Volatility Smile

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Black Scholes Framework

Black Scholes SDE

The stock price follows a geometric Brownian motion with constant drift and volatility.

$$dS_t = \mu S\,dt + \sigma S\,dW_t$$

- Under the risk neutral pricing measure $\mathbb{Q}$ we have $\mu = r_f$
- One can perfectly hedge an option by buying and selling the underlying asset and the bank account dynamically

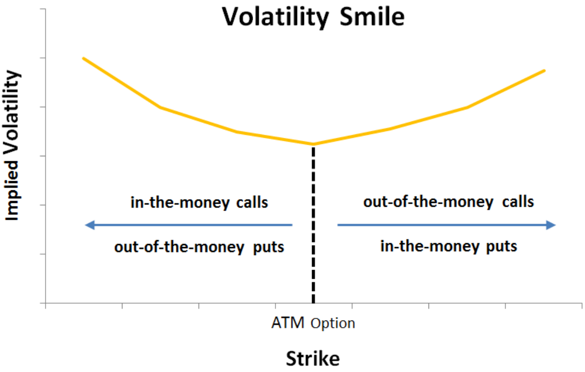
The BSM option's value is a *monotonic increasing* function of implied volatility c.p.

$$C_t = S_t\Phi\left(\frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}\Phi\left(\frac{\ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right)$$

Black Scholes Implied Volatility

The **implied volatility** σ_{imp} is that the Black Scholes option model price C^{BS} equals the option's market price C^{mkt}.

$$C^{BS}(S, K, \sigma_{imp}, r_f, t, T) = C^{mkt}$$



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Definition

Stochastic Volatility Model

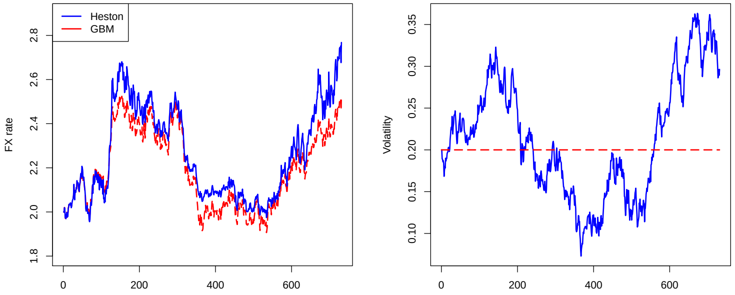
$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{\nu_t} S_t dW_t^S \\ d\nu_t &= \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t} dW_t^\nu \\ dW_t^S dW_t^\nu &= \rho dt \end{aligned}$$

The parameters in this model are:

- μ the drift of the underlying process
- κ the speed of mean reversion for the variance
- θ the long term mean level for the variance
- σ the volatility of the variance
- ν_0 the initial variance at $t = 0$
- ρ the correlation between the two Brownian motions

Sample Paths

Path simulation of the Heston model and the geometric Brownian motion.



Derivation of the Heston Model

As we know the payoff of a European plain vanilla call option to be

$$C_T = (S_T - K)^+$$

we can generally write the price of the option to be at any time point $t \in [0, T]$:

$$\begin{aligned} C_t &= e^{-r(T-t)}\mathbb{E}\left[(S_T - K)^+ \middle| \mathcal{F}_t\right] \\ &= e^{-r(T-t)}\mathbb{E}\left[(S_T - K)\mathbf{1}_{(S_T>K)} \middle| \mathcal{F}_t\right] \\ &= \underbrace{e^{-r(T-t)}\mathbb{E}\left[S_T\mathbf{1}_{(S_T>K)} \middle| \mathcal{F}_t\right]}_{=:(*)} - \underbrace{e^{-r(T-t)}K\mathbb{E}\left[\mathbf{1}_{(S_T>K)} \middle| \mathcal{F}_t\right]}_{=:(**)} \end{aligned}$$

With constant interest rates the stochastic discount factor using the bank account B_t then becomes $1/B_t = e^{-\int_0^t r_s ds} = e^{-rt}$. We now need to perform a *Radon-Nikodym* change of measure.

$$\mathbb{Z}_t = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{F}_t = \frac{S_t}{B_t}\frac{B_T}{S_T}$$

Thus the first term $(*)$ gets

$$\begin{aligned}
(*) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{P}}\left[S_T \mathbf{1}_{(S_T>K)}\Big|\mathcal{F}_t\right] \\
&= \frac{B_t}{B_T}\mathbb{E}^{\mathbb{P}}\left[S_T \mathbf{1}_{(S_T>K)}\Big|\mathcal{F}_t\right] \\
&= \frac{B_t}{B_T}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{Z}_t S_T \mathbf{1}_{(S_T>K)}\Big|\mathcal{F}_t\right] \\
&= \frac{B_t}{B_T}\mathbb{E}^{\mathbb{Q}}\left[\frac{S_t}{B_t}\frac{B_T}{S_T} S_T \mathbf{1}_{(S_T>K)}\Big|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[S_t \mathbf{1}_{(S_T>K)}\Big|\mathcal{F}_t\right] \\
&= S_t\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{(S_T>K)}\Big|\mathcal{F}_t\right] \\
&= S_t\mathbb{Q}\left(S_T > K|\mathcal{F}_t\right)
\end{aligned}$$

Get the distribution function

How to do ...

- Find the characteristic function
- Fourier Inversion theorem to get the probability distribution function

We apply the *Fourier Inversion Formula* on the characteristic function

$$F_X(x) - F_X(0) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{iux} - 1}{-iu} \varphi_X(u) du$$

and use the solution of *Gil-Pelaez* to get the nicer real valued solution of the transformed characteristic function:

$$\mathbb{P}(X > x) = 1 - F_X(x) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re\left[\frac{e^{-iux}}{iu}\varphi_X(u)\right] du$$

The Heston PDE

We apply the Ito-formula to expand $dU(S,\nu,t)$:

$$dU = U_t dt + U_S dS + U_\nu d\nu + \frac{1}{2}U_{SS}(dS)^2 + U_{S\nu}(dSd\nu) + \frac{1}{2}U_{\nu\nu}(d\nu)^2$$

With the quadratic variation and covariation terms expanded we get

$$\begin{aligned} (dS)^2 &= d\langle S\rangle = \nu S^2 d\left\langle W^S\right\rangle = \nu S^2 dt, \\ (dSd\nu) &= d\langle S,\nu\rangle = \nu S\sigma d\left\langle W^S, W^\nu\right\rangle = \nu S\sigma\rho dt, \text{ and} \\ (d\nu)^2 &= d\langle \nu\rangle = \sigma^2\nu d\langle W^\nu\rangle = \sigma^2\nu dt. \end{aligned}$$

The other terms including $d\langle t\rangle, d\langle t, W^\nu\rangle, d\left\langle t, W^S\right\rangle$ are left out, as the quadratic variation of a finite variation term is always zero and thus the terms vanish. Thus

$$\begin{aligned} dU &= U_t dt + U_S dS + U_\nu d\nu + \frac{1}{2}U_{SS}\nu S dt + U_{S\nu}\nu S\sigma\rho dt + \frac{1}{2}U_{\nu\nu}\sigma^2\nu dt \\ &= \left[U_t + \frac{1}{2}U_{SS}\nu S + U_{S\nu}\nu S\sigma\rho + \frac{1}{2}U_{\nu\nu}\sigma^2\nu\right] dt + U_S dS + U_\nu d\nu \end{aligned}$$

The Heston PDE

As in the BSM portfolio replication also in the Heston model you get your portfolio PDE via dynamic hedging, but we have a portfolio consisting of:

- one option $V(S, \nu, t)$
- a portion of the underlying ΔS_t and
- a third derivative to hedge the volatility $\phi U(S, \nu, t)$.

$$\frac{1}{2}\nu U_{XX} + \rho\sigma\nu U_{X\nu} + \frac{1}{2}\sigma^2\nu U_{\nu\nu} + \left(r - \frac{1}{2}\nu\right) U_X +$$

$$+ \left[\kappa(\theta - \nu_t) - \lambda_0\nu_t\right] U_\nu - rU - U_\tau = 0$$

where $\lambda_0\nu_t$ is the *market price of volatility risk*.

Characteristic Function PDE

Heston assumed the characteristic function to be of the form

$$\varphi^i_{x_\tau}(u) = \exp\left(C_i(u,\tau) + D_i(u,\tau)\nu_t + iux\right)$$

The pricing PDE is always fulfilled irrespective of the terms in the call contract.

- $S = 1, K = 0, r = 0 \quad \Rightarrow \quad C_t = P_1$
- $S = 0, K = 1, r = 0 \quad \Rightarrow \quad C_t = -P_2$

We have to set up the boundary conditions we know to solve the PDE:

$$\begin{aligned} C(T,\nu,S) &= \max(S_T - K, 0) \\ C(t,\infty,S) &= Se^{-r(T-t)} \\ \frac{\partial C}{\partial S}(t,\nu,\infty) &= 1 \\ C(t,\nu,0) &= 0 \\ rC(t,0,S) &= \left[rS\frac{\partial C}{\partial S} + \kappa\theta\frac{\partial C}{\partial \nu} + \frac{\partial C}{\partial t}\right](t,0,S) \end{aligned}$$

The *Feynman-Kac theorem* ensures that then also the characteristic function follows the Heston PDE.

Heston Model Steps

Recall that we have a pricing formula of the form

$$C_t = S_t P_1(S_t, \nu_t, \tau) - e^{-r(T-t)} K P_2(S_t, \nu_t, \tau)$$

where the two probabilities P_j are

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[\frac{e^{-iux}}{iu} \varphi_X^j(u) \right] du$$

with the characteric function being of the form

$$\varphi_j(u) = e^{C_j(\tau,u) + D_j(\tau,u)\nu_t + iux}.$$

FX Black Scholes Framework

The **exchange rate process** Q_t is the price of units of domestic currency for 1 unit of the foreign currency and is described under the actual probability measure $\mathbb{P}$ by

$$dQ_t = \mu Q_t dt + \sigma Q_t dW_t$$

Let us now consider an auxiliary process $Q_t^* := Q_t B_t^f / B_t^d$ which then of course satisfies

$$\begin{aligned} Q_t^* &= \frac{Q_t B_t^f}{B_t^d} \\ &= Q_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t} e^{(r_f - r_d)t} \\ &= Q_0 e^{\left(\mu + r_f - r_d - \frac{\sigma^2}{2}\right)t + \sigma W_t} \end{aligned}$$

Thus we can clearly see that Q_t^* is a martingale under the original measure $\mathbb{P}$ iff $\mu = r_d - r_f$.

FX Option Price

If we now assume that the underlying process (Q_t) is now the exchange rate we still have the final payoff for a Call option of the form

$$FXC_T = \max(Q_T - K, 0)$$

and following the *Garman-Kohlhagen model* we know that the price of the FX option gets

$$FXC_t = e^{-r_f(T-t)} Q_t P_1^{FX}(Q_t, \nu_t, \tau) - e^{-r_d(T-t)} K P_2^{FX}(Q_t, \nu_t, \tau)$$

FX Option Volatility Surface

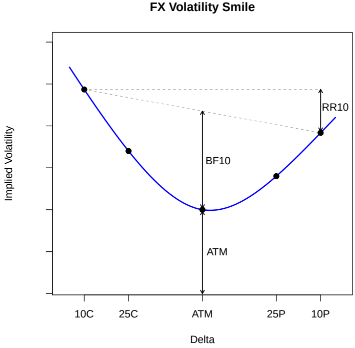
Risk Reversal:
Risk reversal is the difference between the volatility of the call price and the put price with the same moneyness levels.

$$RR_{25} = \sigma_{25C} - \sigma_{25P}$$

Butterfly:
Butterfly is the difference between the avarage volatility of the call price and put price with the same moneyness level and at the money volatility level.

$$BF_{25} = (\sigma_{25C} + \sigma_{25P})/2 - \sigma_{ATM}$$

FX volatility smile with the 3-point market quotation



Bloomberg FX Option Data


EURJPY 107.10 +.56 107.10/107.10
At 17:45 Op 106.54 Hi 107.36 Lo 106.48 Prev 106.54
EURJPY Curncy | 90) Asset | 91) Actions | 92) Settings | Volatility Surface
Bloomberg BGN | Weekdays | As of 19-Apr-2012
1) Vol Table | 2) 3D Surface | 3) Term Analysis | 4) Smile Analysis | 5) Dep and Fwd Rates
RR/BF | Put/Call | Bid/Ask | Mid/Spread

	ATM		25D RR		25D BF		10D RR		10D BF	
Exp	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask
1D	12.214	15.917	-2.553	0.039	-0.317	1.535	-4.650	-0.206	0.404	3.366
1W	11.190	12.590	-1.425	-0.445	0.030	0.730	-2.525	-0.845	0.640	1.760
2W	11.630	12.680	-1.645	-0.910	0.085	0.610	-3.000	-1.740	0.710	1.550
3W	11.695	12.565	-1.765	-1.155	0.135	0.570	-3.210	-2.160	0.755	1.455
1M	11.495	12.345	-1.930	-1.335	0.130	0.555	-3.465	-2.445	0.690	1.370
2M	11.820	12.650	-2.400	-1.820	0.250	0.665	-4.415	-3.415	0.925	1.590
3M	12.120	12.765	-2.795	-2.345	0.310	0.635	-5.160	-4.390	1.250	1.765
6M	12.895	13.720	-3.440	-2.865	0.370	0.780	-6.405	-5.415	1.830	2.490
1Y	13.630	14.430	-4.175	-3.615	0.500	0.900	-7.825	-6.865	2.510	3.150
18M	14.410	15.680	-4.725	-3.840	0.495	1.130	-8.965	-7.440	2.645	3.660
2Y	14.795	16.295	-5.120	-4.070	0.355	1.105	-9.500	-7.700	2.515	3.715
3Y	15.755	17.755	-5.605	-4.205	0.065	1.065	-10.370	-7.970	2.215	3.815
5Y	17.010	19.010	-6.195	-4.795	-0.345	0.655	-11.215	-8.815	1.950	3.550

97) Option Pricing (OVML) 98) Legend Zoom 100%

99) Quick Pricer					Bid	Ask		Mid	Deposit	
Mty	1M	Delta	49.956 C	Vol	11.495	12.345	Fwd	107.09	EUR	0.301%
Exp	21-May-2012	Strike	107.02	EUR Price	1.388%	1.489%	Spot	107.10	JPY	0.144%

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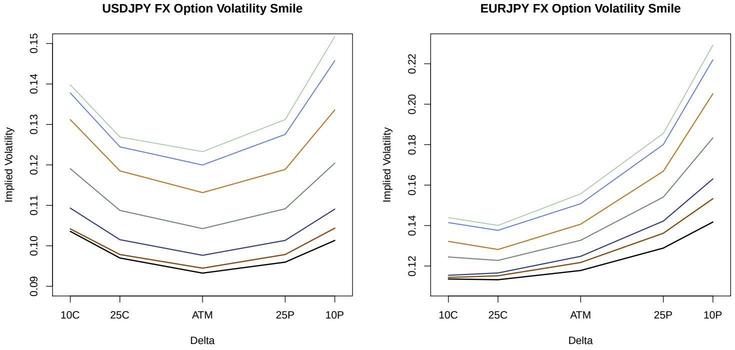
Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2012 Bloomberg Finance L.P.
SN 499738 CEST GMT+2:00 H266-45-1 19-Apr-2012 17:45:10

Bloomberg FX Option Data

USD/JPY and EUR/JPY volatility surface



Calibration to the Implied Volatility Surface

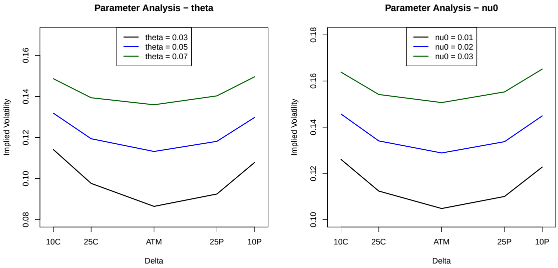
- **Implement the Heston Pricing procedure**
  - Characteristic function
  - Numerical integration algorithm
  - Heston pricer
- **BSM implied volatility from Heston prices**
- **Sum of squared errors minimisation algorithm**
  compare the market implied volatility $\hat{\sigma}$ with the volatility returned by the Heston model $\sigma(\kappa, \theta, \sigma, \nu_0, \rho)$

$$\min_{\theta,\sigma,\rho} \left(\sum_{i,j} \left(\hat{\sigma} - \sigma(\kappa, \theta, \sigma, \nu_0, \rho) \right)^2 \right)$$

Parameter Impacts

Recall

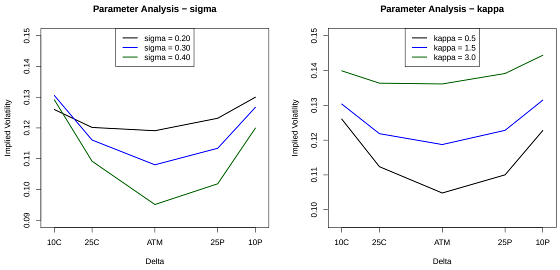
$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{\nu_t} S_t dW_t^S \\
d\nu_t &= \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t} dW_t^\nu \\
dW_t^S dW_t^\nu &= \rho dt
\end{aligned}
$$



$\Rightarrow$ set $\sqrt{\nu_0} = \sigma_{ATM}$.

Parameter Impacts 2

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{\nu_t} S_t dW_t^S \\ d\nu_t &= \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t} dW_t^\nu \\ dW_t^S dW_t^\nu &= \rho dt \end{aligned}$$

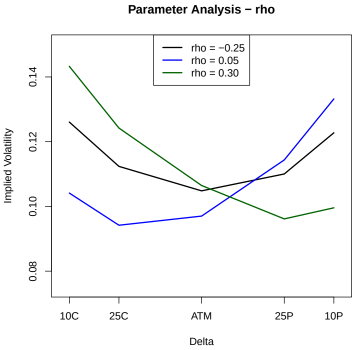


$\Rightarrow$ use for κ fixed values depending on curvature. E.g. 0.5, 1.5, or 3.

Parameter Impacts 3

The skew parameter ρ:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{\nu_t} S_t dW_t^S \\
d\nu_t &= \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t} dW_t^\nu \\
dW_t^S dW_t^\nu &= \rho dt
\end{aligned}
$$

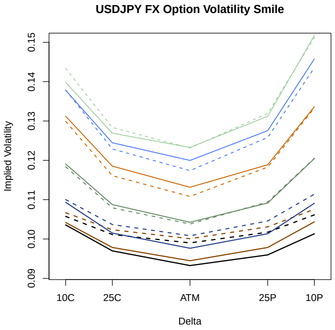


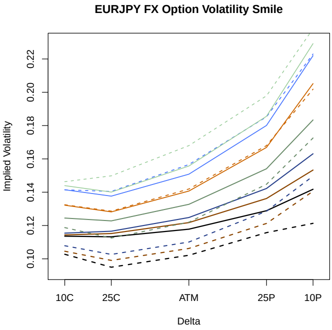
FX Option Data Calibration

USD/JPY and EUR/JPY volatility surface calibration

	optim NM	optim BFGS	nlmin constr.
theta	0.03423300	0.03423542	0.03423272
vol	0.27744796	0.27746901	0.27745127
rho	-0.01206708	-0.01208952	-0.01204884

	optim NM	optim BFGS	nlmin constr.
theta	0.0508903	0.0508923	0.0508911
vol	0.4366006	0.4366059	0.4365979
rho	-0.3715149	-0.3715445	-0.3715368





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Definition

Stochastic Volatility Model

$$d\hat{F} = \hat{\alpha}\hat{F}^{\beta} dW_1, \qquad \hat{F}(0) = f$$
$$d\hat{\alpha} = \nu\hat{\alpha} dW_2, \qquad \hat{\alpha}(0) = \alpha$$
$$dW_1 dW_2 = \rho dt$$

The parameters are

- α the initial variance,
- ν the volatility of variance,
- β the exponent for the forward rate,
- ρ the correlation between the two Brownian motions.

Derivation

The derivation is based on small volatility expansions, $\hat{\alpha}$ and ν, re-written to $\hat{\alpha} \to \epsilon\hat{\alpha}$ and $\nu \to \epsilon\nu$ such that

$$d\hat{F} = \epsilon\hat{\alpha}C(\hat{F})dW_1,$$
$$d\hat{\alpha} = \epsilon\nu\hat{\alpha}dW_2$$

with $dW_1 dW_2 = \rho dt$ in the distinguished limit $\epsilon \ll 1$ and $C(\hat{F})$ generalized. The probability density is defined as

$$p(t, f, \alpha; T, F, A)dFdA = Prob\left\{F < \hat{F}(T) < F + dF, A < \hat{\alpha}(T) < A + dA \right.$$
$$\left. \mid \hat{F}(t) = f, \hat{\alpha}(t) = \alpha\right\}.$$

Then the density at maturity T is defined as

$$p(t, f, \alpha; T, F, A) = \delta(F - f)\delta(A - \alpha) + \int_t^T p_T(t, f, \alpha; T, F, A)dT$$

with

$$p_T = \frac{1}{2}\epsilon^2 A^2 \frac{\partial^2}{\partial F^2} C^2(F)p + \epsilon^2 \rho\nu \frac{\partial^2}{\partial F \partial A} A^2 C^2(F)p + \frac{1}{2}\epsilon^2\nu^2 \frac{\partial^2}{\partial A^2} A^2 p.$$

Derivation

Let $V(t, f, \alpha)$ then be the value of an European call option at t at above defined state of economy:

$$
\begin{aligned}
V(t,f,\alpha) &= \mathbb{E}\left([\hat{F}(T) - K]^+ \mid \hat{F}(t) = f, \hat{\alpha}(t) = \alpha\right) \\
&= \int_{-\infty}^{\infty}\int_{K}^{\infty}(F-K)p(t,f,\alpha;T,F,A)dFdA \\
&= [f-K]^+ + \int_t^T\int_{-\infty}^{\infty}\int_K^{\infty}(F-K)p_T(t,f,\alpha;T,F,A)dT \\
&= [f-K]^+ + \frac{\epsilon^2}{2}\int_t^T\int_{-\infty}^{\infty}\int_K^{\infty}A^2(F-K)\frac{\partial^2}{\partial F^2}C^2(F)p\,dFdAdT \\
&= [f-K]^+ + \frac{\epsilon^2 C^2(K)}{2}\int_t^T\int_{-\infty}^{\infty}A^2 p(t,f,\alpha;T,K,A)dAdT \\
&\;\;\vdots \\
&= [f-K]^+ + \frac{\epsilon^2 C^2(K)}{2}\int_t^{\tau}P(\tau,f,\alpha;K)d\tau
\end{aligned}
$$

Derivation

Where

$$P(t,f,\alpha;T,K) = \int_{-\infty}^{\infty} A^2 p(t,f,\alpha;T,K,A)dA$$

and $P(\tau, f, \alpha; K)$ is the solution of

$$P_\tau = \frac{1}{2}\epsilon^2\alpha^2 C^2(f)\frac{\partial^2 P}{\partial f^2} + \epsilon^2\rho\nu\alpha^2 C(f)\frac{\partial^2 P}{\partial f \partial \alpha} + \frac{1}{2}\epsilon^2\nu^2\alpha^2\frac{\partial^2 P}{\partial \alpha^2}, \quad \text{for } \tau > 0,$$

$$P = \alpha^2\delta(f-K), \quad \text{for } \tau = 0.$$

with $\tau = T - t$.

Given these results one could obtain the option formula directly. However more useful formulas can be derived through

1. Singular perturbation expansion
2. Equivalent normal volatility
3. Equivalent Black volatility
4. Stochastic β model

Singular perturbation expansion

The goal is to use perturbation expansion methods which yield a Gaussian density of the form

$$P = \frac{\alpha}{\sqrt{2\pi\epsilon^2 C^2) K)\tau}} e^{-\frac{(f-K)^2}{2\epsilon^2\alpha^2 C^2(K)\tau}\{1+\cdots\}}.$$

Consiquently, the singular perturbation expansion yields a European call option value

$$V(t,f,\alpha) = [f-K]^+ + \frac{\mid f-K \mid}{4\sqrt{\pi}} \int_{\frac{x^2}{2\tau}-\epsilon^2\theta}^{\infty} \frac{e^{-q}}{q^{3/2}} dq$$

with

$$x = \frac{1}{\epsilon\nu} \log\left(\frac{\sqrt{1-2\epsilon\rho\nu z+\epsilon^2\nu^2 z^2}-\rho+\epsilon\nu z}{1-\rho}\right), \; z = \frac{1}{\epsilon\alpha}\int_K^f \frac{df'}{C(f')},$$

$$\epsilon^2\theta = \log\left(\frac{\epsilon\alpha z}{f-K}\sqrt{B(0)B(\epsilon\alpha z)}\right) + \log\left(\frac{x I^{1/2}(\epsilon\nu z)}{z}\right) + \frac{1}{4}\epsilon^2\rho\nu\alpha b_1 z^2.$$

Equivalent normal volatility

Suppose the previous analysis is repeated under the normal model

$$d\hat{F} = \sigma_N dW, \hat{F}(0) = f.$$

with σ_N constant, not stochastic. The option value would then be

$$V(t,f) = [f-K]^+ + \frac{\mid f-K \mid}{4\sqrt{\pi}} \int_{\frac{(f-K)^2}{2\sigma_N^2 \tau}}^{\infty} \frac{e^{-q}}{q^{3/2}} dq$$

for $C(f) = 1$, $\epsilon\alpha = \sigma_N$ and $\nu = 0$. Integration yields then

$$V(t,f) = (f-K)\Phi\left(\frac{f-K}{\sigma_N \sqrt{\tau}}\right) + \sigma_N \sqrt{\tau} \mathcal{G}\left(\frac{f-K}{\sigma_N \sqrt{\tau}}\right)$$

with the Gaussian density $\mathcal{G}$

$$\mathcal{G}(q) = \frac{1}{\sqrt{2\pi}} e^{-q^2/2}.$$

Equivalent normal volatility

The option price under the normal model matches the option price under the SABR model, iff σ_N is chosen the way that

$$\sigma_N = \frac{f-K}{x}\left\{1+\epsilon^2\frac{\theta}{x^2}\tau+\cdots\right\}$$

through $\mathcal{O}(\epsilon^2)$. Simplifying yields the the implied normal volatility

$$\sigma_N(K) = \frac{\epsilon\alpha(f-K)}{\int_K^f \frac{df'}{C(f')}}\left(\frac{\zeta}{\hat{x}(\zeta)}\right)$$
$$\cdot\left\{1+\left[\frac{2\gamma_2-\gamma_1^2}{24}\alpha^2C^2(f_{av})+\frac{1}{4}\rho\nu\alpha\gamma_1C(f_{av})+\frac{2-3\rho^2}{24}\nu^2\right]\epsilon^2\tau+\cdots\right\}$$

with

$$f_{av}=\sqrt{fK}, \qquad \gamma_1=\frac{C'(f_{av})}{C(f_{av})}, \quad \gamma_2=\frac{C''(f_{av})}{C(f_{av})}$$

$$\zeta=\frac{\nu(f-K)}{\alpha C(f_{av})}, \qquad \hat{x}(\zeta)=\log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho}\right).$$

Equivalent Black volatility

To derive the implied volatility consider again Black's model

$$d\hat{F} = \epsilon\sigma_B \hat{F} dW, \ \hat{F}(0) = f$$

with $\epsilon\sigma_B$ for consistency of the analysis. The implied normal volatility for Black's model for SABR can be obtained by setting $C(f) = f$ and $\nu = 0$ in previous results such that

$$\sigma_N(K) = \frac{\epsilon\sigma_B(f-K)}{\log\frac{f}{K}}\{1 - \frac{1}{24}\epsilon^2\sigma_B^2\tau + \cdots\}.$$

through $\mathcal{O}(\epsilon^2)$. Solving the equation for σ_B yields

$$\sigma_B(K) = \frac{\alpha\log\frac{f}{K}}{\int_K^f \frac{df'}{C(f')}}\left(\frac{\zeta}{\hat{x}(\zeta)}\right) \cdot \left\{1 + \left[\frac{2\gamma_2 - \gamma_1^2 + \frac{1}{f_{av}^2}}{24}\alpha^2 C^2(f_{av}) + \frac{1}{4}\rho\nu\alpha\gamma_1 C(f_{av}) + \frac{2-3\rho^2}{24}\nu^2\right]\epsilon^2\tau + \cdots\right\}.$$

Stochastic β model

Finally, let's look at the original state with $C(f) = f^\beta$. Making the substitutions as previously and following approximations

$$f - K = \sqrt{fK} \log f/K \{1 + \frac{1}{24} \log^2 f/K + \frac{1}{1920} \log^4 f/K + \cdots\},$$

$$f^{1-\beta} - K^{1-\beta} = (1-\beta)(fK)^{(1-\beta)/2} \log f/K \{1 + \frac{(1-\beta)^2}{24} \log^2 f/K + \frac{(1-\beta)^4}{1920} \log^4 f/K + \cdots\},$$

the implied normal volatility reduces to

$$\sigma_N(K) = \epsilon\alpha(fK)^{\beta/2} \frac{1 + \frac{1}{24}\log^2 f/K + \frac{1}{1920}\log^4 f/K + \cdots}{1 + \frac{(1-\beta)^2}{24}\log^2 f/K + \frac{(1-\beta)^4}{1920}\log^4 f/K + \cdots} \left(\frac{\zeta}{\hat{x}(\zeta)}\right)$$
$$\cdot \left\{1 + \left[\frac{-\beta(2-\beta)\alpha^2}{24(fK)^{1-\beta}} + \frac{\rho\alpha\nu\beta}{4(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2\right]\epsilon^2\tau + \cdots\right\}$$

with $\zeta = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2} \log f/K$. Setting $\epsilon = 1$ one gets ...

SABR Implied Volatility - General

The implied volatility $\sigma_B(f, K)$ is given by

$$\sigma_B(K,f) = \frac{\alpha}{(fK)^{(1-\beta)/2}\left\{1+\frac{(1-\beta)^2}{24}\log^2\frac{f}{K}+\frac{(1-\beta)^4}{1920}\log^4\frac{f}{K}\right\}}\cdot\left(\frac{z}{x(z)}\right)\cdot \left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right]T\right\}$$

where z is defined by

$$z = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log\frac{f}{K}$$

and $x(z)$ is given by

$$x(z) = \log\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}.$$

SABR Implied Volatility - ATM

For at-the-money options $(K = f)$ the formula reduces to $\sigma_B(f, f) = \sigma_{ATM}$ such that

$$\sigma_{ATM} = \frac{\alpha \left\{ 1 + \left[\frac{(1-\beta)^2}{24} \frac{\alpha^2}{f^{2-2\beta}} + \frac{1}{4} \frac{\rho\beta\nu\alpha}{f^{(1-\beta)}} + \frac{2-3\rho^2}{24} \nu^2 \right] T \right\}}{f^{(1-\beta)}}.$$

Model Dynamics

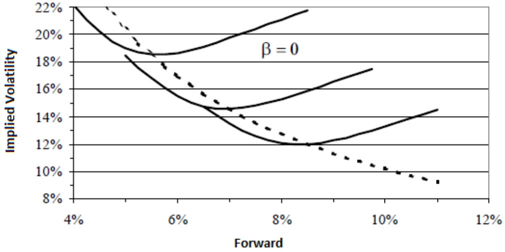
Approximate the model with $\lambda = \frac{\nu}{\alpha} f^{1-\beta}$ such that

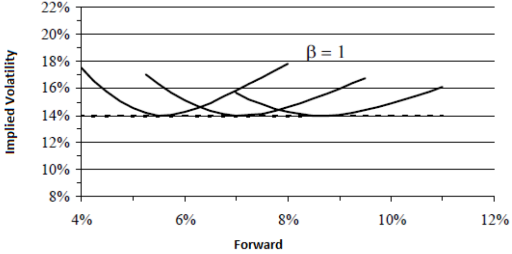
$$\sigma_B(K,f) = \frac{\alpha}{f^{1-\beta}} \left\{ 1 - \frac{1}{2}(1-\beta-\rho\lambda)\log\frac{K}{f} + \frac{1}{12}\left[(1-\beta)^2 + (2-3\rho^2)\lambda^2\right]\log^2\frac{K}{f} \right\},$$

The SABR model is then described with

- *Backbone*: $\frac{\alpha}{f^{1-\beta}}$
- *Skew*: $-\frac{1}{2}(1-\beta-\rho\lambda)\log\frac{K}{f}$, $\frac{1}{12}(1-\beta)^2\log^2\frac{K}{f}$
- *Smile*: $\frac{1}{12}(2-3\rho^2)\log^2\frac{K}{f}$

Backbone


22%
20%
18%
16%
14%
12%
10%
8%
Implied Volatility
β = 0
4%
6%
8%
10%
12%
Forward


22%
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10%
8%
Implied Volatility
β = 1
4%
6%
8%
10%
12%
Forward

Parameter Estimation

For estimation of the SABR model the estimation of β is used as a starting point.

With β estimated, there are two possible choices to continue calibration:

1. Estimate α, ρ and ν directly, or
2. Estimate ρ and ν directly, and infer α from ρ, ν and the at-the-money.

In general, it is more convenient to use the ATM volatility $\sigma_{ATM}, \beta, \rho$ and ν as the SABR parameters instead of the original parameters α, β, ρ and ν.

Estimation of β

For estimation of β the at-the money volatility σ_{ATM} from equation is used

$$
\begin{aligned}
\log \sigma_{ATM} &= \log \alpha - (1-\beta)\log f + \\
&\quad \log\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2-2\beta}} + \frac{1}{4}\frac{\rho\beta\nu\alpha}{f^{(1-\beta)}} + \frac{2-3\rho^2}{24}\nu^2\right]T\right\} \\
&\approx \log \alpha - (1-\beta)\log f
\end{aligned}
$$

Alternatively, β can be chosen from prior beliefs of the appropriate model:

- $\beta = 1$: stochastic log-normal, for FX option markets
- $\beta = 0$: stochstic normal, for markets with zero or negative f
- $\beta = \frac{1}{2}$: CIR model, for interest rate markets

Estimation of α, ρ and ν

Estimation of all three parameters by minimization of the errors between the model and the market volatilities σ_i^{mkt} at identical maturity T.

Using the sum of squared errors (SSE)

$$(\hat{\alpha}, \hat{\rho}, \hat{\nu}) = \arg\min_{\alpha,\rho,\nu} \sum_i \left(\sigma_i^{\text{mkt}} - \sigma_B(f_i, K_i; \alpha, \rho, \nu)\right)^2 .$$

is produced.

Estimation of ρ and ν

The number of parameters can be reduced by extracting α directly from σ_{ATM}. Thus, by inverting the equation the cubic equation is received

$$\left(\frac{(1-\beta)^2 T}{24 f^{2-2\beta}}\right)\alpha^3 + \left(\frac{1}{4}\frac{\rho\beta\nu T}{f^{(1-\beta)}}\right)\alpha^2 + \left(1 + \frac{2-3\rho^2}{24}\nu^2 T\right)\alpha - \sigma_{ATM} f^{(1-\beta)} = 0.$$

As it it possible to receive more than one single real root, it is suggested to select the smallest positive real root.

Given α the SSE

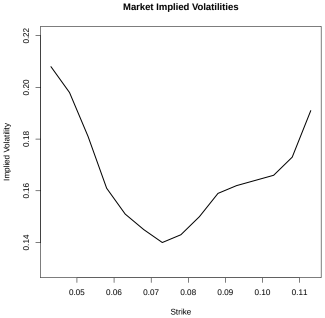
$$(\hat{\alpha}, \hat{\rho}, \hat{\nu}) = \arg\min_{\alpha,\rho,\nu} \sum_i \left(\sigma_i^{\text{mkt}} - \sigma_B(f_i, K_i; \alpha(\rho,\nu), \rho, \nu)\right)^2$$

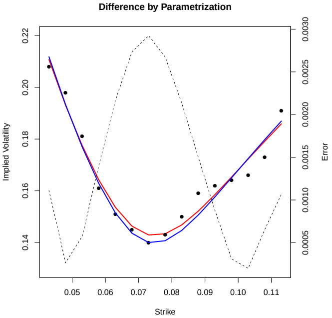
has to be minimized for the ρ and ν.

Calibration

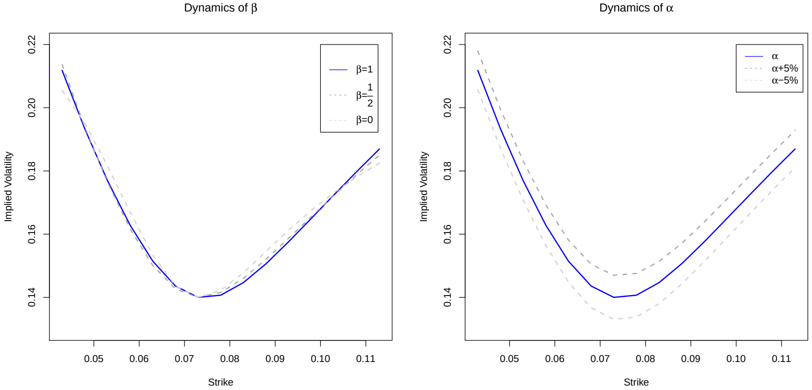
- Calibration for a fictional data set, with 15 implied market volatilities at maturity $T = 1$.

	1.Param.	2.Param.
α	0.139	0.136
ρ	-0.069	-0.064
ν	0.578	0.604
SSE	$2.456 \cdot 10^{-4}$	$2.860 \cdot 10^{-4}$

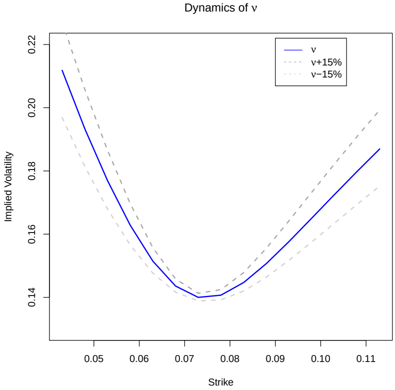


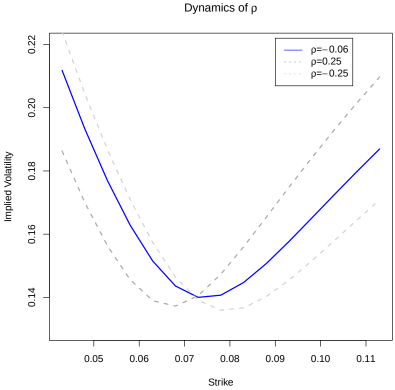


Parameter dynamics - β, α

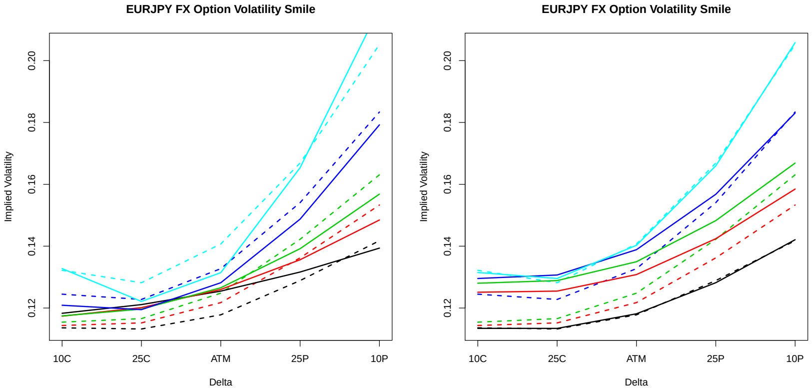


Parameter dynamics - ρ, ν





SABR and FX Options- EUR/JPY



SABR and FX Options - USD/JPY

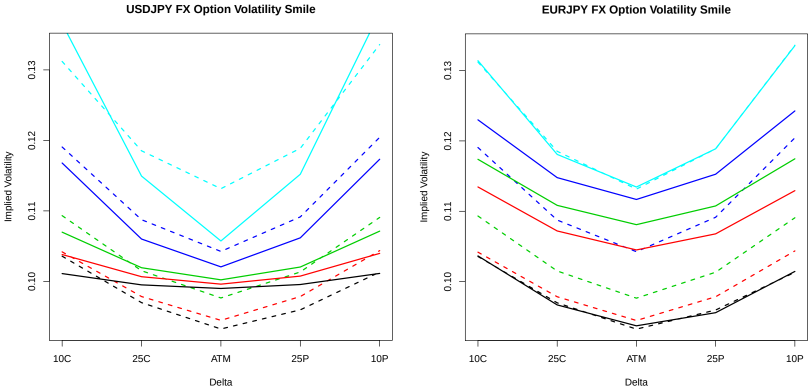


Table of Contents

Observations and Facts

Heston

- its volatility structure permits analytical solutions to be generated for European options
- this model describes important mean-reverting property of volatility
- allows price dynamics to be of non-lognormal probability distributions
- the model does not perform well for short maturities
- parameters after calibration to market data turn out to be non-constant

SABR

- simple stochastic volatility model; as only one formula
- no derivation of prices, comparision directly via implied volatility
- no time dependency implemented
- interpolation errenous and inaccurate (e.g. shifts)