Addenda and Corrigenda for the Publications of Stefan Gerhold

September 18, 2024

I try to keep an up-to-date list of things that are worth knowing when reading my works. The publications appear in reverse chronological order, with triangles ▷ marking errors, and bullets • marking comments. Some of the errors that are listed here have been corrected in the electronic versions of the texts on my homepage (`http://www.fam.tuwien.ac.at/~sgerhold/`).

1. **A converse to the neo-classical inequality with an application to the Mittag-Leffler function** [34]

   • Hino and Namba [38] introduce a probability distribution motivated by the generalized binomial theorem, and apply our inequality (1.7) in their paper.

2. **Asymptotics of some generalized Mathieu series** [31]

   • A class of Dirichlet series related to (2.5) is studied in Cramér's PhD thesis [10].

3. **Large deviations related to the law of the iterated logarithm for Itô diffusions** [30]

   ▷ In (3.6), the first $\tilde{u}$ should be u.

4. **Small time central limit theorems for semimartingales with applications** [32]

   ▷ Replace $t(u)$ by t in (2.13).

   • In the heuristic argument following (2.13), one should keep in mind that ε is constant in (2.13). Therefore, when formally replacing ε by $z\sqrt{t}$ with $t \downarrow 0$, one should think of $z \to \infty$, which explains why the *tail asymptotics* of the Gaussian cdf are found on the right hand side of the second formula after (2.13), and not the Gaussian cdf itself.

5. **Small-maturity digital options in Lévy models: an analytic approach** [29]

   - A follow-up paper by V. P. Knopova [41] presents a more general result on $\mathbb{P}[X_T > 0]$ for $T \downarrow 0$ (with X a Lévy process starting at zero).

6. **Disproof of a conjecture by Rademacher on partial fractions** [11]

   - Concerning the polylogarithm function $\mathrm{Li}_\nu(w)$, we quote from Pickard [49] that "little is known about behavior in the ν-plane except along and near the line $(0, \infty)$" and proceed to prove a bound for $\mathrm{Li}_\nu(w)$ for non-real ν. For this, we use a well-known formula (see (9) in [11]) representing the polylogarithm by the Hurwitz zeta function. Although this (simple) bound seems to be new, it should be mentioned here that other bounds for the Hurwitz zeta function (see [54], and more recently [42, 51]) also yield bounds for $\mathrm{Li}_\nu(w)$ for non-real ν.
   - ▷ On p. 128, replace "z_2 to z_3 to" by "z_2 to z_3 as"
   - ▷ On p. 133, replace "width" by "range"

7. **How to make Dupire's local volatility work with jumps** [18]

   - ▷ In the last formula of the section "The Merton model", the term $\log(\lambda T \delta^2)$ has to be replaced by $\log(\lambda \delta^2)$.[1]
   - Theorem 4 might have been featured more prominently in the paper; the resulting algorithm is better than the one from Theorem 1. It says that, without shifting the local vol function, simulating with stochastic initial value recovers the call prices exactly for all maturities $\geq \varepsilon$. One should also mention that, from the definition of $\tilde{S}^\varepsilon$, we have $\tilde{S}^\varepsilon_t = S^\varepsilon_{t-\varepsilon}$ for $t \geq \varepsilon$.

8. **Can there be an explicit formula for implied volatility?** [28]

   - ▷ p. 21 and 22: Replace "asymptotes" by "asymptotics".
   - The paper [20] is somewhat related.[2] It shows that vanilla option prices in the binomial model do not admit a closed form, in the sense of a sum of a fixed number of hypergeometric terms.

9. **Asymptotics for a variant of the Mittag-Leffler function** [27][3]

   - For *real* z and $0 < \alpha \leq 4$, the asymptotics of $\sum_{n\geq 0} z^n/n!^\alpha$ are also derived in Olver's book [45, p. 307].

[1] I thank Jacopo Corbetta for pointing this out.
[2] I thank Yuan X Chen for bringing it to my attention.
[3] I thank Richard B. Paris for interesting comments on this paper, as well as on [15].

- In the introduction I mention that the discrete Laplace method can be used to get the asymptotics of $\sum_{n\geq 0} z^n/n!^\alpha$ for real $z \to \infty$ and any positive α. This was carried out by Bender and Orszag [5, p. 304].

▷ Concerning integral α, I write that “the asymptotic behavior of the multiple Mittag-Leffler function has not been studied”. This is incorrect, as this function is hypergeometric (a Wright function, in fact), and a full asymptotic expansion is available. It is classically derived via contour integration; see Paris [48] for a derivation by the *discrete* Laplace method.

- Other properties of this generalization of the Mittag-Leffler function are studied in [19] and in the preprint “Some properties of Gerhold-Garra-Polito function” by S.V. Rogosin, M.V. Dubatovskaya, and F. Mainardi.

10. **The Hartman-Watson distribution revisited: asymptotics for pricing Asian options** [26]

- The Dothan model is an interest rate model where the short rate r_t follows geometric Brownian motion. The discount factor $\exp(-\int_0^T r_t dt)$ thus involves the integral of geometric Brownian motion. Pintoux and Privault [50] give a bond pricing formula, completing the original one of Dothan, which involves the Hartman-Watson distribution.

11. **On Refined Volatility Smile Expansion in the Heston Model** [17]

▷ p. 1153, formula after “An elementary computation gives”: The assumption $\rho \leq 0$, which is mentioned right after, is used here already.[4]

▷ p. 1154, last formula before Lemma 2.2: replace s^* by s_+.

▷ p. 1154, last line of the right column: For consistency with the next formula, it would be better to write $O(\tau^2)$ instead of $o(\tau)$.[5]

▷ Formula (3.6): replace t by y in the first line.

▷ Last line of formula (3.9): Add a factor $\frac{1}{2}$ in front of β^{-1}. Same on p. 1162 (twice): 5 lines above (A9), and in the second column (last displayed formula of the proof).[6]

▷ p. 1157, formula for A_1: Add a factor T in front of the square root in the last line, inside the hyperbolic sine.[7]

▷ p. 1157, central formula in the right column: minus sign missing in the lower integration bounds.[8]

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[4] Thanks to Christoph Gerstenecker.
[5] Thanks to Christoph Gerstenecker.
[6] Thanks to Christoph Gerstenecker.
[7] Thanks to Kun Gao, Roger Lee, Stephan Sturm, and Archil Gulisashvili.
[8] Thanks to Christoph Gerstenecker.

- p. 1157, remark 9: The cited proposition from [7] does not include an error estimate. To get the claimed relative error of $\mathrm{O}((\log x)^{-1/2})$, divide the last integral in the proof of [7, Proposition 1.5.10] into $1 \le u \le \log x$ and $\log x < u < \infty$. In the first part, our slowly varying function ℓ satisfies $\ell(ux)/\ell(x) = 1 + \mathrm{O}(\log\log x/\log x)$. Pull this out of the integral. As for the second part, the expression in curly braces is bounded; thus, this part of the integral is $\mathrm{O}((\log x)^{-1/2})$, as our ρ is smaller than $-1/2$.

▷ p. 1162: In Lemma A.3, the equality should be replaced by $\le$, as this is what is proven, and suffices for the application of the lemma.

▷ p. 1163, last formula: ϕ does not have the same meaning as in the rest of the paper, but rather denotes the characteristic function of log-spot.

12. **Counting Finite Languages by Total Word Length** [25][9]

▷ p. 1: Insert z^3 before the last + in (2).

13. **Asymptotic Estimates for Some Number Theoretic Power Series** [24]

- In [37], some more information on this "fake asymptotics" is found, with additional references.

14. **A Generalization of Panjer's Recursion and Numerically Stable Risk Aggregation** [33][10]

▷ p. 91, after (4.9): replace S_N by X_N and $N_{\tilde{N}}$ by $X_{\tilde{N}}$.

▷ p. 94, last formula: replace the upper summation limit n by $n-1$.

▷ Line 3 of Section 5.5 (p. 107): Replace "For k" by "For i".

▷ Line 8 of Section 5.5 (p. 107): Replace N_K by N_r.

▷ p. 109, line 1: Replace $n = 0$ by $m = 0$.

15. **Lindelöf Representations and (Non-)Holonomic Sequences** [15]

▷ Last formula on p. 25: Replace "$n \ge 0$" by "$n \ge 1$".

- The analytic continuation (but not the asymptotics) of some of the functions we consider, e.g. $\sum_{n\ge1} e^{-1/n} z^n$, has been obtained by le Roy [43, p. 356], using a different integral representation.
- The related entire function $\sum_{n=1}^{\infty} e^{-n^\alpha} z^n$, $1 < \alpha < 2$, which we do not consider, has been analyzed asymptotically by Evgrafov [13, p. 294f].

[9] Thanks to Bill Smyth.

[10] Thanks to Cordelia Rudolph.

- The series on page 4 shows that a minor perturbation of the Taylor series of $\exp(-z)$ may destroy the exponential decrease of the function, which is not surprising, due to the huge cancellations that occur in the series. A detailed discussion of several such perturbations has been done by Paris [47].
- The map $\varphi(s) = \exp(-s^\theta)$, with $\theta \in (0, 1)$, is sometimes called *stretched exponential function*[11]. It is a special case of the function that defines the coefficients of our $\mathrm{E}(z; c, \theta)$.
- The discrete probability distribution with probability mass function proportional to $k \mapsto \exp(-k^\theta)$, $k \in \mathbb{N}$, with parameter $\theta \in (0, 1)$, is known as *stretched exponential distribution.* See [12, 44] for some asymptotic properties of this distribution. In particular, the paper [12] uses the Lindelöf integral technique.

16. **On Turán's Inequality for Legendre Polynomials** [1]

- It was known before that the normalized Turán determinant $f_n(x)$ increases w.r.t. x [53, 55]. This was noted by C. Berg and R. Szwarc [6]. So our paper brings a new (very short und computer-assisted) proof of a result that was (implicitly) already in the literature.

17. **On the Positivity Set of a Linear Recurrence Sequence** [3]

- Lines after (3): The result was known for any number of dominating roots, see the first comment concerning my paper "Point Lattices and Oscillating Recurrence Sequences" below.
- The proof of Theorem 2 can be shortened, since it is a corollary of Theorem 1 and a result from complex analysis [8, p. 242] that generalizes Pringsheim's theorem. This result implies that the index set of the negative coefficients of an analytic function with radius of convergence R cannot have zero density if R is not a singularity.

▷ After the first formula of Section 2: The sequence (r_n) itself is in general not a linear recurrence sequence, but $(n^D r_n)$ is one. Same for the sequence (s_n) on page 4. This does not affect the proofs of course, since, for $n > 0$, $n^D r_n > 0$ iff $r_n > 0$.

- We do not claim that the (easy) Lemma 5 is new; indeed, it can be found in Jurek and Mason [39, Lemma 3.8.4], and probably in earlier references as well.
- The sequence (7) is positive for $\theta = \sqrt[3]{2}$. This can be shown using Liouville's theorem (Géza Kós, os Lóczi; private communication). To make the problem more difficult, let θ be transcendental.
- An algorithm to compute the density of the positivity set is presented in [40].

[11] `https://en.wikipedia.org/wiki/Stretched_exponential_function`

18. **On the non-holonomic character of logarithms, powers and the nth prime function** [14]

    - Theorem 3 is classical in the real case, and the extension to complex sectors is easy. In our paper we sketch a proof of the latter fact, without claiming originality. Indeed, the complex version can be found in a paper of Sedletskii [52].

19. **Combinatorial Sequences: Non-Holonomicity and Inequalities (Ph.D. thesis)** [22]

    ▷ p. 16: Move the definition of $g(z, s)$ from line -6 to line 9, right after (14).

    - p. 21: The determination of the $c_i(z)$ here is redundant, since Subsection 1.6.5 gives not only a check, but a determination of the quadratic approximation.

    ▷ p. 22: Replace lines 14 – 16 by

    $$\text{“}\frac{1}{\sin \pi s_0} = \tfrac{1}{\pi} L^{1/2} \Big(1 + \mathrm{O}(L^{1/2-\alpha})\Big) .$$

    Summing up, f equals

    $$f(z, t) = \log \frac{z^{s_0} \mathrm{e}^{1/s_0}}{\sin \pi s_0}\text{”}.$$

    - p. 23: The computation in (1.29) can be slightly simplified by removing the integration constant $\log 2L$.

    ▷ p. 44: Replace a_i by a_k in 8.25.

    ▷ p. 60, line -6: Replace “density the” by “density of the”.

    ▷ p. 61 ff: Same (inconsequential) error concerning r_n as in the paper [3] in which this result appeared, see below.

    ▷ p. 61, line 14: Insert + in front of $\mathbb{Z}\theta_l$.

20. **Point Lattices and Oscillating Recurrence Sequences** [23]

    - This paper should be viewed as a paper about lattice points in squares rather than about recurrence sequences, since Conjecture 1.2 is an easy consequence of Pringsheim's theorem [16]. This argument appears in Győri, Ladas [35, Theorem 7.1.1].

      For a different argument, see Braverman [9], where a proof is only sketched. Ouaknine et al. [46, Prop. 3] extend this to a detailed proof.[12]

      There is also a short unpublished proof of Conjecture 1.2 by P. Folet. See also the comments for my paper “On the Positivity Set of a Linear Recurrence Sequence (2007)” above.

[12] Thanks to Joel Ouaknine.

- In [3] we show that, under the assumptions of Conjecture 1.2, the positivity set and the negativity set have positive density, which provides a sharpening.
- A somewhat related question was posed as problem 2699 in the American Mathematical Monthly (solution in the April 1979 issue): If $1 = \theta_0 > \cdots > \theta_k > 0$ and $\lim_{n\to\infty} \sum_{i=0}^{k} a_i \cos n\theta_i\pi = 0$, then $a_i = 0$ for all i.
- The lattices $L_g(a_1, a_2)$ are known as *lattice rules* in the field of Quasi-Monte Carlo methods.
- A somewhat related problem is studied in [2].

▷ Lemma 5.5: Remove the condition "$\gcd(r_1, r_2) = 1$". Same in the proof of Proposition 5.1. In the preliminaries about lattices in Section 2, include the sentence "A lattice point $\mathbf{x} \in \Lambda$ is *primitive* if there is no integer $l > 1$ with $l^{-1}\mathbf{x} \in \Lambda$."

▷ p. 526: Replace B by $\mathcal{B}$ in line -1.

- The arguments in Section 6 are unnecessarily complicated [3, Lemma 8].

▷ p. 533: Remove "equation" in line 6.

▷ Reference [15] should read like reference [36] below.

21. **On Some Non-Holonomic Sequences** [21]

- All problems in "Open Problems" are solved now [14].
- Theorem 1 has been generalized to complex non-integral exponents in subsequent work with J.P. Bell, M. Klazar, and F. Luca [4].
- For some more information on the Galois theory argument used at the top of p. 5 and related results, see "Linear independence of nth roots" at `https://www-users.cse.umn.edu/~garrett/m/v/`

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