Difference equation theory meets mathematical finance

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Abstract This paper is an extension of Stefan Gerhold's talk at Peter Paule's birthday conference (RISC, May 2018). The main topics of this talk were two applications of Pringsheim's theorem and two applications of the saddle point method. In this paper, we add a section on Hankel contour asymptotics. This method is well-known in analytic combinatorics from Flajolet and Odlyzko's singularity analysis of generating functions, and is applied here to a problem from mathematical finance.

1 Introduction

"Difference equations" and "mathematical finance" appearing in one sentence may evoke the association of numerical derivative pricing by discretizing PDEs. This is *not* what this paper is about. Rather, it deals with a 19th century result from complex analysis (Pringsheim's theorem) and two asymptotic methods (saddle point asymptotics; Hankel contour asymptotics) that have been applied to some problems from the theory of difference equations, and more recently in financial mathematics. Sections 2 and 3 of the present paper are surveys of articles that have appeared elsewhere, whereas most of Section 4 has not been published in a journal, but only in Arpad Pinter's PhD thesis [29]. The reader might be a bit surprised that the content of this paper is only peripherally related to Peter Paule's research interests. The reason is that my (Gerhold's) research during my PhD studies soon started to deviate from symbolic summation towards asymptotics and other problems, followed by a switch to mathematical finance. I am very grateful to Peter for tolerating this as my supervisor, for sparking my interest in combinatorics with his marvellous lectures and lively weekly seminar, and for many useful pieces of advice.

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2 Pringsheim's theorem: oscillations and the rough Heston model

A part of the PhD thesis [14] is devoted to proving inequalities by computer algebra. The proving method presented there and in [1, 18, 19] has received further attention, e.g. in [27, 28]. Here, we recall another problem on inequalities that was investigated in [14] and the subsequent paper [2]. With difference equations being a very common topic in Peter Paule's lectures and research seminar, it seemed to be a natural question to study the positivity of solutions of the simplest kind of linear difference equations: those with constant (real) coefficients, whose solutions are commonly referred to as *recurrence sequences*. In [2], the following result was established in this direction:

Theorem 1 (Bell, Gerhold 2007)

Let $(f_n)_{n\in\mathbb{N}}$ be a nonzero recurrence sequence with no positive dominating characteristic root. Then the sets $\{n \in \mathbb{N} \colon f_n > 0\}$ and $\{n \in \mathbb{N} \colon f_n < 0\}$ have positive density.

The dominating characteristic roots are the roots of maximal modulus among the roots of the characteristic polynomial. They occur in the leading term of the well-known explicit representation of recurrence sequences. Applying the following theorem to the generating function $\sum_{n=1}^{\infty} f_n z^n$ immediately implies the weaker statement that these index sets are both infinite. This has been noted in Theorem 7.1.1 of [21].

Theorem 2 (Pringsheim's theorem)

Suppose that the power series $F(z) = \sum_{n=0}^{\infty} a_n z^n$ has positive finite radius of convergence R, and that all the coefficients are non-negative real numbers. Then F has a singularity at R.

Alfred Pringsheim (1850–1941), father in law of Thomas Mann, proved this result in 1894. For a proof, see Remmert [30], p. 235, or Flajolet and Sedgewick [10], p. 240. I (Gerhold) must admit that I was not aware of Pringsheim's theorem when writing [2]. Shortly after the paper was published in 2007, Alan Sokal informed my coauthor Jason Bell and me that our proof of Theorem 1 can be shortened, since it is a corollary of Theorem 1 in [2] and a generalized version of Pringsheim's theorem (see p. 242 in [4]). This shortcut did not make our paper obsolete, since the *existence* of the densities in Theorem 1 is a non-trivial fact, and moreover there are further results in [2]. See also [15] and, for more recent results on the sign of recurrence sequences, [26].

We now switch to an apparently completely unrelated topic. In mathematical finance, continuous time stochastic processes are used to model the unknown future behavior of assets such as stocks, FX rates, and others. In recent years, so-called *rough* models have received a lot of attention. We just mention that rough refers to the "low" Hölder continuity of the paths, and that these models feature excellent statistical properties when applied to real market data, while their numerical treatment poses some challenges. One particular such model is El Euch and Rosenbaum's rough Heston model [6], with parameters $\rho \in (-1,1)$, and $\lambda, \xi, \bar{v} > 0$, $\alpha \in (\frac{1}{2}, 1)$. It is defined by the SDE (stochastic differential equation)

$$\begin{aligned} dS_t &= S_t \sqrt{V_t}\, dW_t, \quad S_0 > 0, \\ V_t &= V_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda(\bar{v} - V_s)\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \xi \sqrt{V_s}\, dZ_s, \qquad (1) \\ d\langle W, Z\rangle_t &= \rho\, dt. \end{aligned}$$

Here, W and Z are correlated Brownian motions. The process S models an asset price, and $\sqrt{V}$ its stochastic volatility. in [17], we investigated the *moment explosion time* of the rough Heston model. Briefly, this amounts to finding the domain of the map $(u,t) \mapsto \mathbb{E}[S_t^u]$. Knowing the time, depending on u, at which the moment $\mathbb{E}[S_t^u]$ ceases to exist is important when implementing option pricing, as we elucidate in the introduction of [17]. In [6], it was shown that $\mathbb{E}[S_t^u]$ can be expressed by the solution of a fractional Riccati equation:

$$\mathbb{E}[S_t^u] = \exp\left(\bar{v}\lambda I_t^1 \psi(u,t) + v_0 I_t^{1-\alpha} \psi(u,t)\right),$$

where ψ satisfies

$$D_t^\alpha \psi(u,t) = R(u, \psi(u,t)) \qquad (2)$$

with initial condition $I_t^{1-\alpha}\psi(u,0) = 0$. Here, D and I denote the Riemann-Liouville fractional derivative resp. integral, and R is a certain polynomial whose coefficients depend on the model parameters. In [17] a fractional power series

$$\sum_{n=1}^{\infty} a_n(u) t^{\alpha n} \qquad (3)$$

representing this solution was found. Thus, the problem of finding the moment explosion time is transferred to finding the explosion time of (3). The radius of convergence of the power series

$$\sum_{n=1}^{\infty} a_n(u) z^n \qquad (4)$$

can be easily computed, because the fractional ODE (2) yields a recurrence that allows to compute the coefficients $a_n(u)$. However, a priori this need not yield the explosion time. To wit, the explosion time is related to the smallest singularity of (4) *on the positive real axis,* whereas there might be singularities closer to the origin that are negative or non-real, and therefore practically meaningless. This is where Pringsheim's therem enters the stage. Under some restrictions on the parameters, we could show that $a_n(u) \geq 0$ holds, and so Theorem 2 guarantees that the explosion time can be computed from the radius of convergence of (4). Thus, we can determine the domain of finiteness of $\mathbb{E}[S_t^u]$, which is the basis for efficiently evaluating integrals needed to price options in the rough Heston model.

3 Saddle point asymptotics: non-holonomic sequences and the Heston model

A holonomic sequence is a sequence of numbers that satisfies a linear difference equation with polynomial coefficients. This class of sequences, and their generating functions, has received a lot of attention, in particular from the viewpoint of automatic identity proving. Among a very large number of papers, we just cite [5, 23, 31]. When looking for problems to solve during my PhD thesis, I (Gerhold) started to think about the theoretical question of proving the *non*-holonomicity of certain sequences [3, 7, 13]. Asymptotic expansions are a very useful tool for this, because a holonomic generating function satisfies a linear ODE with polynomial coefficients, and it is well known that functions of this kind have a very restricted asymptotic behavior. This method was applied to a good deal of examples in [7]. In 2005, I sent an email to Philippe Flajolet, asking whether the approach could be used to prove that the sequence $e^{1/n}$ is not holonomic. I quote from his response:

This is interesting and here's a way we think it can be done. We didn't reflect too much about it however and didn't work out details. Take $f_n = \exp(1/n)$ *and let* $F(z) = \sum f_n(-z)^n$ *be the corresponding OGF, taken for convenience with alternating signs. We want to prove, right in line with our joint paper, that there is some nonholonomic element in* $F(z)$ *as* $z \to \infty$*. Start from the Lindelöf integral*

$$F(z) = \frac{1}{2i\pi} \int \exp(1/s) z^s \frac{\pi}{\sin(\pi s)} ds,$$

taken along $1/2 - i\infty$ *to* $1/2 + i\infty$*. [Proved by residues upon closing by a large semicircle on the right, seems to work well here.] Then, move the integration line close to* $\mathrm{Re}(s) = 0$ *where the integrand blows up. There's a saddle point, a function of* z*, at*

$$s_0 = 1/\sqrt{\log z}$$

roughly. Then, $F(z)$ *should behave more or less like* $\exp(2\sqrt{\log z})$ *as* $z \to +\infty$*. This is nonholonomic. The full argument [to contradict the structure theorem] needs making sure we can shake the argument of* z *a little, betwen some* $[-\varepsilon, +\varepsilon]$*, but usually the* sin *in the denominator of the integrand plays in your favour.*

This proof strategy worked, of course, although it took us some years of intermittent work to finish the corresponding paper [8], which contains several other asymptotic results (see also Section 4 of the present paper). Concerning the generating function of $e^{1/n}$, the above saddle point approach yields

$$\sum_{n\geq 1} e^{1/n}(-z)^n \sim -\frac{e^{2\sqrt{\log z}}}{2\sqrt{\pi}(\log z)^{1/4}}, \quad z \to \infty, \tag{5}$$

where the left hand side is to be understood in the sense of analytic continuation. This shows that $e^{1/n}$ is a non-holonomic sequence, since the right hand side cannot be asymptotically equivalent to any holonomic function. The latter statement follows from a well-known result on the asymptotic behavior of ODE solutions, summarized in Theorem 2 of [7]. Moreover, using (5), we evaluated the alternating sum

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k e^{1/k} \sim -\frac{e^{2\sqrt{\log n}}}{2\sqrt{\pi}(\log n)^{1/4}}$$

asymptotically for $n \to \infty$, which would be hard by elementary methods.

Again, we now jump to mathematical finance. Among the many asset price models that have been suggested and studied, sending $\alpha \to 1$ in the model from the previous section yields a particularly well-known one: The classical (non-rough) *Heston model.* Since its introduction in 1993 (see [22]), it has been used by many practitioners and studied by many researchers. The main advantages of this model are its explicit characteristic function, which allows for fast and easy option price computation, and its reasonable fit to market data. Its dynamics are as in (1), but with $\alpha = 1$, which removes the weakly singular kernel $(t-s)^{\alpha-1}$. This dramatically improves the regularity of the processes S, V and the numerical tractability of the model, at the price of a less satisfactory fit to financial market data.

My (Gerhold's) work on the Heston model began in 2009 at the ÖMG-DMV congress in Graz, when Peter Friz (TU Berlin) asked my colleague Friedrich Hubalek (TU Wien) about applying the saddle point method to option prices. Friedrich Hubalek directed Peter Friz to me, and we started to analyze option prices given by the Heston model asymptotically. A call option gives the option holder the right, but not the obligation, to buy a unit of the underlying asset at time T for the strike price K, where T and K are fixed. The payoff of this option at maturity T is $(S_T - K)^+$, because a share price $S_T > K$ yields a profit of $S_T - K$, whereas the option becomes worthless in the case $S_T \leq K$. At time zero, the price of the call option is

$$C(K,T) = \mathbb{E}[(S_T - K)^+].$$

We assumed zero interest rate here, and skipped the subtle but very important point that the expectation is to be taken under a special – so-called risk neutral – probability measure that does *not* conicide with the "real world" probability. The question we dealt with in [12] is the asymptotic behavior of $C(K,T)$ for large strike K, if S_T has the probability distribution given by the Heston model. The call price can be recovered by Fourier inversion from the moment generating function:

$$C(K,T) = \frac{K}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} K^{-u} \frac{\mathbb{E}[S_T^u]}{u(u-1)} du. \tag{6}$$

Implementing this numerically requires knowing the domain of the characteristic function (equivalently, of the moment generating function), because this yields the possible values of β, the real part of the integration contour. This is the question we mentioned in Section 2 for the *rough* Heston model. For classical Heston, this domain is well-known, because the characteristic function has an explicit expression. It turns out that, at the border of this domain, it has a singularity of the form "exponential of a pole". Thus, a saddle point analysis with some similarities to the one above could be applied to the problem of approximating (6) (see also [16]). While the tail estimates are quite different, the local expansion, yielding the dominant term, is very similar. The formulas are somewhat tedious, and so we refer to [12] for details. We just mention that there are constants c_i, positive for $i = 1,2,3$, such that the Heston call price satisfies

$$C(K,T) \sim c_1 K^{-c_2} e^{c_3 \sqrt{\log K}} (\log K)^{c_4}, \quad K \to \infty,$$

for $T > 0$ fixed. The dominating factor K^{-c_2} was known before and follows quite easily from the explicit moment generating function. The sub-polynomial factor $e^{c_3 \sqrt{\log K}}$ was the main contribution of [12], improving numerical accuracy significantly. We recall here the role of the asymptotic factor $e^{2\sqrt{\log z}}$ in (5), which came from a very similar saddle point analysis, and proves the non-holonomicity of the function on the left hand side of (5).

4 Hankel contour asymptotics: non-holonomic sequences and the 3/2–model

4.1 Setup

At the beginning of Section 3, we described an asymptotic evaluation of the generating function of $e^{1/n}$. In [8], we studied the natural extension e^{cn^θ} with parameters c and θ, and also more general sequences and their generating functions.[1] We quote here the following result:

$$\sum_{n\geq 1} e^{\pm\sqrt{n}} (-z)^n = -1 \mp \frac{1}{\sqrt{\pi \log z}} + O\big((\log z)^{-3/2}\big). \quad z \to \infty. \tag{7}$$

As above, (7) not only proves non-holonomicity of $e^{\pm\sqrt{n}}$, but also approximations such as

[1] Needless to say, Philippe Flajolet and Bruno Salvy needed no help from a PhD student to set up the various asymptotic methods used in [8], but I (Gerhold) was of some use working out the technical estimates. Among many episodes worth remembering, I vividly recall Philippe's statement after having written the introduction of [8]: "We need brains (pointing at Bruno Salvy), we need strength (pointing at me), and we need blah-blah (pointing at himself)."

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k e^{\pm\sqrt{k}} \sim -\frac{\pm 1}{\sqrt{\pi \log n}}, \quad n \to \infty.$$

The proof of (7) again starts with the Lindelöf representation

$$\sum_{n=1}^{\infty} e^{\pm\sqrt{n}} (-z)^n = -\frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} e^{\pm\sqrt{s}} z^s \frac{\pi}{\sin \pi s} ds. \tag{8}$$

This time, a saddle point approach is not appropriate, because the singularity of $e^{\pm\sqrt{s}}$ at zero is too "tame", and a somewhat larger integration contour is needed to extract sufficient asymptotic information. The method of choice is to use a contour that goes around the branch cut of $e^{\pm\sqrt{s}}$, and part of which is transformed to a Hankel contour by a substitution. Recall that a well-known application of Hankel contour asymptotics is Flajolet and Odlyzko's singularity analysis of generating functions [9, 10]. We refer to [8] for details on the asymptotic analysis of (8), but use the same method in the present section on a different problem.

Maybe unsurprisingly at this point, the problem we consider comes from mathematical finance. The model we consider is again a stochastic volatility model, which goes under the name of 3/2–model. The logarithmic stock price process $X_t = \log S_t$ in this model solves the SDE (stochastic differential equation)

$$\begin{aligned} dX_t &= -\tfrac{1}{2} V_t \, dt + \sqrt{V_t} \, dW_t, & X_0 &= x_0 \in \mathbb{R}, \\ dV_t &= \kappa V_t (\theta - V_t) \, dt + \xi V_t^{3/2} \, dZ_t, & V_0 &= v_0 > 0, \\ d\langle W, Z \rangle_t &= \rho \, dt, \end{aligned}$$

with correlated Brownian motions W and Z and parameters $\kappa > 0$, $\theta > 0$, $\xi > 0$ and $|\rho| < 1$. Define $\bar{\rho} := \sqrt{1-\rho^2}$ and $\bar{\kappa} := 2\kappa + \xi^2$. The moment-generating function (mgf) of X_T for $T > 0$ can be computed as

$$M(u,T) := \mathbb{E}[e^{uX_T}] = e^{ux_0} \frac{\Gamma(\mu_u - \alpha_u)}{\Gamma(\mu_u)} z_T^{\alpha_u} \, {}_1F_1(\alpha_u, \mu_u, -z_T), \tag{9}$$

at least for all $u \in \mathbb{C}$ in the vertical strip $a < \mathrm{Re}(u) < b$ with $a \le 0$ and $b \ge 1$, and with the confluent hypergeometric function ${}_1F_1$ and the auxiliary functions

$$\begin{aligned} \alpha_u &:= \frac{1}{\xi^2}(\gamma_u - \chi_u), & \gamma_u &:= \sqrt{\chi_u^2 - \xi^2 u(u-1)}, \\ \mu_u &:= \frac{1}{\xi^2}(\xi^2 + 2\gamma_u), & \chi_u &:= \tfrac{1}{2}\bar{\kappa} - \rho\xi u, \\ z_T &:= \frac{2}{\xi^2 \beta_T}, & \beta_T &:= \frac{v_0}{\kappa\theta}(e^{\kappa\theta T} - 1). \end{aligned} \tag{10}$$

Without loss of generality, from now on, we assume $x_0 = 0$. Define the two real numbers

$$u_\pm := \frac{1}{2\xi\bar{\rho}^2} \left(\xi - \rho\bar{\kappa} \pm \sqrt{(\xi - \rho\bar{\kappa})^2 + \bar{\kappa}^2\bar{\rho}^2} \right), \tag{11}$$

which are the unique roots of the quadratic term under the square root of γ. After factorization of the polynomial, we have the following representation of γ

$$\gamma_u = \xi\bar{\rho}\sqrt{(u_+ - u)(u - u_-)}. \tag{12}$$

Throughout, we make the technical assumption

$$\mu_{u_+} - \alpha_{u_+} > 0$$

which is always satisfied if $\rho < 0$. Under this assumption, the right boundary b of the vertical strip, where equation (9) holds, can be extended until $b = u_+$. Note that the mgf has a branch cut along $[u_+, +\infty)$ due to the branch cut of (12). For further information on the 3/2–model, see e.g. Lewis [24].

4.2 Tail asymptotics of the density

We are interested in tail asymptotics of the density function $\varphi(k,T) := \varphi_{X_T}(k)$ of X_T for $T > 0$, i.e., the asymptotic behaviour as $k \to \infty$ for fixed $T > 0$. The density function φ can be expressed via Fourier-transform as

$$\varphi(k,T) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{-ku} M(u,T)\, du, \quad k \in \mathbb{R}, \tag{13}$$

with $a \in (u_-, u_+)$. For the analysis, we adjust the integration path in (13) similarly to Friz and Gerhold [11] and split it into two parts, the critical path $C(k)$ and the neglectable path $\mathcal{N}(k)$, depending on the strike parameter $k \geq 1$. The critical contour $C(k)$ embraces the critical moment u_+, see the left panel of Figure 1. The critical path $C(k)$ starts at $u_+ + 2\log(k)/k - i/k$, goes horizontally to $u_+ - i/k$, then clockwise along the half-circle with center u_+ and radius $1/k$ until it reaches $u_+ + i/k$, and again horizontally to the end point $u_+ + 2\log(k)/k + i/k$. The remaining part, denoted by $\mathcal{N}(k)$, starts at the points $u_+ + 2\log(k)/k \pm i/k$ and goes straight to $u_+ + 2\log(k)/k + i\infty$ resp. $u_+ + 2\log(k)/k - i\infty$. This allows us to write the density function as

$$\varphi(k,T) = \frac{1}{2\pi i}\int_{C(k)\cup\mathcal{N}(k)} e^{-ku} M(u,T)\, du, \quad k \in \mathbb{R}. \tag{14}$$

Theorem 3 (Tail asymptotics)

Assume $\mu_{u_+} - \alpha_{u_+} > 0$. Then the first term in the tail expansion of the density function of X_T in the 3/2–model, with $T > 0$ fixed, is given by

$$\varphi(k,T) \sim c\frac{e^{-ku_+}}{k^{3/2}}, \quad k \to \infty, \tag{15}$$

where $c = -m_1/(2\sqrt{\pi})$ with m_1 defined in (30).

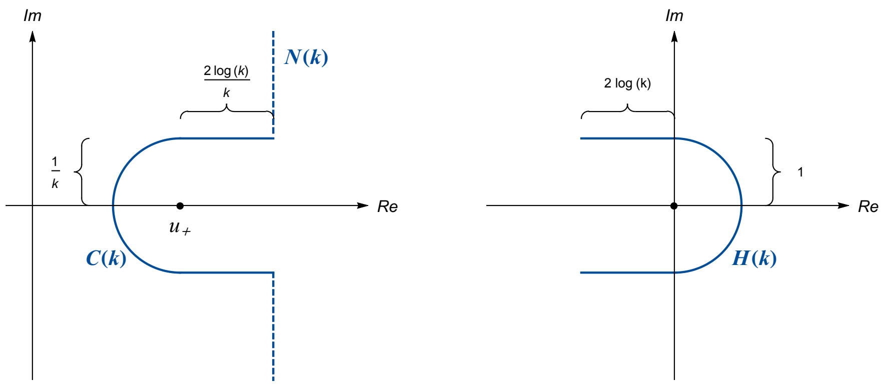

Fig. 1 In the left panel, the critical path $C(k)$ and the neglectable path $\mathcal{N}(k)$ (dashed line) are illustrated in the complex plane, whereas the right panel displays the transformed path $\mathcal{H}(k)$ after the transformation $w \mapsto u_+ - \frac{w}{k}$.

Proof The integral over $\mathcal{N}(k)$ in (14) is negligible; this will be proved in Lemma 1 below. Now consider the integral over the critical part $C(k)$ in (14). The change of variables $u = u_+ - w/k$ yields, as $k \to \infty$,

$$\frac{1}{2\pi i}\int_{C(k)} e^{-ku} M(u,T)\, du = \frac{e^{-ku_+}}{k}\left(\frac{1}{2\pi i}\int_{\mathcal{H}(k)} e^{w} M\left(u_+ - \frac{w}{k},T\right) dw\right), \quad (16)$$

where $\mathcal{H}(k)$ is the transformed path of $C(k)$, see the right panel of Figure 1. In Lemma 2 below we give an expansion of M which yields

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\mathcal{H}(k)} e^{w} M\left(u_+ - \frac{w}{k},T\right) dw \\
&\qquad = M(u_+,T)\underbrace{\frac{1}{2\pi i}\int_{\mathcal{H}(k)} e^{w}\, dw}_{O\left(\frac{1}{k^2}\right)} + \frac{m_1}{\sqrt{k}}\underbrace{\frac{1}{2\pi i}\int_{\mathcal{H}(k)} e^{w} w^{1/2}\, dw}_{\to 1/\Gamma\left(-\frac{1}{2}\right) = -\frac{1}{2\sqrt{\pi}}} \\
&\qquad\quad + \underbrace{\frac{1}{2\pi i}\int_{\mathcal{H}(k)} e^{w} O\left(\frac{w}{k}\right) dw}_{O\left(\frac{1}{k}\right)}.
\end{aligned}$$

The first integral is an easy computation. In the second and third integral, we used Hankel's integral representation for the gamma function, see [25]. Therefore,

$$\frac{1}{2\pi i}\int_{C(k)} e^{-ku} M(u,T)\, du \sim c\,\frac{e^{-ku_+}}{k^{3/2}}, \quad k \to \infty,$$

for $c = \frac{-m_1}{2\sqrt{\pi}}$. □

While we established just first order asymptotics in Theorem 3, we note that the same method easily yields further terms in the asymptotic expansion, if desired.

Lemma 1 *The integral over* $\mathcal{N}(k)$ *in* (14) *satisfies*

$$\frac{1}{2\pi i}\int_{\mathcal{N}(k)} e^{-ku} M(u,T)\,du = o\left(e^{-ku_+}k^{-3/2}\right), \quad k \to \infty.$$

Proof By symmetry, it suffices to consider only the integral over the upper part of the contour $\mathcal{N}(k)$. We define the path $u_k(t) := u_+ + 2\log k/k + it$ with $t \in [1/k, \infty)$,

$$\frac{1}{2\pi i}\int_{u_k} e^{-ku} M(u,T)\,du = \frac{e^{-ku_+}}{k^2}\left(\frac{1}{2\pi}\int_{1/k}^{\infty} e^{-itk} M(u_k(t),T)\,dt\right).$$

By showing the boundedness of the latter integral, the proof is finished. We use the triangular inequality for integrals and split the integral into two parts,

$$\left|\int_{1/k}^{\infty} e^{-itk} M(u_k(t),T)\,dt\right| \le \int_{1/k}^{t_1} |M(u_k(t),T)|\,dt + \int_{t_1}^{\infty} |M(u_k(t),T)|\,dt, \quad (17)$$

where $t_1 \ge 1$ will be determined later. For the first integral in (17), note that $2\log k/k \in [0,1]$ for any $k \ge 1$. Recall that $M(\cdot,T)$ has a branch cut along $[u_+,\infty)$, but a continuous extension $\tilde{M}$ of M exists on the half-plane $\Im(s) \ge 0$. Hence $|M(\cdot,T)|$ attains a maximum value on $[u_+, u_+ + 1] + i(0,t_1]$,

$$\int_{1/k}^{t_1} |M(u_k(t),T)|\,dt \le t_1 \max_{u \in [u_+,u_++1]+i(0,t_1]} |M(u,T)| < \infty$$

In order to show the boundedness of the second integral in (17) and to determine $t_1 \ge 1$, we have to take a closer look at the mgf and the auxiliary functions defined in (10). The fact $2\log k/k \in [0,1]$ for $k \ge 1$ ensures $u_k(t) = it + O(1)$ for $t \to \infty$ uniformly for all $k \ge 0$. Thus, the following asymptotic expansions of the auxiliary functions χ and γ in (10) hold

$$\begin{aligned}
\chi(u_k(t)) &= -i\xi\rho t + O(1),\\
\gamma(u_k(t)) &= \sqrt{-\xi^2\rho^2t^2 + \xi^2t^2 + O(t)} = \xi\bar{\rho}t + O(1),
\end{aligned}$$

and simple computations then yield

$$\begin{aligned}
\alpha(u_k(t)) &= \tfrac{1}{\xi}(\bar{\rho} + i\rho)t + O(1), &(18)\\
\mu(u_k(t)) &= \tfrac{2}{\xi}\bar{\rho}t + O(1),\\
\mu(u_k(t)) - \alpha(u_k(t)) &= \tfrac{1}{\xi}(\bar{\rho} - i\rho)t + O(1), &(19)
\end{aligned}$$

for $t \to \infty$ uniformly for all $k \ge 1$. Due to (18), (19) and $\bar{\rho} > 0$, there exists $t_0 \ge 1$, such that $\mathrm{Re}(\mu(u_k(t)) - \alpha(u_k(t))) > 1$ and $\mathrm{Re}(\alpha(u_k(t))) > 1$ for all $k \ge 1$ and $t \ge t_0$. In particular, in this region we have

$$\mathrm{Re}(\mu(u_k(t))) > \mathrm{Re}(\alpha(u_k(t))) > 0,$$

and so we can use the representation (34) of the confluent hypergeometric function, which reduces the mgf to

$$M(u_k(t),T) = \frac{z_T^{\alpha(u_k(t))}}{\Gamma(\alpha(u_k(t)))} \int_0^1 e^{-z_T y} y^{\alpha(u_k(t))-1}(1-y)^{\mu(u_k(t))-\alpha(u_k(t))-1}\, dy. \tag{20}$$

Note that the absolute value of the integral is bounded by 1. Furthermore, we have uniformly for all $k \geq 1$

$$|z_T^{\alpha(u_k(t))}| = \exp\left(\tfrac{1}{\xi}\bar{\rho}\log(z_T)t(1+o(1))\right), \quad t \to \infty. \tag{21}$$

Our choice $\mathrm{Re}(\alpha(u_k(t))) > 1$ guarantees $|\arg(\alpha(u_k(t)))| < \frac{\pi}{2}$ and Stirling's formula (35) is applicable to $\Gamma(\alpha(u_k(t)))$ for all $t \geq t_0$ and all $k \geq 1$. Combining with (18) we have, uniformly for all $k \geq 1$,

$$\begin{aligned}
|\Gamma(\alpha(u_k(t)))| &\sim \sqrt{2\pi}|e^{-z}z^z z^{-1/2}|_{z=\frac{1}{\xi}(\bar{\rho}+i\rho)t} \\
&= \sqrt{2\pi\xi}\ x^{-1/2}\ \exp\left(\tfrac{1}{\xi}\bar{\rho}t\log(\tfrac{t}{\xi}) - \tfrac{1}{\xi}\rho\arg(\bar{\rho}+i\rho)t - \tfrac{1}{\xi}\bar{\rho}t\right) \\
&= \exp\left(\tfrac{1}{\xi}\bar{\rho}t\log t(1+o(1))\right), \quad t \to \infty.
\end{aligned} \tag{22}$$

Putting (21) and (22) back into formula (20), we can find a sufficiently large $t_1 \geq t_0$ such that

$$|M(u_k(t),T)| \leq \exp\left(-(1+\varepsilon)\tfrac{1}{\xi}\bar{\rho}t\log t\right) \tag{23}$$

for all $t \geq t_1$ and all $k \geq 1$, with a constant $\varepsilon > 0$. The integrability of the right-hand side of (23) proves that the third integral in (17) is bounded. □

Lemma 2 *Assume $\mu_{u_+} - \alpha_{u_+} > 0$. Near the critical moment u_+, the following expansion of the mgf holds uniformly for all $w \in \mathcal{H}(k)$,*

$$M\left(u_+ - \frac{w}{k}, T\right) = M(u_+,T) + m_1\sqrt{\frac{w}{k}} + O\left(\frac{w}{k}\right), \quad k \to \infty,$$

where m_1 is defined in (30)*.*

Proof First, we expand the functions χ and γ in a neighbourhood of u_+. Using the representation (12) of γ we only have to expand $\sqrt{u-u_-} = \sqrt{(u_+-u_-)-(u_+-u)}$ near u_+. Thus, as $u \to t_+$,

$$\gamma_u = \xi\bar{\rho}\sqrt{u_+-u_-}(u_+-u)^{1/2} + O\left((u_+-u)^{3/2}\right), \tag{24}$$

$$\chi_u = \chi_{u_+} + \rho\xi(u_+-u). \tag{25}$$

With these results, expansions for α and μ near u_+ can easily be computed,

$$\alpha_u = \alpha_{u_+} + \frac{\bar\rho}{\xi}\sqrt{u_+ - u_-}(u_+ - u)^{1/2} + O(u_+ - u), \quad u \to u_+ \tag{26}$$

$$\mu_u = \mu_{u_+} + 2\frac{\bar\rho}{\xi}\sqrt{u_+ - u_-}(u_+ - u)^{1/2} + O\left((u_+ - u)^{3/2}\right), \quad u \to u_+. \tag{27}$$

Define $u_k(w) := u_+ - \frac{w}{k}$, $w \in \mathcal{H}(k)$, for $k \geq 1$. From the uniform convergence $\sup_{w\in\mathcal{H}(k)} |u_k(w) - u_+| \to 0$ for $k \to \infty$, we have

$$\Delta\alpha := \alpha(u_k(w)) - \alpha_{u_+} = \frac{\bar\rho}{\xi}\sqrt{u_+ - u_-}\left(\frac{w}{k}\right)^{1/2} + O\left(\frac{w}{k}\right), \quad k \to \infty \tag{28}$$

$$\Delta\mu := \mu(u_k(w)) - \mu_{u_+} = 2\frac{\bar\rho}{\xi}\sqrt{u_+ - u_-}\left(\frac{w}{k}\right)^{1/2} + O\left(\left(\frac{w}{k}\right)^{3/2}\right), \quad k \to \infty, \tag{29}$$

uniformly for all $w \in \mathcal{H}(k)$. Define the function

$$\tilde M(\alpha,\mu) := \frac{\Gamma(\mu-\alpha)}{\Gamma(\mu)}(z_T)^{\alpha}\,{}_1F_1(\alpha,\mu,-z_T),$$

for all $(\alpha,\mu) \in \mathbb{C}^2$ where $\mu - \alpha, \mu \notin \mathbb{Z}_0^-$. In this region $\tilde M$ is jointly analytic in both variables. Note the relation $M(u,T) = \tilde M(\alpha_u, \mu_u)$. Since $\mu_{u_+} = 1$ and $\mu_{u_+} - \alpha_{u_+} > 0$, we can make a Taylor expansion of $\tilde M$ at the point $(\alpha_{u_+}, \mu_{u_+})$. Combining this with (28) and (29) gives us, uniformly for all $w \in \mathcal{H}(k)$,

$$\begin{aligned} M(u_k(w),T) &= \tilde M(\alpha(u_k(w)), \mu(u_k(w))) \\ &= \tilde M(\alpha_{u_+},\mu_{u_+}) + \Delta\alpha\frac{\partial}{\partial\alpha}\tilde M(\alpha_{u_+},\mu_{u_+}) + \Delta\mu\frac{\partial}{\partial\mu}\tilde M(\alpha_{u_+},\mu_{u_+}) \\ &\qquad + O\left((\Delta\alpha)^2\right) + O\left((\Delta\mu)^2\right) \\ &= M(u_+,T) + \underbrace{\left(\frac{\partial\tilde M}{\partial\alpha} + 2\frac{\partial\tilde M}{\partial\mu}\right)(\alpha_{u_+},\mu_{u_+})\frac{\bar\rho}{\xi}\sqrt{u_+ - u_-}}_{=:m_1}\left(\frac{w}{k}\right)^{1/2} + O\left(\frac{w}{k}\right), \quad k \to \infty. \end{aligned} \tag{30}$$

4.3 Large-strike asymptotics for the implied volatility

From tail asymptotics for the density function, it is possible to obtain large strike asymptotics for the implied volatility, see Gulisashvili [20] and Friz, Gerhold, Gulisashvili and Sturm [12]. Recall that the implied volatility is the volatility parameter that has to be used in the Black-Scholes model to recover given option prices. The statement is, that if the density function φ satisfies, for fixed $T > 0$,

$$c_1 k^{-\xi} h(k) \leq \varphi(k) \leq c_2 k^{-\xi} h(k),$$

for all sufficiently large k, with $\xi > 2$, h slowly varying and constants $c_1, c_2 > 0$, then the implied volatility $\sigma_{\mathrm{imp}}(K,T)$ satisfies

$$\sigma_{\mathrm{imp}}(K,T)\frac{\sqrt{T}}{\sqrt{2}} = \sqrt{\log K + \log\frac{1}{K^{2-\xi}h(K)} + \tfrac{1}{2}\log\log\frac{1}{K^{2-\xi}h(K)}} - \sqrt{\log\frac{1}{K^{2-\xi}h(K)} + \tfrac{1}{2}\log\log\frac{1}{K^{2-\xi}h(K)}} + O\big((\log K)^{-1}\big), \tag{31}$$

as $K \to \infty$. In Theorem 3, we have established tail asymptotics for the density φ_{X_T} of the log-price $X_T = \log(S_T)$ in the $3/2$–model. Because the density φ_{S_T} of S_T is given by

$$\varphi_{S_T}(K) = \frac{\varphi_{X_T}(\log K)}{K}, \quad K > 0,$$

we clearly have the tail asymptotics for φ_{S_T}

$$\varphi_{S_T}(K) \sim cK^{-(u_+ + 1)}h(K), \quad K \to \infty, \tag{32}$$

with the slowly varying function $h(K) = (\log K)^{-3/2}$. Note that the critical moment always satisfies $u_+ \geq 1$, and $u_+ = 1$ if and only if $2\xi\rho = \bar{\kappa}$. Hence, the previous statement is applicable.

Theorem 4 *Assume $\mu_{u_+} - \alpha_{u_+} > 0$ and $2\xi\rho \neq \bar{\kappa}$. The large-strike expansion of the implied volatility function in the 3/2–model, with $T > 0$ fixed, is given by, as $K \to \infty$,*

$$\sigma_{\mathrm{imp}}(K,T)\frac{\sqrt{T}}{\sqrt{2}} = \left(\sqrt{u_+} - \sqrt{u_+ - 1}\right)\sqrt{\log K} + \frac{1}{2}\left(\frac{1}{\sqrt{u_+}} - \frac{1}{\sqrt{u_+ - 1}}\right)\frac{\log\log K}{\sqrt{\log K}} + O\left(\frac{\log\log\log K}{\sqrt{\log K}}\right). \tag{33}$$

Proof A straightforward calculation, using (31) and (32), shows

$$
\begin{aligned}
\sigma_{\mathrm{imp}}(K,T)\frac{\sqrt{T}}{\sqrt{2}} &= \sqrt{\log K - \log(K^{1-u_+}h(K)) - \tfrac{1}{2}\log\left(-\log(K^{1-u_+}h(K))\right)} \\
&\quad - \sqrt{-\log(K^{1-u_+}h(K)) - \tfrac{1}{2}\log\left(-\log(K^{1-u_+}h(K))\right)} \\
&\quad + O\left((\log K)^{-1}\right) \\
&= \sqrt{u_+ \log K + \log\log K + O(\log\log\log K))} \\
&\quad - \sqrt{(u_+ - 1)\log K + \log\log K + O(\log\log\log K)} \\
&\quad + O\left((\log K)^{-1}\right), \\
&= \sqrt{u_+}\sqrt{\log K}\left(1 + \frac{\log\log K}{u_+ \log K} + O\left(\frac{\log\log\log K)}{\log K}\right)\right) \\
&\quad - \sqrt{(u_+ - 1)}\sqrt{\log K}\left(1 + \frac{\log\log K}{(u_+ - 1)\log K} + O\left(\frac{\log\log\log K}{\log K}\right)\right) \\
&\quad + O\left((\log K)^{-1}\right),
\end{aligned}
$$

as $K \to \infty$. This easily yields the statement. □

We state the following lemmas which are used in the proof of the tail asymptotics of the density function. The first lemma describes a representation of the confluent hypergeometric function ${}_1F_1$, whereas the second lemma is the well-known Stirling formula for the Gamma function. For further details, see e.g. [25].

Lemma 3 *If* $\mathrm{Re}(\mu) > \mathrm{Re}(\alpha) > 0$*, then the confluent hypergeometric function* ${}_1F_1$ *has the integral representation*

$$
{}_1F_1(\alpha,\mu,z) = \frac{\Gamma(\mu)}{\Gamma(\alpha)\Gamma(\mu-\alpha)}\int_0^1 e^{zy} y^{\alpha-1}(1-y)^{\mu-\alpha-1}\,dy. \tag{34}
$$

Lemma 4 (Stirling) *The Gamma function satisfies*

$$
\Gamma(z) = \sqrt{2\pi}e^{-z}z^z z^{-1/2}(1+o(1)), \quad z \to \infty \quad \textit{with } |\arg(z)| < \pi - \varepsilon, \tag{35}
$$

where $\varepsilon > 0$ *is arbitrary.*

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