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Convergence Properties of Kemp's q-Binomial Distribution

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Abstract

We consider Kemp's q-analogue of the binomial distribution. Several convergence results involving the classical binomial, the Heine, the discrete normal, and the Poisson distribution are established. Some of them are q-analogues of classical convergence properties. From the results about distributions, we deduce some new convergence results for (q-)Krawtchouk and q-Charlier polynomials. Besides elementary estimates, we apply Mellin transform asymptotics.

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1. **Introduction**

Kemp and Kemp (1991) introduced a q-analogue $KB(n, \theta, q)$ of the binomial distribution. It is well known that for $n \to \infty$ (and fixed θ) Kemp's q-binomial distribution converges to a Heine distribution. We are now interested in sequences of random variables X_n with $X_n \sim KB(n, \theta_n, q)$, where (θ_n) is a sequence of positive real parameters. Our main results contain q-analogues to the convergence of the classical binomial distribution to the Poisson distribution and the normal distribution, and show that the limits $q \to 1$ and $n \to \infty$ can be exchanged. From the limit theorems about distributions we deduce limit relations for q-polynomials that are orthogonal w.r.t. these distributions.

The paper is organised as follows. In Section 2 we give all definitions of q-calculus, q-distributions, and q-polynomials we need in the following; afterwards we sum up some important properties of Kemp's q-binomial distribution in Section 3. Section 4 deals with the case of convergent parameter θ_n, in particular with the case of constant mean. The pertinent limit law is the Heine distribution, and the involved q-polynomials are the q-Krawtchouk and the q-Charlier polynomials. In Section 5 we investigate parameter sequences that tend to infinity. If they do so fast enough, then it turns out that $n - X_n$ is either degenerate in the limit or tends to a Heine distribution. The main result of the paper is concerned with parameter sequences of slower growth, where the law of the normalized X_n converges to a discrete normal distribution. From this property we deduce a limit relation for the q-Krawtchouk and the Stieltjes-Wigert polynomials.

2. Notation and Definitions

Throughout the paper we use the notation of Gasper and Rahman (1990). The q-shifted factorial $(z;q)_n$ and the q-binomial coefficient are defined by

$$(z;q)_n = \prod_{i=0}^{n-1}(1 - zq^i) \qquad \text{and} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

In the limit $q \to 1$ the q-shifted factorial converges to $(1-z)^n$, and the q-binomial coefficient to the binomial coefficient $\binom{n}{k}$. The q-number $[x]_q$ is defined as

$$[x]_q := \frac{1-q^x}{1-q};$$

for $q \to 1$, we have $[x]_q \to x$. Moreover, we will need two analogues of the exponential function:

$$e_q(z) = \frac{1}{(z;q)_\infty}, \qquad z \in \mathbb{C} \setminus \{q^{-i} : \ i = 0, 1, 2, \dots\},$$

and $E_q(z) = (-z;q)_\infty$. Here the limit relations $e_q((1-q)z) \to e^z$ and $E_q((1-q)z) \to e^z$ hold, as $q \to 1$. The basic hypergeometric series ${}_r\phi_s$ is defined by

$$\begin{aligned} &{}_r\phi_s(a_1, a_2, \dots, a_r; b_1, \dots, b_s; q, z) \\ &\qquad = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n. \end{aligned}$$

Our main object of interest is the following q-analogue $KB(n,\theta,q)$ of the binomial distribution, see Kemp and Kemp (1991):

$$\mathbb{P}(X_{KB} = x) = \begin{bmatrix} n \\ x \end{bmatrix}_q \frac{\theta^x q^{x(x-1)/2}}{(-\theta;q)_n}, \qquad 0 \le x \le n, \ 0 < \theta, \ 0 < q < 1.$$

The Heine distribution (Benkherouf and Bather, 1988, Kemp, 1992a, 1992b) $H(\theta)$ is defined by

$$\mathbb{P}(X_H = x) = \frac{q^{x(x-1)/2}\theta^x}{(q;q)_x} e_q(-\theta), \qquad x \geq 0,\ 0 < q < 1.$$

If $q \to 1$, then $H((1-q)\theta) \to P(\theta)$, where $P(\theta)$ denotes the Poisson distribution with parameter θ.

For details about the properties of the following q-polynomials, we refer to the encyclopaedic report by Koekoek and Swarttouw (1998) and the references therein. The q-Krawtchouk polynomials are given by

$$K_n(q^{-x}; p, N; q) = {}_3\phi_2(q^{-n}, q^{-x}, -pq^n; q^{-N}, 0; q, q), \quad n = 0, \ldots, N.$$

The q-Charlier polynomials are defined as

$$C_n(q^{-x}; a; q) = {}_2\phi_1\left(q^{-n}, q^{-x}; 0; q, -\frac{q^{n+1}}{a}\right),$$

and the Stieltjes-Wigert polynomials as

$$S_n(x; q) = \frac{1}{(q;q)_n} {}_1\phi_1(q^{-n}; 0; q, -q^{n+1}x).$$

3. **Basic Properties**

In this section we recall some of the properties of Kemp's q-binomial distribution (see Johnson, Kemp, and Kotz, 2005, Kemp, 2002, 2003, Kemp and Newton, 1990). In the limit $q \to 1$, the Kemp distribution $KB(n, \theta, q)$ converges to a binomial distribution:

$$KB(n, \theta, q) \to B\left(n, \frac{\theta}{1+\theta}\right),$$

whereas for $n \to \infty$ we obtain a Heine distribution $H(\theta)$. The random variable $X_{KB} \sim KB(n, \theta_n, q)$ can be written as the sum of independent Bernoulli random variables (Kemp and Newton, 1990), which leads to the expressions

$$\mu = \sum_{i=0}^{n-1} \frac{\theta q^i}{1+\theta q^i} \qquad \text{and} \qquad \sigma^2 = \sum_{i=0}^{n-1} \frac{\theta q^i}{(1+\theta q^i)^2} \tag{3.1}$$

for the mean and variance. Furthermore, the random variable $n - X_{KB}$ has the law $KB(n, \theta^{-1}q^{1-n}, q)$. We note in passing that Kemp (2003) deduced a characterization result for the KB distribution from a theorem of Rao and Shanbhag (1994).

4. **Convergent Parameter**

Our first result is a mild generalization of the convergence to the Heine distribution mentioned in the preceding section.

PROPOSITION 4.1. *Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with limit $\theta \geq 0$. Then the sequence of Kemp's q-binomial distributions $KB(n, \theta_n, q)$ converges for $n \to \infty$ to a Heine distribution $H(\theta)$.*

PROOF. Note that

$$\mathbb{P}(X_n = x) = \begin{bmatrix} n \\ x \end{bmatrix}_q \theta_n^x \frac{q^{x(x-1)/2}}{\prod_{i=0}^{n-1}(1+\theta_n q^i)}.$$

The q-binomial coefficient tends to $1/(q;q)_x$. As for the product in the denominator, apply the dominated convergence theorem to its logarithm to see that it tends to $E_q(\theta)$. □

EXAMPLE 4.1. Let λ be a real number with $0 < \lambda < n$, and put $\theta_n(q) = \lambda/[n-\lambda]_q$. Then, as $n \to \infty$, the sequence of Kemp's q-binomial distributions $KB(n, \theta_n(q), q)$ converges to a Heine distribution $H((1-q)\lambda)$. Thus the following diagram is commutative:

$$\begin{array}{ccc} KB(n,\theta_n(q),q) & \xrightarrow{n\to\infty} & H((1-q)\lambda) \\ {\scriptstyle q\to 1}\big\downarrow & & \big\downarrow{\scriptstyle q\to 1} \\ B\left(n,\frac{\lambda}{n}\right) & \xrightarrow[n\to\infty]{} & P(\lambda) \end{array}$$

The two preceding results yield limit relations for orthogonal polynomials. The orthogonal polynomials for Kemp's q-binomial, the Heine, and the binomial distribution, are, respectively, the q-Krawtchouk, the q-Charlier, and the Krawtchouk polynomials.

COROLLARY 4.1. *(i) Let θ_n be as in Proposition 4.1. Then, as $n \to \infty$, the q-Krawtchouk polynomial $K_k(q^{-x}; q^{-n}\theta_n^{-1}, n; q)$ converges to the q-Charlier polynomial $C_k(q^{-x}; \theta; q)$.*

(ii) For the special parameter sequence $\theta_n(q) = \lambda/[n-\lambda]_q$, as $q \to 1$, the q-Krawtchouk polynomial $K_k(q^{-x}; q^{-n}\theta_n(q)^{-1}, n; q)$ converges to the Krawtchouk polynomial $K_k(x; \lambda/n, n)$.

The classical convergence of the binomial distribution with constant mean to the Poisson distribution has the following q-analogue.

THEOREM 4.1. *Fix $\mu > 0$ and choose the parameter $\theta_n = \theta_n(\mu, q)$ of Kemp's q-binomial distribution such that $\mu_n = \mu$. Then we have:*

(i) The sequence $KB(n, \theta_n, q)$ converges for $n \to \infty$ to a Heine distribution $H(\theta)$, where $\theta = \lim_{n\to\infty} \theta_n$.

(ii) For fixed n, $KB(n, \theta_n, q)$ tends to a binomial distribution $B\left(n, \frac{\mu}{n}\right)$ in the limit $q \to 1$.

(iii) For $q \to 1$, the Heine distribution $H(\theta)$ converges to a Poisson distribution with parameter μ.

So we obtain the following commutative diagram:

$$\begin{array}{ccc} KB(n, \theta_n(\mu, q), q) & \xrightarrow{n\to\infty} & H(\theta(\mu, q)) \\ {\scriptstyle q\to 1}\big\downarrow & & \big\downarrow{\scriptstyle q\to 1} \\ B\left(n, \frac{\mu}{n}\right) & \xrightarrow[n\to\infty]{} & P(\mu) \end{array}$$

PROOF. First we check that for given μ, q and large n, there is a unique θ_n such that $\mu_n(\theta_n, q) = \mu$. The function $\mu_n(\theta, q)$ is strictly increasing in θ, and $\mu_n(0, q) = 0$. Since

$$\mu_n(q^{-n+1}, q) \geq \sum_{i=0}^{n-1} \frac{q^{i-n+1}}{2q^{i-n+1}} = \frac{n}{2},$$

and $\mu_n(\theta, q)$ is continuous in θ, there exists a unique solution θ_n of $\mu_n(\theta, q) = \mu$ for each $n \geq 2\mu$. An easy continuity argument (de Haan and Ferreira, 2006, Lemma 1.1.1) shows that $\lim_{n\to\infty} \theta_n = \theta$, with θ the unique solution of $\mu_\infty(\theta, q) = \mu$. Thus $KB(n, \theta_n, q) \to H(\theta)$ by Proposition 4.1.

Again by Lemma 1.1.1 of de Haan and Ferreira (2006), we get $\theta_n \to \frac{\mu}{n-\mu}$ for $q \to 1$. Hence $KB(n, \theta_n, q) \to B\left(n, \frac{\mu}{n}\right)$.

It remains to check that $\theta/(1-q)$ converges to μ for $q \to 1$, which yields $H(\theta) \to P(\mu)$. The value $\theta/(1-q)$ is the unique solution of $\mu_\infty((1-q)\theta, q) = \mu$. Moreover, $\lim_{q\to 1} \mu_\infty((1-q)\theta, q) = \theta$, because $H((1-q)\theta) \to P(\theta)$. □

Analogously to Corollary 4.1, Theorem 4.1 implies the following result about (q-)Krawtchouk polynomials.

COROLLARY 4.2. *Let θ_n and θ be as in Theorem 4.1. Then the q-Krawtchouk polynomial $K_k(q^{-x}; q^{-n}\theta_n^{-1}, n; q)$ converges to the Krawtchouk polynomial $K_k(x; \mu/n, n)$ for $q \to 1$.*

5. Increasing Parameter

If we consider fast growing parameter sequences, in the sense that $\theta_n = q^{-n-g(n)}$ with $g(n) \to \infty$ or convergent, we obtain the corresponding limit distribution easily.

Corollary 5.1. *Let* $X_n \sim KB(n, q^{-n-g(n)}, q)$.

(i) If $g(n)$ *converges to a limit* g_0*, then the distribution of* $n - X_n$ *tends to the Heine distribution* $H(q^{1+g_0})$ *as* $n \to \infty$.

(ii) If $g(n) \to \infty$ *for* $n \to \infty$*, then the distribution of* $n - X_n$ *tends to the point measure* δ_0 *as* $n \to \infty$.

Proof. As remarked in Section 3, $n - X_n \sim KB(n, \tau, q)$ with $\tau = q^{g(n)+1}$. Applying Proposition 4.1 yields the result. □

It follows from Corollary 5.1(i) that the q-Krawtchouk polynomials converge to the alternative q-Charlier polynomials, which is a known result, see (4.15.1) of Koekoek and Swarttouw (1998).

Now we turn to the main result of the paper, where we assume that

$$\theta_n = q^{-f(n)}, \quad \text{with} \quad f(n) \to \infty \quad \text{and} \quad n - f(n) \to \infty \quad \text{for} \quad n \to \infty. \tag{5.1}$$

This assumption on θ_n will be in force throughout the section. Since the sequence of means tends to infinity, we will normalize our random variables to $(X_n - \mu_n)/\sigma_n$. Still, this sequence does not converge in distribution without further assumptions on $f(n)$. It turns out that the fractional part $\{f(n)\} = \{-\log \theta_n / \log q\}$ has to be constant, which induces convergence to discrete normal distributions.

Theorem 5.1. *Suppose that* $X_n \sim KB(n, \theta_n, q)$*, such that the sequence* θ_n *satisfies* (5.1) *and* $\{f(n)\} = \beta$ *is constant. Then* $(X_n - \mu_n)/\sigma_n$ *converges for* $n \to \infty$ *to a limit* X*, with, for* $x \in \mathbb{Z}$,

$$\mathbb{P}\left(X = -(\beta + c)\frac{1}{\sigma} + \frac{1}{\sigma}x\right) = e_q(q)e_q(-q^{\beta})e_q(-q^{1-\beta})q^{(x-1)(x-2\beta)/2}, \tag{5.2}$$

where $c = c(\beta, q)$ *is a constant and* $\sigma = \lim_{k\to\infty} \sigma_{n_k}$.

The discrete normal distribution is defined by

$$\mathbb{P}(X = x) = \frac{q^{-x\alpha}q^{x^2/2}}{\sum_{k=-\infty}^{\infty} q^{-k\alpha}q^{k^2/2}} \qquad \alpha \in \mathbb{R}, x \in \mathbb{Z}.$$

So the limit distributions in the preceding theorem are (scaled and shifted) discrete normal distributions with parameters

$$\begin{array}{ll} \alpha = \frac{1}{2} + \beta & \text{if } \beta < \frac{1}{2} \\ \alpha = -\frac{1}{2} + \beta & \text{if } \beta > \frac{1}{2} \\ \alpha = 0 & \text{if } \beta = \frac{1}{2} \end{array} .$$

For $q \to 1$, they converge to the standard normal distribution, see Szablowski (2001). Therefore, as in Proposition 4.1 and Theorem 4.1, the limits $q \to 1$ and $n \to \infty$ can be exchanged. Indeed, for $q \to 1$, the distribution of X_n in Theorem 5.1 tends to the binomial distribution $B(n, \frac{1}{2})$. The latter converges to the standard normal distribution after normalization.

To prepare the proof of Theorem 5.1, we will carry out a rather fine analysis of the asymptotics of the sequence of means μ_n. The following two propositions clarify its asymptotics up to order o(1).

PROPOSITION 5.1. *Let $X_n \sim KB(n, \theta_n, q)$ with θ_n as in* (5.1). *Then there is a function c such that, for $n \to \infty$,*

$$\mu_n = f(n) + c(\{f(n)\}, q) + \mathrm{o}(1). \tag{5.3}$$

In other words, the O(1) term of μ_n is constant if $\{f(n)\}$ is constant.

PROOF. We start from

$$\mu_n = \sum_{i=0}^{n-1} \frac{q^{i-f(n)}}{1+q^{i-f(n)}} = \sum_{i=0}^{n-1} \frac{1}{1+q^{f(n)-i}} \tag{5.4}$$

and split the sum into two parts (w.l.o.g. $f(n) < n$): By expanding the denominator as a geometric series and changing the order of summation we find

$$\begin{aligned} \sum_{i=0}^{\lfloor f(n) \rfloor -1} \frac{1}{1+q^{f(n)-i}} &= \sum_{\ell \geq 0} (-1)^\ell q^{\ell f(n)} \sum_{i=0}^{\lfloor f(n) \rfloor -1} q^{-\ell i} \\ &= \lfloor f(n) \rfloor - \sum_{\ell \geq 1} \frac{(-1)^\ell q^{\ell \{f(n)\}}}{1-q^{-\ell}} + \mathrm{O}\left(q^{f(n)}\right). \end{aligned}$$

Expanding the denominator as a geometric series and changing the order of summation again yields

$$\sum_{i=0}^{\lfloor f(n) \rfloor -1} \frac{1}{1+q^{f(n)-i}} = \lfloor f(n) \rfloor - \sum_{j \geq 0} \frac{1}{1+q^{-j-1-\{f(n)\}}} + \mathrm{O}\left(q^{f(n)}\right). \tag{5.5}$$

For the upper portion of the sum, we find

$$\sum_{i=\lfloor f(n)\rfloor+1}^{n-1} \frac{1}{1+q^{f(n)-i}} = \sum_{i=\lfloor f(n)\rfloor+1}^{\infty} \frac{1}{1+q^{f(n)-i}} + \mathrm{O}\left(q^{n-f(n)}\right)$$
$$= \sum_{i=0}^{\infty} \frac{1}{1+q^{\{f(n)\}-i-1}} + \mathrm{O}\left(q^{n-f(n)}\right). \tag{5.6}$$

The result now follows from (5.5) and (5.6). □

We next determine the Fourier series of the O(1)-term from Proposition 5.1, which shows that it is a $\frac{1}{2}$-periodic function of $\{f(n)\}$.

PROPOSITION 5.2. *Let $X_n \sim KB(n, \theta_n, q)$ with $\theta_n = q^{-f(n)}$. Then, as $n \to \infty$,*

$$\mu_n = f(n) + \frac{1}{2} + \sum_{k>0} \frac{2\pi \sin(2kf(n)\pi)}{\sinh\left(\frac{2k\pi^2}{\log q}\right)\log q} + \mathrm{o}(1). \tag{5.7}$$

PROOF. We write

$$\mu_n = \sum_{i=0}^{n-1} \frac{1}{1+q^{f(n)-i}} = \sum_{i=0}^{\infty} \frac{1}{1+q^{f(n)-i}} + \mathrm{O}\left(q^{n-f(n)}\right)$$

and apply the Mellin transformation (see Flajolet, Gourdon and Dumas, 1995) to the function

$$h(t) = \sum_{i=0}^{\infty} \frac{1}{1+tq^{-i}}.$$

The linearity of the Mellin transformation $\mathcal{M}$ and its properties $\mathcal{M}\left(\frac{1}{1+t}\right) = \frac{\pi}{\sin \pi s}$ and $\mathcal{M}h(\alpha t)(s) = \alpha^{-s}\mathcal{M}(h)(s)$ give

$$\mathcal{M}(h)(s) = \int_0^\infty x^{-s} h(x)\mathrm{d}x$$
$$= \sum_{i=0}^{\infty} \left(q^{-i}\right)^{-s} \frac{\pi}{\sin \pi s} = \frac{1}{1-q^s}\frac{\pi}{\sin \pi s}.$$

From the inverse transformation formula we get

$$h\left(q^{f(n)}\right) = \frac{1}{2\pi \mathrm{i}} \int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty} q^{-f(n)s} \frac{1}{1-q^s}\frac{\pi}{\sin \pi s}\mathrm{d}s. \tag{5.8}$$

Upon pushing the line of integration to the left, each pole of the integrand yields a term in the asymptotic expansion of h at infinity, see Flajolet, Gourdon and Dumas (1995). If k is a non-zero integer, then $1/(1-q^s)$ has a simple pole at $2\pi \mathrm{i}k/\log q$ with residue

$$\frac{\mathrm{i}\pi e^{-2\mathrm{i}f(n)k\pi}}{\sinh\left(\frac{2k\pi^2}{\log q}\right)\log q}. \tag{5.9}$$

The residue at the double pole at zero is easily computed, too, and equals $f(n) + \frac{1}{2}$. Merging the terms (5.9) corresponding to k and $-k$ gives the result. □

The following two lemmas complete our analysis of the means μ_n and prepare the proof of the main result of this section, viz. Theorem 5.1. Recall that (5.1) is assumed to hold throughout the present section.

LEMMA 5.1. *If the fractional part $\{f(n)\} = \beta$ is constant, then:*

(i) $c(0, q) = c(1/2, q) = 1/2$

(ii) $c(\beta, q) + c(-\beta, q) = 1$

(iii) $\lfloor c(\beta, q) + \beta \rfloor = \begin{cases} 0 & \textit{if} \quad 0 \le \beta < 1/2 \\ 1 & \textit{if} \quad 1/2 \le \beta < 1 \end{cases}$

PROOF. For (i) and (ii), use (5.7) and simple properties of sin. Part (iii) follows easily from (i), (ii), and the fact that the quantity $c(\{f(n)\}, q) - 1 + \{f(n)\}$ increases w.r.t. $\{f(n)\}$, which in turn is readily seen from the proof of Proposition 5.1. □

We can now evaluate the integer part of the means μ_n.

LEMMA 5.2. *Suppose again that the fractional part $\{f(n)\} = \beta$ is constant.*

(i) If $\beta \neq \frac{1}{2}$, then $f(n) + c(\beta, q) \notin \mathbb{Z}$. Thus

$$\lfloor \mu_n \rfloor = \lfloor f(n) + c(\beta, q) \rfloor = \lfloor f(n) \rfloor + \lfloor \beta + c(\beta, q) \rfloor.$$

(ii) For $\beta = \frac{1}{2}$,

$$\mu_n > f(n) + \frac{1}{2}, \text{ if } 2f(n) \le n - 1 \quad \text{and} \quad \mu_n < f(n) + \frac{1}{2}, \text{ if } 2f(n) \ge n.$$

Thus

$$\lfloor \mu_n \rfloor = f(n) + \frac{1}{2}, \qquad \text{if } 2f(n) \leq n-1$$

and

$$\lceil \mu_n \rceil = f(n) + \frac{1}{2}, \qquad \text{if } 2f(n) \geq n.$$

Proof. Part (i) is proved similarly to the preceding lemma. As for part (ii), assume $2f(n) \leq n-1$ first. Then μ_n equals

$$\begin{aligned}\sum_{i=0}^{n-1} \frac{q^{i-f(n)}}{1+q^{i-f(n)}} &= \sum_{i=0}^{f(n)-\frac{1}{2}} \frac{q^{i-f(n)}}{1+q^{i-f(n)}} + \sum_{i=f(n)+\frac{1}{2}}^{2f(n)} \frac{q^{i-f(n)}}{1+q^{i-f(n)}} \\ &\quad + \sum_{2f(n)+1}^{n-1} \frac{q^{i-f(n)}}{1+q^{i-f(n)}} \\ &= \sum_{i=0}^{f(n)-\frac{1}{2}} \frac{q^{-i-\frac{1}{2}}}{1+q^{-i-\frac{1}{2}}} + \sum_{i=0}^{f(n)-\frac{1}{2}} \frac{q^{i+\frac{1}{2}}}{1+q^{i+\frac{1}{2}}} + \mathrm{o}(1) \\ &= f(n) + \frac{1}{2} + \mathrm{o}(1).\end{aligned}$$

The o(1)-term is non-negative (and vanishes only for $2f(n) = n-1$). If $2f(n) \geq n$, then the third sum vanishes and the second sum just runs up to $n-1 < 2f(n)$, so $\mu_n < f(n) + \frac{1}{2}$. □

Now we are in a position to establish the announced convergence of the normalized X_n to a discrete normal random variable.

Proof of Theorem 5.1. Note that σ_n converges, which follows from the identity

$$\sigma_n^2 = \sum_{i=0}^{n} \frac{q^{i-f(n)}}{(1+q^{i-f(n)})^2} = \sum_{i=0}^{\lfloor f(n) \rfloor} \frac{q^{i-f(n)}}{(1+q^{i-f(n)})^2} + \sum_{i=\lfloor f(n) \rfloor+1}^{n} \frac{q^{i-f(n)}}{(1+q^{i-f(n)})^2}.$$

First we consider the case $\beta \neq 1/2$. To evaluate the product in the denominator of

$$\mathbb{P}(X_n = \lfloor \mu_n \rfloor + x) = \begin{bmatrix} n \\ \lfloor \mu_n \rfloor + x \end{bmatrix}_q \frac{q^{-(\lfloor \mu_n \rfloor + x) f(n) + (\lfloor \mu_n \rfloor + x)(\lfloor \mu_n \rfloor + x - 1)/2}}{\prod_{i=0}^{n-1} \left(1 + \frac{q^i}{q^{f(n)}}\right)}, \tag{5.10}$$

we split it into two parts:

$$\prod_{i=0}^{n-1}\left(1+\frac{q^i}{q^{f(n)}}\right)=\prod_{i=0}^{\lfloor f(n)\rfloor}\left(1+\frac{q^i}{q^{f(n)}}\right)\prod_{i=\lfloor f(n)\rfloor+1}^{n-1}\left(1+\frac{q^i}{q^{f(n)}}\right).$$

Using Relation I.3 of Gasper and Rahman (1990), we obtain

$$\prod_{i=0}^{n-1}\left(1+\frac{q^i}{q^{f(n)}}\right)=q^{-f(n)(\lfloor f(n)\rfloor+1)+(\lfloor f(n)\rfloor+1)\lfloor f(n)\rfloor/2}\times \left(-q^{\beta};q\right)_{\lfloor f(n)\rfloor+1}\left(-q^{-\beta+1};q\right)_{n-\lfloor f(n)\rfloor-2}. \quad (5.11)$$

The last two terms in (5.11) tend to $e_q\left(-q^{\beta}\right)$ and $e_q\left(-q^{-\beta+1}\right)$, respectively. The q-binomial coefficient in (5.10) tends to $e_q(q)$. By Lemma 5.2, we can simplify the exponent of q resulting from (5.10) and (5.11) to

$$\frac{1}{2}(x-1+\delta)(\delta-2\beta+x),$$

where $c=c(\beta,q)$ and

$$\delta=\lfloor\beta+c\rfloor=\begin{cases}0 & \beta<1/2\\ 1 & \beta>1/2\end{cases}$$

by Lemma 5.1 (iii). Putting things together, we obtain

$$\mathbb{P}(X_n=\lfloor\mu_n\rfloor+x)\to e_q(q)e_q\left(-q^{\beta}\right)e_q\left(-q^{-\beta+1}\right)q^{\frac{(\delta+x-1)(\delta+x-2\beta)}{2}}.$$

By normalizing X_n we get (5.2).

For $\beta=1/2$ define

$$G(\mu_n):=\begin{cases}\lfloor\mu_n\rfloor & \text{if } 2f(n)\le n-1\\ \lceil\mu_n\rceil & \text{if } 2f(n)\ge n.\end{cases}$$

Then

$$\mathbb{P}\left(X_n=G(\mu_n)+x\right)=\begin{bmatrix}n\\ G(\mu_n)+x\end{bmatrix}_q\frac{q^{-(G(\mu_n)+x)f(n)+(G(\mu_n)+x)(G(\mu_n)+x-1)/2}}{\prod_{i=0}^{n-1}\left(1+\frac{q^i}{q^{f(n)}}\right)}.$$

The q-binomial-coefficient tends to $e_q(q)$, and the product can be transformed as above. This time the exponent of q equals $\frac{x^2}{2}$. So we have

$$\mathbb{P}\left(X_n=G(\mu_n)+x\right)\to e_q(q)e_q\left(-q^{\frac{1}{2}}\right)^2q^{\frac{x^2}{2}},$$

from which (5.2) follows by normalizing X_n. □

With little extra effort one can see that the limit distribution is symmetric if and only if $\beta = 0$ or $\beta = 1/2$.

Again, the convergence of the distributions in Theorem 5.1 yields a convergence property of the corresponding orthogonal polynomials. The orthogonal polynomials for the discrete normal distribution are the Stieltjes-Wigert polynomials $S_k(x; q)$, see Christiansen and Koelink (2006), Koekoek and Swarttouw (1998).

Corollary 5.2. *Let x be a real number, and $f(n)$ as in* (5.1)*, with $\{f(n)\}$ constant. Then, as $n \to \infty$, the q-Krawtchouk polynomial given by $K_k(q^{-x-f(n)+o(1)}; q^{f(n)-n}, n; q)$ tends to $(q; q)_k \times S_k(q^{-x}; q)$.*

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References

Benkherouf, L. and Bather, J.A. (1988). Oil exploration: sequential decisions in the face of uncertainty. *J. Appl. Probab.*, **25**, 529–543.

Christiansen, J.S. and Koelink, E. (2006). Self-adjoint difference operators and classical solutions to the Stieltjes-Wigert moment problem. *J. Approx. Theory*, **140**, 1–26.

Flajolet, P., Gourdon, X. and Dumas, P. (1995). Mellin transforms and asymptotics: harmonic sums. *Theoret. Comput. Sci.*, **144**, 3–58.

Gasper, G. and Rahman, M. (1990). *Basic Hypergeometric Series*. With a foreword by Richard Askey. Encyclopedia of Mathematics and its Applications, vol. **35**. Cambridge University Press, Cambridge.

DE HAAN, L. and FERREIRA, A. (2006). *Extreme Value Theory. An Introduction.* Springer, New York.

JOHNSON, N.L., KEMP, A.W. and KOTZ, S. (2005). *Univariate Discrete Distributions.* Third edition. Wiley Series in Probability and Statistics. Wiley-Interscience, Hoboken, NJ.

KEMP, A. and KEMP, C. (1991). Weldon's dice data revisited. *Amer. Statist.*, **45**, 216–222.

KEMP, A.W. (1992a). Heine-Euler extensions of the Poisson distribution. *Comm. Statist. Theory Methods*, **21**, 571–588.

KEMP, A.W. (1992b). Steady-state Markov chain models for the Heine and Euler distributions. *J. Appl. Probab.*, **29**, 869–876.

KEMP, A.W. (2002). Certain q-analogues of the binomial distribution. *Sankhyā Ser. A*, **64**, 293–305. Selected articles from San Antonio Conference in honour of C. R. Rao (San Antonio, TX, 2000).

KEMP, A.W. (2003). Characterizations involving $U|(U + V = m)$ for certain discrete distributions. *J. Statist. Plann. Inference*, **109**, 31–41.

KEMP, A.W. and NEWTON, J. (1990). Certain state-dependent processes for dichotomised parasite populations. *J. Appl. Probab.*, **27**, 251–258.

KOEKOEK, R. and SWARTTOUW, R.F. (1998). The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue. Technical Report, 98-17, Delft University of Technology. Available at `http://fa.its.tudelft.nl/~koekoek/askey.html`.

RAO, C.R. and SHANBHAG, D.N. (1994). *Choquet-Deny Type Functional Equations with Applications to Stochastic Models.* John Wiley & Sons Ltd., Chichester.

SZABŁOWSKI, P. (2001). Discrete normal distribution and its relationship with Jacobi theta functions. *Statist. Probab. Lett.*, **52**, 289–299.

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