance (average squared deviation) of \(Y\) is:

\[
V(Y) = p_1(y_1 - E)^2 + p_2(y_2 - E)^2 + ... + p_n(y_n - E)^2
\]  

(2.4)

Standard deviation is denoted by \(\sigma = \sqrt{V}\).

\(R = (R_1, ..., R_n)\) are also random variables and \(R_i\) s the linear combination of the \(R_i\).

\[
R = a_1R_1 + a_2R_2 + ... + a_nR_n
\]  

(2.5)

The mean is of \(R\) is:

\[
E[R] = a_1E[R_1] + a_2E[R_2] + ... + a_nE[R_n]
\]  

(2.6)

The covariance between \(R_i\) and \(R_j\) is \(^2\):

\[
\sigma_{ij} = E[(R_i - E[R_i])(R_j - E[R_j])]
\]  

(2.7)

Covariance may be defined in terms of correlation coefficient \(\rho_{ij}\).

\[
\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j
\]  

(2.8)

The the variance is:

\[
V(R) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_ia_j\sigma_{ij}
\]  

(2.9)

\(^2\text{e.g. J.V.Uspensky, Introduction to Mathematical Probability (New York: McGraw Hill 1937), chapter 9, pp. 161-81\)
The expected value of portfolio is:

\[ E = \sum_{i=1}^{n} X_i \mu_i \]  \hspace{1cm} (2.10)

And the variance of portfolio is:

\[ V = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} X_i X_j \]  \hspace{1cm} (2.11)

Where \( \mu \) is the expected value of \( R_i \) and \( \sigma_{ij} \) is the covariance between \( R_i \) and \( R_j \).

Markowitz stated that the investor has a choice of combinations of \( E \) and \( V \) depending on his choice of portfolio.

In 1959, Markowitz has written a paper related to this topic where he stated that the investor should maximize his returns while minimizing his risk.

\[ \text{maximize} (\alpha R - (1 - \alpha) V) \]  \hspace{1cm} (2.12)

where \( \alpha \) represents an indicator how investor immune to the risk is.

Related works to this topic are written by Francis and Archer (1979), Dong and Nagurney (2001) and Doumpos et al. (2000).
Chapter 3

General Financial Equilibrium
Problems

Equilibrium is per definition the stable state of the system. In finance, it means that the price is stable which is consequence of the rough equality of supply and demand.

In this section, we are going to observe financial systems in which there is more than a single decision maker. We are going to make a theoretical framework for the formulation, analysis and computation of financial equilibria using variational inequality theory. This theory was firstly introduced by Nagurney et al. in 1992.

Financial system as a network has its roots in the work of Francois Quesnay (1758).

In 1976 A.Charnes and W.Cooper stated that systems of linked accounts could be represented as a network. The nodes represented the balance sheets and the links represented the credit and debit entries.


A.R.Ferguson and G.B.Dantzing (1956) as well as G.B.Dantzing and A.Madanasky (1961) introduced the approach utilized two-stage linear programs.

In 1975 Storoy et al. presented how decomposition theory could be used for the computation of equilibrium.

In 1980 Thore further studied network models of linked portfolios as well as financial intermediation and decomposition theory. However computational techniques were not so developed to handle such problems. Later, in 1984, Thore made a proposal for
an international financial network for the Euro Dollar market observed as a logistical system.

In the systems where there is more than one decision maker, equilibrium was a central concept with the role of prices in the equilibrating mechanism. K.J.Arrow (1951) and G.Debreu (1951) proposed rigorous approaches for formulation of equilibrium and price determination (Arrow-Debreu model).

As already mentioned, Nagurney et al. (1992) introduced use of finite dimensional variational inequality theory. Nagurney stated that general financial equilibrium problems could be large scale in practice, since we have multiple decision makers. Decomposition algorithms that solve such large scale problems into simpler sub-problems are especially appealing. Nagurney has written in her work:

"A variational inequality decomposition algorithm is proposed, based on the modified projection method which in many applications yields network sub-problems which not only can be solved using equilibrium algorithms but can also be implemented on parallel architectures."

Hughes and Nagurney (1992) and Nagurney and Hughes (1992) stated that the formulation and solution of estimation of financial flow of funds accounts could be represented as a network optimization problem.

For the first time, in 1997 Nagurney and Skios presented an international financial equilibrium problem using finite dimensional variational inequality theory.

Variational inequality theory is a methodology for studying the equilibrium problems (apart from financial equilibrium problems also traffic network equilibrium problems, migration equilibrium problems, knowledge network problems, spatial price equilibrium problems).

**Definition 1 Variational Inequality Problem**

The finite dimensional variational inequality problem, $VI(F, K)$, is to determine a vector $x^* \in K \subset \mathbb{R}^n$, such that $F(x^*)^T(x - x^*) \geq 0$ for all $x \in K$ or, equivalently $\langle F(x^*)^T, x - x^* \rangle \geq 0$ for all $x \in K$ where $F$ is a given continuous function from $K$ to $\mathbb{R}^n$, $K$ is a given closed convex set, and $\langle \cdot, \cdot \rangle$ the inner product in $n$ dimensional space.
3.1 A Multisector, Multi-Instrument Financial Equilibrium Model

In order to illustrate how powerful tool finite dimensional variational inequality theory is, it is used multi sector, multi instrument model and an extension due to Nagurney et al. (1992) and Nagurney (1994).

In the work of Markowitz (1959), we have seen the classical mean-variance model. Now, we are going to observe the model with \( m \) sectors where the typical sector is denoted by \( i \) and with \( n \) instruments where the typical instrument is denoted by \( j \). Examples of sectors are households, businesses, state and local governments, banks etc. Nagurney denoted in her work the volume of instrument \( i \) held in sector \( j \)'s portfolio as an asset by \( X^j_i \) and the volume of instrument \( i \) held in sector \( j \)'s portfolio as a liability by \( Y^j_i \).

The assets in the portfolio of sector \( j \) are grouped into column vector \( X^j \in \mathbb{R}^n_+ \) and the assets of all sectors in the economy into column vector \( X \in \mathbb{R}^{mn}_+ \). The liabilities in the portfolio of sector \( j \) are grouped into with column vector \( Y^j \in \mathbb{R}^n_+ \) and liabilities of all sectors in the economy are grouped into column vector \( Y \in \mathbb{R}^{mn}_+ \). The (non-negative) price of instrument \( i \) is denoted by \( r_i \) and the prices of all instruments are grouped into the column vector \( r \). The financial volume held by sector \( j \) is denoted by \( S^j \).

Explicit recognition of both sides of the sectoral balance sheet is included here. It is done in order to maintain the strategic distinction between acquisitions net of sales and issues net of paybacks. Balance sheet is a financial report that contains the status of company’s assets, liabilities and owner’s equity at specific time. The left hand side of a balance sheet demonstrates the assets that a sector holds and the right hand side of a balance sheet demonstrates the liabilities and owner’s equity. Due to accounting principles, the sum of all assets is equal to the sum of all liabilities and owner’s equity. The total volume of each balance sheet side of each sector is exogenous. Let’s suppose that competition is perfect, then will the each sector behave as if it has no influence on instrument prices or on the behavior of the other sectors.

Sector \( j \) seeks to determine its optimal composition of instruments held as assets and as liabilities, so to minimize the risk while at the same time maximizing the value of its asset holdings and minimizing the value of its liabilities. The portfolio optimization
problems for each sector \( j \) is given by:

\[
\text{minimize } \left( \begin{array}{c} X^j \\ Y^j \end{array} \right)^T Q^j \left( \begin{array}{c} X^j \\ Y^j \end{array} \right) - \sum_{i=1}^{n} (X^i_j - Y^i_j) \tag{3.1}
\]

subject to

\[
\sum_{i}^{n} X^i_j = S^j \tag{3.2}
\]

\[
\sum_{i}^{n} Y^i_j = S^j \tag{3.3}
\]

\[
X^i_j \geq 0, Y^i_j \geq 0, i = 1, 2, ..., n \tag{3.4}
\]

\( Q^j \) is symmetric 2nx2n variance-covariance matrix associated with assets and liabilities of sector \( j \). In addition, each sector’s uncertainty or assessment of risk is based on a variance-covariance matrix with the respect to the future value of the portfolio. Partition the matrix \( Q^j \) as

\[
Q^j = \begin{pmatrix} Q_{11}^j & Q_{12}^j \\ Q_{21}^j & Q_{22}^j \end{pmatrix} \tag{3.5}
\]

where \( Q_{11}^j \) and \( Q_{22}^j \) are the variance-covariance matrices for only the assets and only
the liabilities, respectively, of sector $j$. Each of sub-matrices have dimension $n \times n$. The sub-matrices $Q^j_{12}$ and $Q^j_{21}$ are identical since $Q^j$ is symmetric.

Expectations of each sector are formed by reference to current market activity and according to that sector utility maximization can be written in terms of optimizing the current portfolio. Sectors can trade, liquidate or issue holdings in order to optimize their portfolio compositions.

### 3.1.1 A Sector’s Portfolio Optimization Problem

In the chapter 2, the minimization of a portfolio’s risk is done through the use of variance-covariance matrix. In multisector problem, we are applying the same method:

$$
\text{minimize } \left( \begin{array}{c}
X^j \\
Y^j
\end{array} \right)^T Q^j \left( \begin{array}{c}
X^j \\
Y^j
\end{array} \right) - \sum_{i=1}^{n} (X_i^j - Y_i^j) 
$$

subject to

$$
\sum_{i}^n X_i^j = S^j 
$$

$$
\sum_{i}^n Y_i^j = S^j 
$$

for $X_i^j \geq 0$ and $Y_i^j \geq 0$, $i = 1, ..., n$.

$Q^j$ is a symmetric $2n \times 2n$ variance-covariance matrix associated with assets and liabilities of sector $j$.

$$
Q^j = \begin{pmatrix}
Q^j_{11} & Q^j_{12} \\
Q^j_{21} & Q^j_{22}
\end{pmatrix} 
$$

$Q^j_{(\alpha,\beta)}$ denotes the $i$-th column of $Q^j_{(\alpha,\beta)}$ with $\alpha = 1, 2$ and $\beta = 1, 2$. 
3.1.2 Optimality Conditions

In order to have an optimal portfolio the following system of equalities and inequalities must be satisfied for each instrument $i$:

\[
2(Q_{(11)i})^T \cdot X^j - 2(Q_{(21)i})^T \cdot Y^j - r^*_i - \mu^*_j \geq 0 \quad (3.10)
\]

\[
2(Q_{(22)i})^T \cdot Y^j - 2(Q_{(12)i})^T \cdot X^j + r^*_i - \mu^*_j \geq 0 \quad (3.11)
\]

\[
X^j \cdot [2(Q_{(11)i})^T \cdot X^j - 2(Q_{(21)i})^T \cdot Y^j - r^*_i - \mu^*_j] = 0 \quad (3.12)
\]

\[
Y^j \cdot [2(Q_{(22)i})^T \cdot Y^j + 2(Q_{(12)i})^T \cdot X^j + r^*_i - \mu^*_j] = 0 \quad (3.13)
\]

where $(X^j, Y^j) \in K_j$ is the vector of assets and liabilities for sector $j$. $r^*_i$ is the price for instrument $i$. $K_j$ is an achievable set for sector $j$ given by (3.7)-(3.8). $\mu^*_j$ and $\mu^*_j$ are Lagrange multipliers associated with (3.7)-(3.8). The achievable set for all asset and liability holdings of all the sectors and prices of the instruments is denoted by $K$ where $K = \{K \times R^n\}$ and $K = \sum_{i=1}^{m} K_i$.

Similar system of inequalities and equalities will hold for each of $m$ sectors.

3.1.3 Economic System Conditions

The economic system conditions must be satisfied in order to insure market clearance. For each instrument $i$, $i = 1, ..., n$:

\[
\sum_{j=1}^{m} (X^j - Y^j) \begin{cases} = 0 & , r^*_i > 0 \\ \geq 0 & , r^*_i = 0 \end{cases}
\]

According to above, the market must clear for the instrument, if the price is positive. On the other hand, if there is an excess of supply of an instrument, the price must be equal to zero. The combination of optimality conditions and economic system conditions we get the following definition of an equilibrium:

**Definition 2 Multisector, Multi-instrumental Financial Equilibrium**

A vector $(X^*, Y^*, r^*) \in K$ is an equilibrium of financial model if and only if it satisfies the optimality conditions and the economic system condition, for all sectors $j$, $j = 1, ..., m$ and for all instruments $i$, $i = 1, ..., n$. 
Chapter 3 General Financial Equilibrium Problems

The variational inequality formulation of the equilibrium conditions is according to Nagurney et al. (1992) given by the following:

**Theorem 1 Variational Inequality Formulation of Financial Equilibrium** A vector of assets and liabilities of the sectors and instrument prices, is financial equilibrium if and only if it satisfies the variational inequality problem:

\[
\begin{align*}
\sum_{j=1}^{m} \sum_{i=1}^{n} & \left(2(Q_{(11)j})^T \cdot X^j i^* \right) x \left[ X_i^j - X_i^* \right] \\
+ \sum_{j=1}^{m} \sum_{i=1}^{n} & \left(2(Q_{(22)j})^T \cdot Y^j i^* \right) x \left[ Y_i^j - Y_i^* \right] \\
+ \sum_{j=1}^{m} \sum_{i=1}^{n} & \left[ X_i^j - Y_i^* \right] x \left[ r_i^j - r_i^* \right] \geq 0
\end{align*}
\]  

Let \( Z = (X, Y, r) \in \mathcal{K} \) be a \( N \)-dimensional column vector and \( F(Z) \) a \( N \)-dimensional column vector such that:

\[
F(Z) = D \begin{pmatrix} X \\ Y \\ r \end{pmatrix}
\]

where

\[
D = \begin{pmatrix} 2Q & B \\ -B^T & 0 \end{pmatrix}
\]

and

\[
B^T = (-I... - II...I)_{n \times mn}
\]

and \( I \) is the \( n \times n \)-dimensional identity matrix. The variational inequality problem from above (3.14)-(3.16) can be put in standard variational inequality form (Definition 1):

Determine \( Z^* \in \mathcal{K} \): \( \langle F(Z^*)^T, Z - Z^* \rangle \geq 0 \) for all \( Z \in \mathcal{K} \).
3.2 Model with Utility Functions

Each sector wants to maximize its utility where utility function is given by:

\[ U^j(X^j, Y^j, r) = u^j(X^j, Y^j) + r^T(X^j - Y^j) \]  \hspace{1cm} (3.20)

Where:

\[ u^j(x^j, y^j) = -\left( \begin{array}{c} x^j \\ y^j \end{array} \right)^T Q^j \left( \begin{array}{c} x^j \\ y^j \end{array} \right) \]  \hspace{1cm} (3.21)

The optimization problem for sector \( j \) described with:

\[ \text{maximize} \sum_{j=1}^{m} u^j(X^j, Y^j) \]  \hspace{1cm} (3.22)

subject to

\[ \sum_{j=1}^{m} (X^j - Y^j) = 0 \]  \hspace{1cm} (3.23)

for \( i = 1, \ldots, n \) and \( (X^j, Y^j) \in K_j, j = 1, \ldots, m \).

3.3 Computation of Financial Equilibria

In this section, we are going to see the algorithm for the computation of solutions of financial equilibrium problems. The modified projection method (Korpelovich, 1977) is represented in following way:

**Step 0: Initialization**
Set \( Z^0 \in K \). Let \( \theta = 0 \) and \( \gamma \) be a scalar such that \( 0 < \gamma \geq \frac{1}{L} \). \( L \) is the Lipschitz constant.

**Step 1: Computation**
Compute \( Z^{\theta + 1} \) through the solution of the variational inequality problem:

\[ \langle (Z^{\theta} + \gamma F(Z^{\theta})^T - Z^{\theta})^T, Z - Z^{\theta} \rangle \geq 0 \]  \hspace{1cm} (3.24)

**Step 2: Adaptation**
Compute \( Z^{\theta + 1} \) through the solution of the variational inequality problem:

\[ \langle (Z^{\theta + 1} + \gamma F(Z^{\theta})^T - Z^{\theta})^T, Z - Z^{\theta + 1} \rangle \geq 0 \]  \hspace{1cm} (3.25)
Step 3: Convergence Verification

If $\max|Z_{b}^{\theta+1} - Z_{b}^{\theta}| \geq \epsilon$ for all $b$ and $\epsilon > 0$ for all $b$ and $\epsilon > 0$ then stop, else set $\theta = \theta + 1$ and go back to the step 1.
Chapter 3 General Financial Equilibrium Problems

Now we are going to expand $F(Z)$ to our specific model:

**Step 0: Initialization**
Set $(X^0, Y^0, r^0) \in \mathcal{K}$. Let $\theta = 0$ and $0 < \gamma \geq \frac{1}{L}$.

**Step 1: Computation**
Compute $(\bar{X}^\theta, \bar{Y}^\theta, \bar{r}^\theta) \in \mathcal{K}$ through the solution of the variational inequality problem:

$$
\sum_{j=1}^{J} \sum_{i=1}^{I} \left[ \bar{X}_i^\theta + \gamma \left( - \frac{\partial U^j(X^\theta, Y^\theta, r^\theta)}{\partial X^j_i} \right) - X_i^\theta \right] \times [X_i^j - \bar{X}_i^\theta] \quad (3.26)
$$

$$
+ \sum_{j=1}^{J} \sum_{i=1}^{I} \left[ \bar{Y}_i^\theta + \gamma \left( - \frac{\partial U^j(X^\theta, Y^\theta, r^\theta)}{\partial Y^j_i} \right) - Y_i^\theta \right] \times [Y_i^j - \bar{Y}_i^\theta] \quad (3.27)
$$

$$
+ \sum_{i=1}^{I} \left[ \bar{r}_i^\theta + \gamma \left( \sum_{j=1}^{J} (X_i^j - Y_i^j) \right) - r_i^\theta \right] \times [r_i - \bar{r}_i^\theta] \quad (3.28)
$$

**Step 2: Adaptation**
Compute $(X^{\theta+1}, Y^{\theta+1}, r^{\theta+1}) \in \mathcal{K}$ through the solution of the variational inequality problem:

$$
\sum_{j=1}^{J} \sum_{i=1}^{I} \left[ \bar{X}_i^{\theta+1} + \gamma \left( - \frac{\partial U^j(X^\theta, Y^\theta, r^\theta)}{\partial X^j_i} \right) - X_i^\theta \right] \times [X_i^j - \bar{X}_i^{\theta+1}] \quad (3.29)
$$

$$
+ \sum_{j=1}^{J} \sum_{i=1}^{I} \left[ \bar{Y}_i^{\theta+1} + \gamma \left( - \frac{\partial U^j(X^\theta, Y^\theta, r^\theta)}{\partial Y^j_i} \right) - Y_i^\theta \right] \times [Y_i^j - \bar{Y}_i^{\theta+1}] \quad (3.30)
$$

$$
+ \sum_{i=1}^{I} \left[ \bar{r}_i^{\theta+1} + \gamma \left( \sum_{j=1}^{J} (X_i^j - Y_i^j) \right) - r_i^\theta \right] \times [r_i - \bar{r}_i^{\theta+1}] \quad (3.31)
$$

**Step 3: Convergence**
If $\max|X_i^{\theta+1} - X_i^\theta| \geq \epsilon$, $\max|Y_i^{\theta+1} - Y_i^\theta| \geq \epsilon$, $\max|r_i^{\theta+1} - r_i^\theta| \geq \epsilon$ for all $i = 1, ..., I$ and $j = 1, ..., J$ and $\epsilon > 0$ then stop, else set $\theta = \theta + 1$ and go back to the step 1.
Convergence results led us to the following theorem \(^1\):

**Theorem 2** The modified projection method is guaranteed to converge to the solutions of variational inequality.

According to Nagurney and Skios (1996), the proposed model bring us the optimal composition of assets and liabilities in the portfolio. This model counts in every sector of each country and currency. As we have already seen, the equilibrium conditions are presented in order to satisfy a variational inequality problem. Then, the mathematical framework is used in order to derive properties of the equilibrium asset, liability, instrument price and exchange rate. At the end, the modified projection algorithm together with the convergence results is proposed for computation.

\(^1\)Nagurney and Skios (1996)
Bibliography


