Volatility Smile

Heston, SABR

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- Calibration of the FX Heston Model
Black Scholes SDE
The stock price follows a geometric Brownian motion with constant drift and volatility.

\[ dS_t = \mu S \, dt + \sigma S \, dW_t \]

- Under the risk neutral pricing measure \( \mathbb{Q} \) we have \( \mu = r_f \)
- One can perfectly hedge an option by buying and selling the underlying asset and the bank account dynamically

The BSM option’s value is a \textit{monotonic increasing} function of implied volatility c.p.

\[ C_t = S_t \Phi \left( \frac{\ln \left( \frac{S}{K} \right) + (r + \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}} \right) - Ke^{-r(T-t)} \Phi \left( \frac{\ln \left( \frac{S}{K} \right) + (r - \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}} \right) \]
The **implied volatility** $\sigma_{imp}$ is that the Black Scholes option model price $C^{BS}$ equals the option’s market price $C^{mkt}$.

$$C^{BS}(S, K, \sigma_{imp}, r_f, t, T) = C^{mkt}$$
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Definition

Stochastic Volatility Model

\[
\begin{align*}
\quad dS_t &= \mu S_t dt + \sqrt{\nu_t} S_t dW_t^S \\
\quad d\nu_t &= \kappa (\theta - \nu_t) dt + \sigma \sqrt{\nu_t} dW_t^\nu \\
\quad dW_t^S \, dW_t^\nu &= \rho dt
\end{align*}
\]

The parameters in this model are:

- \(\mu\) the drift of the underlying process
- \(\kappa\) the speed of mean reversion for the variance
- \(\theta\) the long term mean level for the variance
- \(\sigma\) the volatility of the variance
- \(\nu_0\) the initial variance at \(t = 0\)
- \(\rho\) the correlation between the two Brownian motions
Sample Paths

Path simulation of the Heston model and the geometric Brownian motion.
As we know the payoff of a European plain vanilla call option to be

\[ C_T = (S_T - K)^+ \]

we can generally write the price of the option to be at any time point \( t \in [0, T] \):

\[
C_t = e^{-r(T-t)} \mathbb{E}\left[ (S_T - K)^+ \bigg| \mathcal{F}_t \right] \\
= e^{-r(T-t)} \mathbb{E}\left[ (S_T - K) \mathbf{1}_{S_T>K} \bigg| \mathcal{F}_t \right] \\
= e^{-r(T-t)} \mathbb{E}\left[ S_T \mathbf{1}_{S_T>K} \bigg| \mathcal{F}_t \right] - e^{-r(T-t)} K \mathbb{E}\left[ \mathbf{1}_{S_T>K} \bigg| \mathcal{F}_t \right] \\
= :(*), \quad \underbrace{=} :(**)
\]
With constant interest rates the stochastic discount factor using the bank account $B_t$ then becomes $1/B_t = e^{-\int_0^t r_s \, ds} = e^{-rt}$. We now need to perform a *Radon-Nikodym* change of measure.

$$Z_t = \frac{dQ}{dP} |_F = \frac{S_t}{B_t} \frac{B_T}{S_T}$$

Thus the first term $(\ast)$ gets

$$(\ast) = e^{-r(T-t)} \mathbb{E}^P \left[ S_T 1(S_T > K) | F_t \right]$$

$$= \frac{B_t}{B_T} \mathbb{E}^P \left[ S_T 1(S_T > K) | F_t \right]$$

$$= \frac{B_t}{B_T} \mathbb{E}^Q \left[ Z_t S_T 1(S_T > K) | F_t \right]$$

$$= \frac{B_t}{B_T} \mathbb{E}^Q \left[ \frac{S_t}{B_t} \frac{B_T}{S_T} S_T 1(S_T > K) | F_t \right]$$

$$= \mathbb{E}^Q \left[ S_t 1(S_T > K) | F_t \right]$$

$$= S_t \mathbb{E}^Q \left[ 1(S_T > K) | F_t \right]$$

$$= S_t Q(S_T > K | F_t)$$
Get the distribution function

How to do ...

- Find the characteristic function
- Fourier Inversion theorem to get the probability distribution function

We apply the *Fourier Inversion Formula* on the characteristic function

\[
F_X(x) - F_X(0) = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{iux} - 1}{-iu} \varphi_X(u) \, du
\]

and use the solution of *Gil-Pelaez* to get the nicer real valued solution of the transformed characteristic function:

\[
P(X > x) = 1 - F_X(x) = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \Re \left[ \frac{e^{-iux}}{iu} \varphi_X(u) \right] \, du
\]
The Heston PDE

We apply the Ito-formula to expand $dU(S, \nu, t)$:

$$dU = U_t dt + U_S dS + U_\nu d\nu + \frac{1}{2} U_{SS} (dS)^2 + U_{S\nu} (dS d\nu) + \frac{1}{2} U_{\nu \nu} (d\nu)^2$$

With the quadratic variation and covariation terms expanded we get

$$(dS)^2 = d \langle S \rangle = \nu S^2 d \left\langle W^S \right\rangle = \nu S^2 dt,$$

$$(dS d\nu) = d \langle S, \nu \rangle = \nu S \sigma d \left\langle W^S, W^\nu \right\rangle = \nu S \sigma \rho dt,$$

and

$$(d\nu)^2 = d \langle \nu \rangle = \sigma^2 \nu d \left\langle W^\nu \right\rangle = \sigma^2 \nu dt.$$ 

The other terms including $d \langle t \rangle, d \langle t, W^\nu \rangle, d \langle t, W^S \rangle$ are left out, as the quadratic variation of a finite variation term is always zero and thus the terms vanish. Thus

$$dU = U_t dt + U_S dS + U_\nu d\nu + \frac{1}{2} U_{SS} \nu S dt + U_{S\nu} \nu S \sigma \rho dt + \frac{1}{2} U_{\nu \nu} \sigma^2 \nu dt$$

$$= \left[ U_t + \frac{1}{2} U_{SS} \nu S + U_{S\nu} \nu S \sigma \rho + \frac{1}{2} U_{\nu \nu} \sigma^2 \nu \right] dt + U_S dS + U_\nu d\nu$$
As in the BSM portfolio replication also in the Heston model you get your portfolio PDE via dynamic hedging, but we have a portfolio consisting of:

- one option \( V(S, \nu, t) \)
- a portion of the underlying \( \Delta S_t \) and
- a third derivative to hedge the volatility \( \phi U(S, \nu, t) \).

\[
\frac{1}{2} \nu U_{XX} + \rho \sigma \nu U_{X\nu} + \frac{1}{2} \sigma^2 \nu U_{\nu\nu} + \left( r - \frac{1}{2} \nu \right) U_X + \\
+ \left[ \kappa(\theta - \nu_t) - \lambda_0 \nu_t \right] U_\nu - r U - U_\tau = 0
\]

where \( \lambda_0 \nu_t \) is the *market price of volatility risk.*
Heston assumed the characteristic function to be of the form

$$\varphi^i_{x,\tau}(u) = \exp \left( C_i(u, \tau) + D_i(u, \tau) \nu_t + iux \right)$$

The pricing PDE is always fulfilled irrespective of the terms in the call contract.

- $S = 1, K = 0, r = 0 \Rightarrow C_t = P_1$
- $S = 0, K = 1, r = 0 \Rightarrow C_t = -P_2$

We have to set up the boundary conditions we know to solve the PDE:

$$C(T, \nu, S) = \max(S_T - K, 0)$$
$$C(t, \infty, S) = Se^{-r(T-t)}$$
$$\frac{\partial C}{\partial S}(t, \nu, \infty) = 1$$
$$C(t, \nu, 0) = 0$$
$$rC(t, 0, S) = \left[ rS\frac{\partial C}{\partial S} + \kappa \theta \frac{\partial C}{\partial \nu} + \frac{\partial C}{\partial t} \right](t, 0, S)$$

The *Feynman-Kac theorem* ensures that then also the characteristic function follows the Heston PDE.
Recall that we have a pricing formula of the form

$$C_t = S_t P_1(S_t, \nu_t, \tau) - e^{-r(T-t)} K P_2(S_t, \nu_t, \tau)$$

where the two probabilities $P_j$ are

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[ \frac{e^{-iux}}{iu} \varphi^j_X(u) \right] du$$

with the characteristic function being of the form

$$\varphi_j(u) = e^{C_j(\tau,u) + D_j(\tau,u)\nu_t + iux}.$$
The exchange rate process $Q_t$ is the price of units of domestic currency for 1 unit of the foreign currency and is described under the actual probability measure $\mathbb{P}$ by

$$dQ_t = \mu Q_t dt + \sigma Q_t dW_t$$

Let us now consider an auxiliary process $Q^*_t := Q_t B^f_t / B^d_t$ which then of course satisfies

$$Q^*_t = \frac{Q_t B^f_t}{B^d_t}$$

$$= Q_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t} e^{(r_f - r_d)t}$$

$$= Q_0 e^{\left(\mu r_f - r_d - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

Thus we can clearly see that $Q^*_t$ is a martingale under the original measure $\mathbb{P}$ iff $\mu = r_d - r_f$. 
If we now assume that the underlying process $(Q_t)$ is now the exchange rate we still have the final payoff for a Call option of the form

$$FXC_T = \max(Q_T - K, 0)$$

and following the Garman-Kohlhagen model we know that the price of the FX option gets

$$FXC_t = e^{-r_f(T-t)} Q_t P_{1}^{FX}(Q_t, \nu_t, \tau) - e^{-r_d(T-t)} KP_{2}^{FX}(Q_t, \nu_t, \tau)$$
**Risk Reversal:**
Risk reversal is the difference between the volatility of the call price and the put price with the same moneyness levels.

\[ RR_{25} = \sigma_{25C} - \sigma_{25P} \]

**Butterfly:**
Butterfly is the difference between the average volatility of the call price and put price with the same moneyness level and at the money volatility level.

\[ BF_{25} = (\sigma_{25C} + \sigma_{25P})/2 - \sigma_{ATM} \]
Bloomberg FX Option Data
Bloomberg FX Option Data

USD/JPY and EUR/JPY volatility surface
Calibration to the Implied Volatility Surface

- **Implement the Heston Pricing procedure**
  - Characteristic function
  - Numerical integration algorithm
  - Heston pricer

- **BSM implied volatility from Heston prices**

- **Sum of squared errors minimisation algorithm**
  compare the market implied volatility $\hat{\sigma}$ with the volatility returned by the Heston model $\sigma(\kappa, \theta, \sigma, \nu_0, \rho)$

$$
\min_{\theta, \sigma, \rho} \left( \sum_{i,j} \left( \hat{\sigma} - \sigma(\kappa, \theta, \sigma, \nu_0, \rho) \right)^2 \right)
$$
Parameter Impacts

Recall

\[
\begin{align*}
\frac{dS_t}{S_t} &= \mu_t dt + \sqrt{\nu_t} S_t dW_t^S \\
\frac{d\nu_t}{\nu_t} &= \kappa (\theta - \nu_t) dt + \sigma \sqrt{\nu_t} dW_t^\nu \\
\rho dW_t^S dW_t^\nu &= \rho dt
\end{align*}
\]

\[\Rightarrow \quad \text{set} \quad \sqrt{\nu_0} = \sigma_{ATM}.\]
Parameter Impacts 2

\[
\begin{align*}
    dS_t &= \mu S_t dt + \sqrt{\nu_t} S_t dW^S_t \\
    d\nu_t &= \kappa (\theta - \nu_t) dt + \sigma \sqrt{\nu_t} dW^\nu_t \\
    dW^S_t dW^\nu_t &= \rho dt
\end{align*}
\]

⇒ use for \( \kappa \) fixed values depending on curvature. E.g. 0.5, 1.5, or 3.
The skew parameter $\rho$:

\[
\begin{align*}
    dS_t &= \mu S_t dt + \sqrt{\nu_t} S_t dW_t^S \\
    d\nu_t &= \kappa (\theta - \nu_t) dt + \sigma \sqrt{\nu_t} dW_t^{\nu} \\
    dW_t^S dW_t^{\nu} &= \rho dt
\end{align*}
\]

Parameter Analysis – rho

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USD/JPY and EUR/JPY volatility surface calibration

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USDJPY FX Option Volatility Smile

EURJPY FX Option Volatility Smile
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4. Conclusion

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Volatility Smile
The parameters are

- \( \alpha \) the initial variance,
- \( \nu \) the volatility of variance,
- \( \beta \) the exponent for the forward rate,
- \( \rho \) the correlation between the two Brownian motions.
The derivation is based on small volatility expansions, $\hat{\alpha}$ and $\nu$, re-written to $\hat{\alpha} \rightarrow \epsilon \hat{\alpha}$ and $\nu \rightarrow \epsilon \nu$ such that
\[
d\hat{F} = \epsilon \hat{\alpha} C(\hat{F}) dW_1, \\
d\hat{\alpha} = \epsilon \nu \hat{\alpha} dW_2
\]
with $dW_1 dW_2 = \rho dt$ in the distinguished limit $\epsilon \ll 1$ and $C(\hat{F})$ generalized. The probability density is defined as
\[
p(t, f, \alpha; T, F, A) dF dA = \text{Prob} \left\{ F < \hat{F}(T) < F + dF, A < \hat{\alpha}(T) < A + dA \\
| \hat{F}(t) = f, \hat{\alpha}(t) = \alpha \right\}.
\]
Then the density at maturity $T$ is defined as
\[
p(t, f, \alpha; T, F, A) = \delta(F - f) \delta(A - \alpha) + \int_t^T p_T(t, f, \alpha; T, F, A) dT
\]
with
\[
p_T = \frac{1}{2} \epsilon^2 A^2 \frac{\partial^2}{\partial F^2} C^2(F)p + \epsilon^2 \rho \nu \frac{\partial^2}{\partial F \partial A} A^2 C^2(F)p + \frac{1}{2} \epsilon^2 \nu^2 \frac{\partial^2}{\partial A^2} A^2 p.
\]
Let $V(t, f, \alpha)$ then be the value of an European call option at $t$ at above defined state of economy:

$$V(t, f, \alpha) = \mathbb{E} \left( [\hat{F}(T) - K]^+ \mid \hat{F}(t) = f, \hat{\alpha}(t) = \alpha \right)$$

$$= \int_{-\infty}^{\infty} \int_{K}^{\infty} (F - K) p(t, f, \alpha; T, F, A) dF dA$$

$$= [f - K]^+ + \int_{t}^{T} \int_{-\infty}^{\infty} \int_{K}^{\infty} (F - K) p_T(t, f, \alpha; T, F, A) dT$$

$$= [f - K]^+ + \frac{\epsilon^2}{2} \int_{t}^{T} \int_{-\infty}^{\infty} \int_{K}^{\infty} A^2(F - K) \frac{\partial^2}{\partial F^2} C^2(F) p dF dAdT$$

$$= [f - K]^+ + \frac{\epsilon^2 C^2(K)}{2} \int_{t}^{T} \int_{-\infty}^{\infty} A^2 p(t, f, \alpha; T, K, A) dAdT$$

$$\vdots$$

$$= [f - K]^+ + \frac{\epsilon^2 C^2(K)}{2} \int_{t}^{\tau} P(\tau, f, \alpha; K) d\tau$$
Where

\[ P(t, f, \alpha; T, K) = \int_{-\infty}^{\infty} A^2 p(t, f, \alpha; T, K, A) dA \]

and \( P(\tau, f, \alpha; K) \) is the solution of

\[
P_{\tau} = \frac{1}{2} \epsilon^2 \alpha^2 C^2(f) \frac{\partial^2 P}{\partial f^2} + \epsilon^2 \rho \nu \alpha^2 C(f) \frac{\partial^2 P}{\partial f \partial \alpha} + \frac{1}{2} \epsilon^2 \nu^2 \alpha^2 \frac{\partial^2 P}{\partial \alpha^2}, \quad \text{for} \ \tau > 0,
\]

\[ P = \alpha^2 \delta(f - K), \quad \text{for} \ \tau = 0. \]

with \( \tau = T - t \).

Given these results one could obtain the option formula directly. However more useful formulas can be derived through

1. Singular perturbation expansion
2. Equivalent normal volatility
3. Equivalent Black volatility
4. Stochastic \( \beta \) model
Singular perturbation expansion

The goal is to use perturbation expansion methods which yield a Gaussian density of the form

$$P = \frac{\alpha}{\sqrt{2\pi \epsilon^2 C^2(K)\tau}} e^{-\frac{(f-K)^2}{2\epsilon^2\alpha^2 C^2(K)\tau}} \{1+\ldots\}.$$ 

Consequently, the singular perturbation expansion yields a European call option value

$$V(t, f, \alpha) = [f - K]^+ + \frac{|f - K|}{4\sqrt{\pi}} \int_{\frac{x^2}{2\tau} - \epsilon^2 \theta}^{\infty} \frac{e^{-q}}{q^{3/2}} dq$$

with

$$x = \frac{1}{\epsilon \nu} \log \left( \frac{\sqrt{1 - 2\epsilon \rho \nu z + \epsilon^2 \nu^2 z^2} - \rho + \epsilon \nu z}{1 - \rho} \right), \quad z = \frac{1}{\epsilon \alpha} \int_K^f \frac{df'}{C(f')},$$

$$\epsilon^2 \theta = \log \left( \frac{\epsilon \alpha z}{f - K} \sqrt{B(0)B(\epsilon \alpha z)} \right) + \log \left( \frac{x I^{1/2}(\epsilon \nu z)}{z} \right) + \frac{1}{4} \epsilon^2 \rho \nu \alpha b_1 z^2.$$
Equivalent normal volatility

Suppose the previous analysis is repeated under the normal model

\[ d\hat{F} = \sigma_N \, dW, \quad \hat{F}(0) = f. \]

with \( \sigma_N \) constant, not stochastic. The option value would then be

\[
V(t, f) = [f - K]^+ + \left| f - K \right| \int_0^\infty \frac{e^{-q/2}}{\sqrt{\pi} \, q^{3/2}} \, dq
\]

for \( C(f) = 1, \epsilon \alpha = \sigma_N \) and \( \nu = 0 \). Integration yields then

\[
V(t, f) = (f - K) \Phi \left( \frac{f - K}{\sigma_N \sqrt{\tau}} \right) + \sigma_N \sqrt{\tau} \mathcal{G} \left( \frac{f - K}{\sigma_N \sqrt{\tau}} \right)
\]

with the Gaussian density \( \mathcal{G} \)

\[
\mathcal{G}(q) = \frac{1}{\sqrt{2\pi}} e^{-q^2/2}.
\]
The option price under the normal model matches the option price under the SABR model, iff $\sigma_N$ is chosen the way that

$$\sigma_N = \frac{f - K}{x} \left\{ 1 + \epsilon^2 \frac{\theta}{x^2} \tau + \cdots \right\}$$

through $O(\epsilon^2)$. Simplifying yields the implied normal volatility

$$\sigma_N(K) = \frac{\epsilon \alpha (f - K)}{\int_K^f \frac{df'}{C(f')}} \left( \frac{\zeta}{\hat{x}(\zeta)} \right)$$

$$\cdot \left\{ 1 + \left[ \frac{2\gamma_2 - \gamma_1^2}{24} \alpha^2 C^2(f_{av}) + \frac{1}{4} \rho \nu \alpha \gamma_1 C(f_{av}) + \frac{2 - 3\rho^2}{24} \nu^2 \right] \epsilon^2 \tau + \cdots \right\}$$

with

$$f_{av} = \sqrt{fK}, \quad \gamma_1 = \frac{C''(f_{av})}{C(f_{av})}, \quad \gamma_2 = \frac{C'''(f_{av})}{C(f_{av})}$$

$$\zeta = \frac{\nu(f - K)}{\alpha C(f_{av})}, \quad \hat{x}(\zeta) = \log \left( \frac{\sqrt{1 - 2\rho \zeta + \zeta^2} - \rho + \zeta}{1 - \rho} \right).$$
Equivalent Black volatility

To derive the implied volatility consider again Black’s model

\[ d\hat{F} = \varepsilon\sigma_B \hat{F}dW, \quad \hat{F}(0) = f \]

with \( \varepsilon\sigma_B \) for consistency of the analysis. The implied normal volatility for Black’s model for SABR can be obtained by setting \( C(f) = f \) and \( \nu = 0 \) in previous results such that

\[ \sigma_N(K) = \frac{\varepsilon\sigma_B (f - K)}{\log \frac{f}{K}} \left\{ 1 - \frac{1}{24} \varepsilon^2 \sigma_B^2 \tau + \cdots \right\}. \]

through \( O(\varepsilon^2) \). Solving the equation for \( \sigma_B \) yields

\[
\begin{align*}
\sigma_B(K) &= \frac{\alpha \log \frac{f}{K}}{\int_{K}^{f} \frac{df'}{C(f')}} \left( \frac{\zeta}{\hat{x}(\zeta)} \right) \\
&\cdot \left\{ 1 + \left[ \frac{2\gamma_2 - \gamma_1^2 + \frac{1}{f_{av}}}{24} \alpha^2 C^2(f_{av}) + \frac{1}{4} \rho \nu \alpha \gamma_1 C(f_{av}) + \frac{2 - 3\rho^2}{24} \nu^2 \right] \varepsilon^2 \tau + \cdots \right\}.
\end{align*}
\]
Finally, let’s look at the original state with $C(f) = f^\beta$. Making the substitutions as previously and following approximations

$$f - K = \sqrt{fK} \log f/K \{1 + \frac{1}{24} \log^2 f/K + \frac{1}{1920} \log^4 f/K + \cdots \},$$

$$f^{1-\beta} - K^{1-\beta} = (1 - \beta)(fK)^{(1-\beta)/2} \log f/K \{1 + \frac{(1 - \beta)^2}{24} \log^2 f/K + \frac{(1 - \beta)^4}{1920} \log^4 f/K + \cdots \},$$

the implied normal volatility reduces to

$$\sigma_N(K) = \epsilon \alpha(fK)^{\beta/2} \left[ 1 + \frac{1}{24} \log^2 f/K + \frac{1}{1920} \log^4 f/K + \cdots \right] \left( \frac{\zeta}{\hat{x}(\zeta)} \right)$$

$$\cdot \left\{ 1 + \left[ -\beta(2 - \beta)\alpha^2 + \frac{\rho\alpha\nu\beta}{24(fK)^{1-\beta}} + \frac{2 - 3\rho^2}{24\nu^2} \right] \epsilon^2 \tau + \cdots \right\}$$

with $\zeta = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2} \log f/K$. Setting $\epsilon = 1$ one gets ...
The implied volatility $\sigma_B(f, K)$ is given by

$$\sigma_B(K, f) = \frac{\alpha}{(fK)^{(1-\beta)/2}} \left\{ 1 + \frac{(1-\beta)^2}{24} \log^2 \frac{f}{K} + \frac{(1-\beta)^4}{1920} \log^4 \frac{f}{K} \right\} \cdot \left( \frac{z}{x(z)} \right) \cdot \left\{ 1 + \left[ \frac{(1-\beta)^2}{24} \frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4} \frac{\rho \beta \nu \alpha}{(fK)^{(1-\beta)/2}} + \frac{2 - 3\rho^2}{24} \nu^2 \right] T \right\}$$

where $z$ is defined by

$$z = \frac{\nu}{\alpha} (fK)^{(1-\beta)/2} \log \frac{f}{K}$$

and $x(z)$ is given by

$$x(z) = \log \left\{ \frac{\sqrt{1 - 2\rho z + z^2 + z - \rho}}{1 - \rho} \right\}.$$
For at-the-money options \((K = f)\) the formula reduces to
\[
\sigma_B(f, f) = \sigma_{ATM}\text{ such that }
\]
\[
\sigma_{ATM} = \alpha \left\{ 1 + \left[ \frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^2-2\beta} + \frac{1}{4} \frac{\rho \beta \nu \alpha}{f(1-\beta)} + \frac{2-3\rho^2}{24} \nu^2 \right] T \right\}. 
\]
Approximate the model with $\lambda = \frac{\nu}{\alpha} f^{1-\beta}$ such that

$$\sigma_B(K, f) = \frac{\alpha}{f^{1-\beta}} \left\{ 1 - \frac{1}{2} (1 - \beta - \rho \lambda) \log \frac{K}{f} ight. \right.$$

$$+ \left. \frac{1}{12} \left[ (1 - \beta)^2 + (2 - 3 \rho^2) \lambda^2 \right] \log^2 \frac{K}{f} \right\},$$

The SABR model is then described with

- **Backbone:** $\frac{\alpha}{f^{1-\beta}}$
- **Skew:** $-\frac{1}{2} (1 - \beta - \rho \lambda) \log \frac{K}{f}$, $\frac{1}{12} (1 - \beta)^2 \log^2 \frac{K}{f}$
- **Smile:** $\frac{1}{12} (2 - 3 \rho^2) \log^2 \frac{K}{f}$
For estimation of the SABR model the estimation of $\beta$ is used as a starting point.

With $\beta$ estimated, there are two possible choices to continue calibration:

1. Estimate $\alpha$, $\rho$ and $\nu$ directly, or
2. Estimate $\rho$ and $\nu$ directly, and infer $\alpha$ from $\rho$, $\nu$ and the at-the-money.

In general, it is more convenient to use the ATM volatility $\sigma_{ATM}$, $\beta$, $\rho$ and $\nu$ as the SABR parameters instead of the original parameters $\alpha$, $\beta$, $\rho$ and $\nu$. 
Estimation of $\beta$

For estimation of $\beta$ the at-the-money volatility $\sigma_{ATM}$ from equation is used

$$\log \sigma_{ATM} = \log \alpha - (1 - \beta) \log f +$$

$$\log \left\{ 1 + \left[ \frac{(1 - \beta)^2}{24} \frac{\alpha^2}{f^{2-2\beta}} + \frac{1}{4} \frac{\rho \beta \nu \alpha}{f^{(1-\beta)}} + \frac{2 - 3 \rho^2}{24} \nu^2 \right] T \right\}$$

$$\approx \log \alpha - (1 - \beta) \log f$$

Alternatively, $\beta$ can be chosen from prior beliefs of the appropriate model:

- $\beta = 1$: stochastic log-normal, for FX option markets
- $\beta = 0$: stochastic normal, for markets with zero or negative $f$
- $\beta = \frac{1}{2}$: CIR model, for interest rate markets
Estimation of $\alpha$, $\rho$ and $\nu$

Estimation of all three parameters by minimization of the errors between the model and the market volatilities $\sigma^\text{mkt}_i$ at identical maturity $T$.

Using the sum of squared errors (SSE)

$$(\hat{\alpha}, \hat{\rho}, \hat{\nu}) = \arg \min_{\alpha, \rho, \nu} \sum_i \left( \sigma^\text{mkt}_i - \sigma_B(f_i, K_i; \alpha, \rho, \nu) \right)^2.$$ 

is produced.
Estimation of $\rho$ and $\nu$

The number of parameters can be reduced by extracting $\alpha$ directly from $\sigma_{ATM}$. Thus, by inverting the equation the cubic equation is received

\[
\left(\frac{(1 - \beta)^2 T}{24 f^2 - 2\beta}\right) \alpha^3 + \left(\frac{1}{4} \frac{\rho \beta \nu T}{f(1-\beta)}\right) \alpha^2 + \left(1 + \frac{2 - 3\rho^2}{24} \nu^2 T\right) \alpha - \sigma_{ATM} f^{(1-\beta)} = 0.
\]

As it is possible to receive more than one single real root, it is suggested to select the smallest positive real root.

Given $\alpha$ the SSE

\[
(\hat{\alpha}, \hat{\rho}, \hat{\nu}) = \arg \min_{\alpha, \rho, \nu} \sum_i \left(\sigma_{i}^{\text{mkt}} - \sigma_B(f_i, K_i; \alpha(\rho, \nu), \rho, \nu)\right)^2
\]

has to be minimized for the $\rho$ and $\nu$. 

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Volatility Smile
Calibration

- Calibration for a fictional data set, with 15 implied market volatilities at maturity $T = 1$.

<table>
<thead>
<tr>
<th>Strike</th>
<th>Implied Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.14</td>
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<tr>
<td>0.06</td>
<td>0.18</td>
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<tr>
<td>0.09</td>
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<tr>
<td>0.11</td>
<td>0.22</td>
</tr>
</tbody>
</table>

- Market Implied Volatilities

- Difference by Parametrization

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<th>Implied Volatility</th>
</tr>
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<td>0.0025</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0030</td>
</tr>
</tbody>
</table>

- Difference by Parametrization

- 1.Param.
  - $\alpha = 0.139$
  - $\rho = -0.069$
  - $\nu = 0.578$
  - $\text{SSE} = 2.456 \cdot 10^{-4}$

  - $\alpha = 0.136$
  - $\rho = -0.064$
  - $\nu = 0.604$
  - $\text{SSE} = 2.860 \cdot 10^{-4}$

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- Volatility Smile
Parameter dynamics - \( \beta, \alpha \)

**Dynamics of \( \beta \)**

- \( \beta = 1 \)
- \( \beta = \frac{1}{2} \)
- \( \beta = 0 \)

**Dynamics of \( \alpha \)**

- \( \alpha \)
- \( \alpha + 5\% \)
- \( \alpha - 5\% \)
Parameter dynamics - $\rho, \nu$

**Dynamics of $\nu$**

- $\nu$
- $\nu + 15\%$
- $\nu - 15\%$

**Dynamics of $\rho$**

- $\rho = -0.06$
- $\rho = 0.25$
- $\rho = -0.25$

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Volatility Smile
SABR and FX Options- EUR/JPY

EURJPY FX Option Volatility Smile

- Delta
- Implied Volatility
- 10C, 25C, ATM, 25P, 10P

0.12, 0.14, 0.16, 0.18, 0.20
SABR and FX Options - USD/JPY

USDJPY FX Option Volatility Smile

EURJPY FX Option Volatility Smile

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Volatility Smile
Introduction

Heston Model

Derivation of the Heston Model

Summary for the Heston Model

FX Heston Model

Calibration of the FX Heston Model

SABR Model

Definition

Derivation

SABR Implied Volatility

Calibration

Conclusio
Observations and Facts

**Heston**
- its volatility structure permits analytical solutions to be generated for European options
- this model describes important mean-reverting property of volatility
- allows price dynamics to be of non-lognormal probability distributions
- the model does not perform well for short maturities
- parameters after calibration to market data turn out to be non-constant

**SABR**
- simple stochastic volatility model; as only one formula
- no derivation of prices, comparision directly via implied volatility
- no time dependency implemented
- interpolation erroneous and inaccurate (e.g. shifts)