UNIMODALITY OF TWO DISTRIBUTIONS RELATED TO THE NEGATIVE BINOMIAL DISTRIBUTION

STEFAN GERHOLD

Abstract. We show that the probability mass function of the minimum negative binomial distribution, also known as the riff-shuffle distribution, is unimodal. Furthermore, we present an asymptotic criterion for the unimodality of the maximum negative binomial distribution.

Keywords: Discrete distribution, unimodal sequence, minimum negative binomial distribution, maximum negative binomial distribution

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1. Introduction

For the purpose of fitting distributions to data or using them to approximate other distributions, it is desirable to have information about their shape, in particular, about their modes. Recall that a discrete distribution $(f_k)_{k\in\mathbb{Z}}$ is called unimodal if there is k_0 such that $f_{k-1} \le f_k$ for $k \le k_0$ and $f_k \ge f_{k+1}$ for $k \ge k_0$. For the statistical and probabilistic significance of unimodality, and advanced techniques for proving it, we refer to Bertin, Cuculescu, and Theodorescu [3], and Dharmadhikari and Joag-Dev [4].

In this note we establish two unimodality results about the minimum (resp. maximum) negative binomial distribution by elementary analysis. Other recent unimodality results for sequences involving binomial coefficients include those by Ghitany, Bouzar, and Aly [5], by Belbachir, Bencherif and Szalay [1], and by Belbachir and Szalay [2].

2. The Minimum Negative Binomial Distribution

The minimum negative binomial (or riff-shuffle) distribution is defined for $0 < p < 1$ and integers $m \ge 1$ by the probability mass function

$$f_k = \binom{m+k-1}{k}(p^m q^k + q^m p^k), \qquad k = 0, \dots, m-1,$$

where $p + q = 1$. If cards are taken with probabilities p and q from two decks of m cards, then the number of cards that have been taken from the remaining deck when one deck has been depleted follows this distribution [6, 7, 8]. Equivalently, it is the distribution of the random variable $X - m$, where X denotes the number of

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Bernoulli trials, with individual success probability p, until either m successes or m failures have been observed.

Uppuluri and Blot [7] gave expressions for the moment generating function, the mean, and the variance in terms of the incomplete Beta function and presented some applications. We will now confirm an observation of Johnson, Kotz, and Kemp [6], who reported that the minimum negative binomial distribution appears to be unimodal.

Theorem 1. *The minimum negative binomial distribution is unimodal. It has either a single peak or a plateau with two elements.*

Proof. For $p = \frac{1}{2}$ it is straightforward to show that $f_0 < f_1 < \cdots < f_{m-2} = f_{m-1}$. By symmetry, we may now assume w.l.o.g. that $0 < p < \frac{1}{2}$. Defining $h(x) := p^m q^x + q^m p^x$, we have

$$\frac{f_{k+1}}{f_k} = \frac{k+m}{k+1} \cdot \frac{h(k+1)}{h(k)}, \qquad k = 0, \ldots, m-2.$$

Therefore, $f_{k+1} \leq f_k$ is equivalent to $g(k) \geq 0$, where the auxiliary function g is defined by

$$g(x) := \frac{h(x)}{h(x+1)} - \frac{x+m}{x+1}.$$

We show that g is concave on $[0, m-2]$. The second derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} \frac{h(x)}{h(x+1)} = \frac{(q-p)(\log p - \log q)^2 p^{m+x} q^{m+x} (p^m q^{x+1} - p^{x+1} q^m)}{(p^m q^{x+1} + p^{x+1} q^m)^3}$$

is negative for $0 \leq x \leq m-2$, because $p < q$ and $(p/q)^{m-x-1} < 1$ implies $p^m q^{x+1} - p^{x+1} q^m < 0$. Thus, $h(x)/h(x+1)$ is concave. Since $(x+m)/(x+1)$ is a convex function of $x \in [0, m-2]$, the function g is indeed concave. At $x = m-2$, the function value

$$\begin{aligned} g(m-2) &= \frac{h(m-2)}{h(m-1)} - \frac{2m-2}{m-1} \\ &= \frac{p^m q^{m-2} + q^m p^{m-2}}{p^m q^{m-1} + q^m p^{m-1}} - 2 \\ &= \frac{1/p^2 + 1/q^2}{1/p + 1/q} - 2 \\ &= \frac{p}{q} + \frac{q}{p} - 2 > 0 \end{aligned}$$

is positive. From this and concavity of g we conclude that, if $g(l) \geq 0$ for some l, then the values $g(k)$, $l < k \leq m-2$, are positive. In other words, as soon as there is a descent $f_{l+1} \leq f_l$, the sequence $(f_k)_{k=l+1}^{m-1}$ strictly decreases. □

3. The Maximum Negative Binomial Distribution

A related distribution is the maximum negative binomial distribution [6, 8], which is defined for positive integral parameters c by

$$f_k = \binom{2c+k-1}{c-1} (p^k + q^k)(pq)^c, \qquad k = 0, 1, \ldots.$$

It is the distribution of the smallest number of Bernoulli trials needed to obtain at least c successes and c failures.

Zhang, Burtness, and Zelterman [8] applied the distribution in the design of a medical experiment. Among other things, they studied approximations and moments of the distribution, and they noticed that it is always unimodal for $c = 1$. For larger c, they found empirically that the distribution is unimodal if p lies in a certain interval around $p = \frac{1}{2}$, the size of which decreases as c grows. We now show that the endpoints of the unimodality region are, asymptotically, given by $p = \frac{1}{2} \pm 1/(2\sqrt{2c})$. Figure 1 illustrates the approximation. It is easy to see that there is always a mode at zero, so that the distribution decreases, if it is unimodal. (Such distributions are sometimes called sesquimodal [6].)

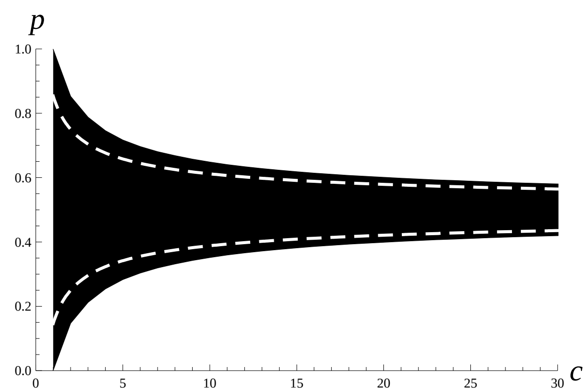

FIGURE 1. The parameter region for which the maximum negative binomial distribution is unimodal (black), and the curves $p = \frac{1}{2} \pm 1/(2\sqrt{2c})$ (dashed).

Theorem 2. *Let ε be a positive real number. If c is sufficiently large, then the maximum negative binomial distribution with parameters p and c is decreasing (hence unimodal) if*

$$\left|p - \frac{1}{2}\right| \le \left(\frac{1}{2\sqrt{2}} - \varepsilon\right) \frac{1}{\sqrt{c}}, \tag{1}$$

and not unimodal if

$$\left|p - \frac{1}{2}\right| \ge \left(\frac{1}{2\sqrt{2}} + \varepsilon\right) \frac{1}{\sqrt{c}}. \tag{2}$$

Proof. Analogously to the proof of Theorem 1, we have that $f_{k+1} \le f_k$ is equivalent to $g(k) \ge 0$, where

$$g(x) = g(x; p, c) = \frac{p^x + q^x}{p^{x+1} + q^{x+1}} - \frac{x + 2c}{x + c + 1}.$$

By symmetry, we may restrict attention to $p \ge \frac{1}{2}$. For fixed $k \ge 1$ and $c \ge 1$, the function g decreases in the interval $\frac{1}{2} \le p < 1$. Indeed, the derivative

$$\frac{\partial g(k; p, c)}{\partial p} = \frac{q^{2k} - p^{2k} - k(pq)^{k-1}(2p - 1)}{(p^{k+1} + q^{k+1})^2}$$

is obviously negative for $p > q$. Therefore, to prove the first assertion of the theorem, it suffices to show that g is positive for large c, where

$$p = \frac{1}{2} + \frac{\alpha}{\sqrt{c}} \qquad \text{with} \qquad \alpha = \frac{1}{2\sqrt{2}} - \varepsilon.$$

Since $(p^k+q^k)/(p^{k+1}+q^{k+1})$ decreases in k (easy to see from the derivative w.r.t. k) and tends to $1/p$ as $k \to \infty$, we have

$$\frac{p^k + q^k}{p^{k+1} + q^{k+1}} > \frac{1}{p}, \qquad k \geq 0.$$

This shows that $g(k)$ is certainly positive if

$$\frac{k+2c}{k+c+1} \leq \frac{1}{p},$$

which is equivalent to

$$k \geq \frac{c(2p-1)-1}{q} = 4\alpha\sqrt{c} + 8\alpha^2 - 2 + \mathrm{O}(c^{-1/2}), \qquad c \to \infty.$$

Furthermore, the value $g(0) = 1/(c+1)$ is always positive, corresponding to the mode at zero. To examine the remaining range $1 \leq k \leq 4\alpha\sqrt{c}$, we put $k = u\sqrt{c}$, so that $u = \mathrm{O}(1)$ as $c \to \infty$. From the equations

$$\begin{aligned} p^k &= \left(\frac{1}{2}\right)^k \left(1 + \frac{2\alpha}{\sqrt{c}}\right)^{u\sqrt{c}} \\ &= \mathrm{e}^{2\alpha u}\left(1 - \frac{2\alpha^2 u}{\sqrt{c}} + \frac{8\alpha^3 u}{3c} + \mathrm{O}\left(\frac{1}{c^{3/2}}\right)\right) \end{aligned}$$

and

$$\begin{aligned} q^k &= \left(\frac{1}{2}\right)^k \left(1 - \frac{2\alpha}{\sqrt{c}}\right)^{u\sqrt{c}} \\ &= \mathrm{e}^{-2\alpha u}\left(1 - \frac{2\alpha^2 u}{\sqrt{c}} - \frac{8\alpha^3 u}{3c} + \mathrm{O}\left(\frac{1}{c^{3/2}}\right)\right), \end{aligned}$$

a tedious, but straightforward calculation yields

$$g(k) = (u - 4\alpha \tanh 2\alpha u)\frac{1}{\sqrt{c}} + \left(2 + 8\alpha^2 - u^2 - \frac{8\alpha^2}{\cosh^2 2\alpha u}\right)\frac{1}{c} + \mathrm{O}\left(\frac{1}{c^{3/2}}\right)$$

as $c \to \infty$, uniformly for $c^{-1/2} < u < 4\alpha$. Since $\tanh x \leq x$, $\cosh^2 x \geq 1$, and $u \leq 4\alpha$, we obtain

$$\begin{aligned} g(k) &\geq \frac{u(1-8\alpha^2)}{\sqrt{c}} + \frac{2-u^2}{c} + \mathrm{O}\left(\frac{1}{c^{3/2}}\right) \\ &\geq \frac{2(1-8\alpha^2)}{c} + \mathrm{O}\left(\frac{1}{c^{3/2}}\right), \qquad 1 \leq k \leq 4\alpha\sqrt{c}. \end{aligned}$$

As $1-8\alpha^2$ is a positive constant, this shows that g is positive if c is sufficiently large. By the definition of g, this means that the probability mass function decreases, so there is no mode besides the one at zero.

To establish the second part of the theorem, we put

$$p = \frac{1}{2} + \frac{\beta}{\sqrt{c}} \qquad \text{with} \qquad \beta = \frac{1}{2\sqrt{2}} + \varepsilon.$$

Our goal is to show that the function g has negative values. By the monotonicity of $g(k;p,c)$ w.r.t. p, negative g-values for this choice of p will induce negative g-values for any larger p, too. Suppose now that $c^{1/10} \le k \le c^{1/6}$, i.e., $c^{-2/5} \le u \le c^{-1/3}$, where we write again $k = u\sqrt{c}$. As above, we have

$$g(k) = (u - 4\beta \tanh 2\beta u)\,\frac{1}{\sqrt{c}} + \mathrm{O}\left(\frac{1}{c}\right).$$

Since $\tanh 2\beta u = 2\beta u + \mathrm{O}(u^3)$, we thus obtain

$$\begin{aligned} g(k) &= \frac{u(1-8\beta^2)}{\sqrt{c}} + \mathrm{O}\left(\frac{1}{c}\right) \\ &\le \frac{1-8\beta^2}{c^{9/10}} + \mathrm{O}\left(\frac{1}{c}\right). \end{aligned}$$

The constant $1 - 8\beta^2$ is negative, so that $g(k)$ is negative for large c and $c^{1/10} \le k \le c^{1/6}$. In particular, for large c there is a positive integer k_0 for which $g(k_0)$ is negative, hence $f_{k_0+1} > f_{k_0}$. There must be a mode of the distribution at or beyond this k_0, besides the mode at zero, hence the distribution is not unimodal. □

The reader might wonder how to guess that $1/\sqrt{c}$ gives the correct growth order in (1) and (2). The answer can be gleaned from the lower end of the distribution, via the function g defined in the proof of Theorem 2. If we fix $k = 1$, the (unique) parameter p that solves $g(1) = 0$ for large c is given by $p = \frac{1}{2} + \frac{1}{2}\sqrt{3/(2c)} + \mathrm{O}(c^{-3/2})$. If p is slightly larger than this, the value $g(1)$ will become negative, which indicates a second mode of the distribution. Similarly, the solution of $g(2) = 0$ satisfies $p = \frac{1}{2} + 1/(2\sqrt{c}) + \mathrm{O}(c^{-5/2})$, and the same reasoning applies.

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Vienna University of Technology, Wiedner Hauptstrasse 8–10, 1040 Vienna, Austria
E-mail address: `sgerhold at fam.tuwien.ac.at`