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Large deviations for fractional volatility models with non-Gaussian volatility driver

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Abstract

We study non-Gaussian fractional stochastic volatility models. The volatility in such a model is described by a positive function of a stochastic process that is a fractional transform of the solution to an SDE satisfying the Yamada–Watanabe condition. Such models are generalizations of a fractional version of the Heston model considered in Bäuerle and Desmettre (2020). We establish sample path and small-noise large deviation principles for the log-price process in a non-Gaussian model. We also illustrate how to compute the second order Taylor expansion of the rate function, in a simplified example.

1. Introduction

In this paper, we introduce and study a general class of non-Gaussian stochastic volatility models. The main building block of the volatility in such a model is a Volterra type integral transform of the solution to a stochastic differential equation satisfying the Yamada–Watanabe condition, while the volatility is described by a positive function of such an integral transform. Interesting special cases of non-Gaussian models are the models in which the kernels appearing in the integral transforms possess certain fractional features. Examples of such kernels are the kernels of fractional Brownian motion, the Riemann–Liouville fractional Brownian motion, or the fractional Ornstein–Uhlenbeck process. We call the corresponding models non-Gaussian fractional stochastic volatility models. Our class of models is related to the fractional Heston model (see [1,11]), as explained in Section 4.

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In a Gaussian model, the stochastic volatility is described by a positive function of a Volterra Gaussian process. Such models have recently become popular objects of study. Numerous examples of Gaussian models are given in [10,12,13]. The non-Gaussian stochastic volatility models are less studied. To our knowledge, the general class of models introduced in the present paper has never been considered before.

The main results obtained in the present paper are Theorems 1.6 and 1.7. In these theorems, small-noise and sample path large deviation principles are established for the log-price process in a non-Gaussian stochastic volatility model. In the proofs of Theorems 1.6 and 1.7, we use on the one hand known techniques form the general theory of large deviations, and on the other hand also employ new techniques. For example, a part of our proof of Theorem 1.7 is based on the results of Chiarini and Fischer (see [5]) concerning small-noise large deviations for Itô processes. Although we cannot use heavy machinery of the theory of Gaussian processes in the non-Gaussian case, we still borrow some techniques employed in [9,12,13] in the proofs of large deviation theorems for Gaussian models. In Section 5 of the present paper, we show how to obtain a Taylor expansion of the rate function in a simplified example.

Recently, there has been a surge of interest in using stochastic Volterra equations for financial modelling. While small-noise large deviations for such equations are well studied in the case of Lipschitz coefficients (see [17,18,20,21]), similar LDPs for equations in which non-Lipschitz functions are used in the description of the dynamics are scarce. In the papers [8] and [11], concrete models with finite-dimensional parameter spaces are considered, whereas [4,9,12–14] deal with large deviation principles for Gaussian models. In the present paper, we assume that the volatility process is a positive function σ of the following process:

$$\hat{V}_t = \int_0^t K(t,s)U(V_s)\,ds, \tag{1.1}$$

where U is a continuous non-negative function, assumptions on the kernel K will be specified below, and V solves a one-dimensional SDE, driven by a Brownian motion B and satisfying the Yamada–Watanabe condition. A (semi-)explicit generating function, as is available in the rough resp. fractional Heston models considered in [8,11], is not required. Also, our process $\hat{V}$ is clearly non-Gaussian in general, which sets our results apart from the related papers with Gaussian drivers mentioned above. While our setup allows a lot of freedom in choosing the diffusion V and the other ingredients, we note that truly rough models are not covered, because (1.1) is a Lebesgue integral and not an integral w.r.t. Brownian motion. However, the models that we are considering may be rough at $t = 0$ (see Remark 4.2). The asset price is given by

$$\begin{aligned} dS_t &= S_t\sigma(\hat{V}_t)(\bar{\rho}\,dW_t + \rho\,dB_t), \quad 0 \le t \le T, \\ S_0 &= 1. \end{aligned} \tag{1.2}$$

Here, B, W are independent standard Brownian motions, $\rho \in (-1,1)$ and $\bar{\rho} = \sqrt{1-\rho^2}$. The extension to arbitrary $S_0 > 0$ is straightforward. We now specify the conditions under which our main results, Theorems 1.6 and 1.7, are valid. Assumptions 1.1, 1.3 and 1.4 formulated below are in force throughout the paper.

Assumption 1.1. Throughout the paper, K is a kernel on $[0,T]^2$ satisfying the following conditions:

(a)

$$\sup_{t\in[0,T]} \int_0^T K(t,s)^2\,ds < \infty. \tag{1.3}$$

(b) The modulus of continuity of the kernel K in the space $L^2[0,T]$ is defined as follows:

$$M(h) = \sup_{\{t_1,t_2\in[0,T]:|t_1-t_2|\le h\}} \int_0^T |K(t_1,s)-K(t_2,s)|^2\,ds, \quad 0\le h\le T. \tag{1.4}$$

It is assumed that there exist constants $c>0$ and $r>0$ such that

$$M(h) \le ch^r \tag{1.5}$$

for all $h\in[0,T]$.

(c) $K(t,s)=0$ for all $0\le t<s\le T$.

The function K is a Volterra kernel in the sense of [12] and [13]. The conditions in Assumption 1.1 have been used earlier; e.g., (b) and (c) are parts of the definition of a Volterra type Gaussian process in [15,16]. It is a standard fact that the associated integral operator

$$\mathcal{K}(h)(t) = \int_0^T K(t,s)h(s)\,ds \tag{1.6}$$

is compact from $L^2[0,T]$ into $C[0,T]$; see e.g. Lemma 2 of [12] for a proof. A standard example of a kernel satisfying Assumption 1.1 is the fractional kernel $\Gamma(H+\frac{1}{2})^{-1}(t-s)^{H-1/2}$, $0\le s\le t$, with Hurst parameter $H\in(0,1)$. We note that Γ denotes the gamma function here, whereas later we will use the letter Γ for the solution map of the ODE (1.16).

Definition 1.2. Let ω be an increasing modulus of continuity on $[0,\infty)$, that is $\omega:\mathbb{R}_+\to\mathbb{R}_+$ is an increasing function such that $\omega(0)=0$ and $\lim_{s\to 0}\omega(s)=0$. A function h defined on $\mathbb{R}$ is called locally ω-continuous, if for every $\delta>0$ there exists a number $L(\delta)>0$ such that for all $x,y\in[-\delta,\delta]$

$$|h(x)-h(y)| \le L(\delta)\omega(|x-y|). \tag{1.7}$$

Assumption 1.3. The function $U:\mathbb{R}_+\to\mathbb{R}_+$ is continuous, and σ is a positive function on $\mathbb{R}_+$ that is locally ω-continuous for some modulus of continuity ω as in Definition 1.2.

The process V in (1.1) is assumed to solve the SDE

$$\begin{aligned} dV_t &= \bar{b}(V_t)\,dt + \bar{\sigma}(V_t)\,dB_t, \quad 0\le t\le T,\\ V_0 &= v_0 > 0, \end{aligned} \tag{1.8}$$

where $\bar{\sigma}$ and $\bar{b}$ satisfy the Yamada–Watanabe condition in Assumption 1.4. A well-known example is the CIR process, where $\bar{\sigma}$ is the square root function.

Assumption 1.4.

(R1) The dispersion coefficient $\bar{\sigma}:\mathbb{R}\to[0,\infty)$ is locally Lipschitz continuous on $\mathbb{R}\backslash\{0\}$, has sub-linear growth at ∞, and $\bar{\sigma}(0)=0$, while $\bar{\sigma}(x)>0$ for all $x\neq 0$. Moreover, there exists a continuous increasing function $\gamma:(0,\infty)\to(0,\infty)$ such that

$$\int_{0+}^{\infty} \frac{du}{\gamma(u)^2} = \infty \tag{1.9}$$

and

$$|\bar{\sigma}(x) - \bar{\sigma}(y)| \le \gamma(|x-y|) \quad \text{for all } x, y \in \mathbb{R},\ x \neq y.$$

Here, the sub-linear growth at ∞ is understood in the sense that for every x_0 there exists a μ such that for all $x > x_0$ we have

$$|\bar{\sigma}(x)|^2 \le \mu(1 + |x|^2).$$

(R2) The drift coefficient $\bar{b} : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, has sub-linear growth at ∞, and $\bar{b}(0) > 0$.

The process V is non-negative (see the remark after Theorem 2.2). Next, introducing a small-noise parameter $\varepsilon > 0$, we define the scaled version V^ε of the process V by

$$\begin{aligned} dV_t^\varepsilon &= \bar{b}(V_t^\varepsilon)\,dt + \sqrt{\varepsilon}\bar{\sigma}(V_t^\varepsilon)\,dB_t, \\ V_0^\varepsilon &= v_0 > 0, \end{aligned} \tag{1.10}$$

and the scaled asset price by

$$dS_t^\varepsilon = \sqrt{\varepsilon} S_t^\varepsilon \sigma(\hat{V}_t^\varepsilon)(\bar{\rho}\,dW_t + \rho\,dB_t). \tag{1.11}$$

Here, we write $\hat{V}^\varepsilon$ for the process

$$\hat{V}_t^\varepsilon = \int_0^t K(t,s)U(V_s^\varepsilon)\,ds. \tag{1.12}$$

The scaled log-price process $X^\varepsilon = \log S^\varepsilon$, which is the process of interest for our large deviations analysis, is now given by

$$X_t^\varepsilon = -\frac{1}{2}\varepsilon \int_0^t \sigma(\hat{V}_s^\varepsilon)^2\,ds + \sqrt{\varepsilon}\int_0^t \sigma(\hat{V}_s^\varepsilon)\,d(\bar{\rho}W_s + \rho B_s), \quad 0 \le t \le T. \tag{1.13}$$

Definition 1.5. In addition to $\mathcal{K}$ from (1.6), we define the integral operators

$$\begin{aligned} \hat{\cdot} &: C[0,T] \to C[0,T], \\ \check{\cdot} &: H_0^1[0,T] \to C[0,T] \end{aligned}$$

by

$$\hat{f}(t) = \int_0^t K(t,s)U(f(s))\,ds, \quad t \in [0,T], \tag{1.14}$$

$$\check{g}(t) = \int_0^t K(t,s)U(v(s))\,ds, \quad t \in [0,T], \tag{1.15}$$

where v is the solution of the ODE

$$\dot{v} = \bar{b}(v) + \bar{\sigma}(v)\dot{g}, \quad v(0) = v_0. \tag{1.16}$$

Clearly, we have $\check{g} = \hat{v}$, where v solves the ODE (1.16). Moreover, $\hat{f} = \mathcal{K}(U \circ f)$ and $\check{g} = \mathcal{K}(U \circ \Gamma(g))$, where Γ maps g to the solution of (1.16). By Assumption 1.1 the integral operators of Definition 1.5 are well-defined. In fact, for our kernel K, we get that $\mathcal{K} : L^2[0,T] \to C[0,T]$. Note that for $h \in H_0^1[0,T]$, we have $h \in C[0,T]$. Further, for $f \in H_0^1[0,T]$ we have $U \circ f \in L^2[0,T]$ and for $g \in H_0^1[0,T]$ we have $U \circ v \in L^2[0,T]$. This can be easily seen using the fact that U is continuous and the input functions are continuous on a bounded interval and hence bounded themselves.

We can now state our main results.

Theorem 1.6. *The family X_T^ε satisfies the small-noise large deviation principle (LDP) with speed ε^{-1} and good rate function I_T given by*

$$I_T(x) = \inf_{f \in H_0^1} \Big[\frac{T}{2} \frac{\big(x - \rho\langle\sigma(\mathcal{K}(U \circ \Gamma(f))), \dot{f}\rangle\big)^2}{\bar{\rho}^2 \langle\sigma(\mathcal{K}(U \circ \Gamma(f)))^2, 1\rangle} + \frac{1}{2}\langle\dot{f}, \dot{f}\rangle \Big] \tag{1.17}$$

for all $x \in \mathbb{R}$, wherever this expression is finite. The validity of the LDP means that for every Borel subset $\mathcal{A}$ of $\mathbb{R}$, the following estimate holds, where $\mathcal{A}^\circ$ and $\bar{\mathcal{A}}$ denote the interior resp. the closure of $\mathcal{A}$:

$$-\inf_{x \in \mathcal{A}^\circ} I_T(x) \le \liminf_{\varepsilon \searrow 0} \varepsilon \log P(X_T^\varepsilon \in \mathcal{A}) \le \limsup_{\varepsilon \searrow 0} \varepsilon \log P(X_T^\varepsilon \in \mathcal{A}) \le -\inf_{x \in \bar{\mathcal{A}}} I_T(x). \tag{1.18}$$

Theorem 1.7. *The family of processes X^ε satisfies the sample path LDP with speed ε^{-1} and good rate function Q given by*

$$Q(g) = \inf_{f \in H_0^1} \Big[\frac{1}{2} \int_0^T \Big(\frac{\dot{g}(t) - \rho\sigma(\mathcal{K}(U \circ \Gamma(f))(t))\dot{f}(t)}{\bar{\rho}\sigma(\mathcal{K}(U \circ \Gamma(f))(t))} \Big)^2 dt + \frac{1}{2} \int_0^T |\dot{f}(t)|^2 \, dt \Big]$$

for all $g \in H_0^1[0, T]$, and by $Q(g) = \infty$, for all $g \in C[0, T] \backslash H_0^1[0, T]$. The validity of the LDP means that for every Borel subset $\mathcal{A}$ of $C[0, T]$, the following estimate holds:

$$-\inf_{g \in \mathcal{A}^\circ} Q(g) \le \liminf_{\varepsilon \searrow 0} \varepsilon \log P(X^\varepsilon \in \mathcal{A}) \le \limsup_{\varepsilon \searrow 0} \varepsilon \log P(X^\varepsilon \in \mathcal{A}) \le -\inf_{g \in \bar{\mathcal{A}}} Q(g). \tag{1.19}$$

Using the definition of $\mathcal{K}$, the rate functions in Theorems 1.6 and 1.7 can be equivalently written as

$$I_T(x) = \inf_{f \in H_0^1} \Bigg[\frac{T}{2} \frac{\big(x - \rho \int_0^T \sigma(\int_0^t K(t, s)U(\Gamma(f)(s)) \, ds)\dot{f}(t) \, dt\big)^2}{\bar{\rho}^2 \int_0^T \sigma(\int_0^t K(t, s)U(\Gamma(f)(s)) \, ds)^2 \, dt} + \frac{1}{2} \int_0^T \dot{f}(t)^2 \, dt \Bigg]$$

and

$$Q(g) = \inf_{f \in H_0^1} \Bigg[\frac{1}{2} \int_0^T \Bigg(\frac{\dot{g}(t) - \rho\sigma(\int_0^t K(t, s)U(\Gamma(f)(s)) \, ds)\dot{f}(t)}{\bar{\rho}\sigma(\int_0^t K(t, s)U(\Gamma(f)(s)) \, ds)} \Bigg)^2 dt + \frac{1}{2} \int_0^T |\dot{f}(t)|^2 \, dt \Bigg],$$

respectively.

The structure of this paper is as follows. In Section 2, we recall small-noise large deviations for SDEs satisfying the Yamada–Watanabe condition. In Section 3, we prove the main results, i.e. the small-noise LDP for the log-price. In Section 4 we clarify the relation of a special case of our setup to fractional Heston models considered in the literature. In Section 5 we compute the coefficients in the second-order Taylor expansion of the rate function in Theorem 1.6 for a special, simplified example. As was mentioned above, Assumptions 1.1, 1.3 and 1.4 are supposed to be satisfied throughout the rest of the paper.

2. LDPs for the driving processes

2.1. Sample path LDP for the diffusion

We apply a result of [5], which is based on a representation formula for functionals of Brownian motion obtained in [3], to obtain an LDP for $(\sqrt{\varepsilon}B, V^\varepsilon)$. While the Yamada–Watanabe condition from Assumption 1.4 covers virtually all one-dimensional diffusions that have been suggested in financial modelling, we note that Assumption 1.4 could still be weakened, if desired, e.g. by inspecting the proof of Theorem 4.3 in [3].

If assumptions (H1)–(H6) of [5] hold, then the family of processes $(\sqrt{\varepsilon}B, V^\varepsilon)$. which satisfy the two-dimensional SDE

$$\begin{pmatrix}\sqrt{\varepsilon}dB_t\\ dV_t^\varepsilon\end{pmatrix} = \begin{pmatrix}0\\ \bar{b}(V_t^\varepsilon)\end{pmatrix}dt + \sqrt{\varepsilon}\begin{pmatrix}1\\ \bar{\sigma}(V_t^\varepsilon)\end{pmatrix}dB_t, \tag{2.1}$$

admits an LDP due to Theorem 1 in [5]. For V^ε, (H1)–(H6) have been checked in [5, pp. 1143–1144]. For $(\sqrt{\varepsilon}B, V^\varepsilon)$, the proofs are similar. The assumptions (H1)–(H3) are clearly satisfied. Let us check condition (H4), namely unique solvability of the control equation (7) in [5]. Here, it is

$$\begin{pmatrix}\varphi_1(t)\\ \varphi_2(t)\end{pmatrix} = \begin{pmatrix}0\\ v_0\end{pmatrix} + \int_0^t \begin{pmatrix}0\\ \bar{b}(\varphi_2(s))\end{pmatrix}ds + \int_0^t \begin{pmatrix}1\\ \bar{\sigma}(\varphi_2(s))\end{pmatrix}f(s)\,ds, \tag{2.2}$$

where $f \in L^2[0,T]$ is the control function. We also have $\varphi_1, \varphi_2 \in C[0,T]$. It follows that the unique solution of (2.2) is given by $\bar{\Gamma}_{v_0}(f) = \begin{pmatrix}\int_0^\cdot f(s)\,ds\\ \varphi_2\end{pmatrix}$, where the function φ_2 is the unique solution of the equation

$$\varphi_2(t) = v_0 + \int_0^t \bar{b}(\varphi_2(s))\,ds + \int_0^t \bar{\sigma}(\varphi_2(s))f(s)\,ds, \quad t \in [0,T], \tag{2.3}$$

which exists, and is positive, by [5, Proposition 1]. This establishes condition (H4) in our setting. Note at this point, that the ODE (2.3) is formulated for $f \in L^2[0,T]$ to match the notation of [5]. Alternatively it can also be written, with a $g \in H_0^1$, and $\dot{g}$ instead of f, see (1.16). Condition (H5) for the second component of $\bar{\Gamma}_{v_0}$ was checked in [5, p. 1144]. For the first component, (H5) is true by the following simple fact.

Lemma 2.1. *The map $f \mapsto \int_0^\cdot f(s)\,ds$ is continuous from $\mathcal{B}_r$ into $C[0,T]$, where $\mathcal{B}_r$ is the closed ball of radius $r > 0$ in $L^2[0,T]$ endowed with the weak topology.*

Proof. If $f_n \in \mathcal{B}_r$ converges weakly to f, then the convergence is uniform on compact subsets of $L^2[0,T]$. Since $\{\mathbb{1}_{[0,t]} : 0 \le t \le T\}$ is compact, we have

$$\sup_{t\in[0,T]}\left|\int_0^t f(u)\,du - \int_0^t f_n(u)\,du\right| \to 0, \quad n \to \infty. \quad \square \tag{2.4}$$

The tightness assumption (H6) can be established as in [5]. The verification, which is based on the sub-linear growth of $\bar{b}$ and $\bar{\sigma}$ and the uniform moment estimate in Lemma A.2 of [5], is found on pp. 1137–1138 of [5]. See also Section 4.2 of [5]. Now, Theorem 1 of [5] implies the following assertion, in fact a Laplace principle. But since the rate function is a good rate function (which is shown in [5]), we also get an LDP with the same rate function. See Theorems 1.2.1 and 1.2.3 of [7].

Theorem 2.2. *The family of processes $(\sqrt{\varepsilon}B, V^\varepsilon)$ satisfies an LDP in the space $C[0,T]^2$ with speed ε^{-1} and good rate function $I : C[0,T]^2 \to [0,\infty]$ given by*

$$I(\varphi_1,\varphi_2) = \inf_{\left\{f\in L^2[0,T]:\, \bar{\Gamma}_{v_0}(f)=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\}} \frac{1}{2}\int_0^T f(t)^2\,dt, \tag{2.5}$$

whenever $\left\{f \in L^2[0,T] : \bar{\Gamma}_{v_0}(f) = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\} \neq \emptyset$, *and* $I(\varphi_1,\varphi_2) = \infty$ *otherwise. Here,* $\bar{\Gamma}_{v_0}(f)$ *maps* f *to the solution of* (2.2).

Note that the positivity of the solution of (2.3) shows that $I(\varphi_1,\varphi_2) = \infty$ whenever φ_2 is negative at some point. Thus, Theorem 2.2 implies that V is a non-negative process, as noted after Assumption 1.4.

The condition $\bar{\Gamma}_{v_0}(f) = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$ implies that $\int_0^t f(s)\,ds = \varphi_1(t)$, or $f(t) = \dot{\varphi}_1(t)$. Therefore

$$\dot{\varphi}_2(t) = \bar{b}(\varphi_2(t)) + \bar{\sigma}(\varphi_2(t))\dot{\varphi}_1(t),$$

and hence (recall that φ_2 is positive by [5, Proposition 1])

$$\dot{\varphi}_1(t) = \frac{\dot{\varphi}_2(t) - \bar{b}(\varphi_2(t))}{\bar{\sigma}(\varphi_2(t))}. \tag{2.6}$$

Therefore, the following statement holds:

Corollary 2.3. *For every* φ_2 *that is absolutely continuous on* $[0,T]$ *with* $\varphi_2(0) = v_0$

$$I\Big(\int_0^{\cdot} \frac{\dot{\varphi}_2(t) - \bar{b}(\varphi_2(t))}{\bar{\sigma}(\varphi_2(t))}\,dt, \varphi_2\Big) = \frac{1}{2}\int_0^T \Big(\frac{\dot{\varphi}_2(t) - \bar{b}(\varphi_2(t))}{\bar{\sigma}(\varphi_2(t))}\Big)^2 dt, \tag{2.7}$$

if the integral is finite, and $I(\varphi_1,\varphi_2) = \infty$ *in all the remaining cases.*

2.2. Sample path LDP for $(\sqrt{\varepsilon}B, \hat{V}^{\varepsilon})$

In this subsection we lift the sample path LDP in Theorem 2.2 to one for the family of processes we get when applying the "hat" operator defined in (1.12) to V^{ε}.

Lemma 2.4. *The mapping* $f \mapsto \hat{f}$ *is continuous from the space* $C[0,T]$ *into itself.*

Proof. For $f \in C[0,T]$ and all $t_1, t_2 \in [0,T]$,

$$|\hat{f}(t_1) - \hat{f}(t_2)| \le M(|t_1 - t_2|)^{\frac{1}{2}}\Big(\int_0^T U(f(s))^2\,ds\Big)^{\frac{1}{2}} \le C_f|t_1 - t_2|^{\frac{r}{2}}.$$

The number r in the exponent of the last term comes from an estimate for the modulus of continuity of the kernel given by (1.5). Here we used the local boundedness of the continuous function U, and also (1.4). Now, it is clear that the function $\hat{f}$ is continuous on $[0,T]$. It remains to prove the continuity of the mapping $f \mapsto \hat{f}$ on $C[0,T]$. Suppose $f_k \to f$ in $C[0,T]$. Then we have

$$\|\hat{f} - \hat{f}_k\|_{C[0,T]} \le \Big(\int_0^T |U(f(s)) - U(f_k(s))|^2\,ds\Big)^{\frac{1}{2}} \sup_{t\in[0,T]}\Big(\int_0^T K(t,s)^2\,ds\Big)^{\frac{1}{2}}. \tag{2.8}$$

Moreover,

$$C_0 = \max\big\{\|f\|_{C[0,T]}, \sup_k \|f_k\|_{C[0,T]}\big\} < \infty.$$

It follows from Assumption 1.1 and (2.8) that there exists a constant C_1 for which

$$\|\hat{f} - \hat{f}_k\|_{C[0,T]} \le C_1 \sup_{s\in[0,T]}\big|U(f(s)) - U(f_k(s))\big|, \tag{2.9}$$

and the previous expression converges to zero by the uniform continuity of U on $[-C_0, C_0]$. This completes the proof. □

The next assertion establishes the LDP for $(\sqrt{\varepsilon}B, \hat{V}^\varepsilon)$.

Theorem 2.5. *The family of processes $(\sqrt{\varepsilon}B, \hat{V}^\varepsilon)$ satisfies an LDP in the space $C[0,T]^2$ with speed ε^{-1} and good rate function given by*

$$\tilde{I}\big(\psi_1, \mathcal{K}(U \circ \Gamma(\psi_1))\big) = \frac{1}{2}\int_0^T \dot{\psi}_1(t)^2\, dt, \tag{2.10}$$

if the expression in (2.6) exists, and $\tilde{I}(\psi_1, \psi_2) = \infty$ otherwise. As above, Γ is the solution map of the one-dimensional ODE (1.16), which means that $\varphi = \Gamma(\psi_1)$ solves the ODE $\dot{\varphi} = \bar{b}(\varphi) + \bar{\sigma}(\varphi)\dot{\psi}_1$.

Proof. We know that $(\sqrt{\varepsilon}B, V^\varepsilon)$ satisfies the LDP in Theorem 2.2. The mapping $(\varphi_1, \varphi_2) \mapsto (\varphi_1, \hat{\varphi}_2)$ of $C[0,T]^2$ into itself is continuous due to Lemma 2.4. Hence, we can use the contraction principle, which gives

$$\tilde{I}(\psi_1, \psi_2) = \inf_{\{(\varphi_1,\varphi_2)\in C[0,T]^2 :\, (\psi_1,\psi_2)=(\varphi_1,\hat{\varphi}_2)\}} I(\varphi_1, \varphi_2) = \inf_{\hat{\varphi}_2=\psi_2} I(\psi_1, \varphi_2).$$

The necessary condition under which we have $I(\psi_1, \varphi_2) < \infty$ is $\dot{\psi}_1 = \frac{\dot{\varphi}_2 - \bar{b}(\varphi_2)}{\bar{\sigma}(\varphi_2)}$ (see Corollary 2.3). □

Since B and W are independent, the following result is an immediate consequence of Theorem 2.5 and Schilder's theorem.

Corollary 2.6.

(i) *The family $(\sqrt{\varepsilon}W_T, \sqrt{\varepsilon}B, \hat{V}^\varepsilon)$ satisfies an LDP with speed ε^{-1} and rate function*

$$\hat{I}\big(y, \psi_1, \mathcal{K}(U \circ \Gamma(\psi_1))\big) = \frac{T}{2}y^2 + \frac{1}{2}\int_0^T \dot{\psi}_1^2(t)dt, \tag{2.11}$$

for $y \in \mathbb{R}$ and $\psi_1 \in H_0^1[0,T]$, if all the expressions are finite, and $\hat{I}(y, \psi_1, \psi_2) = \infty$ otherwise.

(ii) *The family of processes $(\sqrt{\varepsilon}W, \sqrt{\varepsilon}B, \hat{V}^\varepsilon)$ satisfies an LDP with speed ε^{-1} and rate function*

$$\hat{I}\big(\psi_0, \psi_1, \mathcal{K}(U \circ \Gamma(\psi_1))\big) = \frac{1}{2}\int_0^T \dot{\psi}_0(t)^2\, dt + \frac{1}{2}\int_0^T \dot{\psi}_1^2(t)dt, \tag{2.12}$$

for $\psi_0, \psi_1 \in H_0^1[0,T]$, if all the expressions are finite, and $\hat{I}(\psi_0, \psi_1, \psi_2) = \infty$ otherwise.

3. Proof of the LDP for the log-price

3.1. *Proof of Theorem 1.6 (one-dimensional LDP)*

It is clear that the one-dimensional LDP in Theorem 1.6 is a special case of the sample path LDP in Theorem 1.7. For the reader's convenience, though, it seemed better to us to first prove Theorem 1.6, and then refer to some parts of this proof in the proof of Theorem 1.7.

We build on some ideas of [12]. To match the notation there, we note that $\varepsilon^H \hat{B}$ from [12] corresponds to our process $\hat{V}^\varepsilon$ as defined in (1.12). In the original proof of [12] the author first supposes $T = 1$. Here, for convenience, we immediately allow a general $T > 0$. By the following lemma, it suffices to prove an LDP for the drift-less process

$$d\hat{X}_t^\varepsilon = \sqrt{\varepsilon}\sigma(\hat{V}_t^\varepsilon)(\bar{\rho}\, dW_t + \rho\, dB_t), \quad 0 \le t \le T. \tag{3.1}$$

Lemma 3.1. *The families $(X_T^\varepsilon)_{\varepsilon>0}$ and $(\hat{X}_T^\varepsilon)_{\varepsilon>0}$ are exponentially equivalent, i.e. for every $\delta > 0$, the following equality holds:*

$$\limsup_{\varepsilon \searrow 0} \varepsilon \log P(|X_T^\varepsilon - \hat{X}_T^\varepsilon| > \delta) = -\infty. \tag{3.2}$$

Proof. By the same reasoning as in Section 5 of [12], there is a strictly increasing continuous function $\eta : [0,\infty) \to [0,\infty)$ with $\lim_{u \nearrow \infty} \eta(u) = \infty$ and $\bar{\sigma}(u)^2 \le \eta(u)$ for all $u \in \mathbb{R}$. Let $\eta^{-1} : [0,\infty) \to [0,\infty)$ be the inverse function. Replacing $\sqrt{\varepsilon}\hat{B}$ in [12] by $\hat{V}^\varepsilon$, we get the estimate

$$\begin{aligned} P(|X_T^\varepsilon - \hat{X}_T^\varepsilon| > \delta) &= P\Big(\frac{1}{2}\varepsilon \int_0^T \sigma(\hat{V}_s^\varepsilon)^2\, ds > \delta\Big) \le P\Big(\frac{1}{2}\varepsilon \int_0^T \eta(\hat{V}_s^\varepsilon)\, ds > \delta\Big) \\ &\le P\Big(\frac{1}{2}\varepsilon \int_0^T \eta(\sup_{0\le t\le T} |\hat{V}_t^\varepsilon|)\, ds > \delta\Big) = P\Big(\frac{1}{2}\varepsilon T \eta(\sup_{0\le t\le T} |\hat{V}_t^\varepsilon|) > \delta\Big) \\ &= P\Big(\eta(\sup_{0\le t\le T} |\hat{V}_t^\varepsilon|) > \frac{2\delta}{\varepsilon T}\Big) = P\Big(\sup_{0\le t\le T} |\hat{V}_t^\varepsilon| > \eta^{-1}(\frac{2\delta}{\varepsilon T})\Big) \\ &\le \exp\Big(-\frac{\varepsilon^{-1}}{2} J(A)\Big), \end{aligned} \tag{3.3}$$

where J is the rate function of $\sup_{0\le t\le T} |\hat{V}_t^\varepsilon|$, and $A = (\eta^{-1}(\frac{2\delta}{\varepsilon T}), \infty)$. Since J is a good rate function, we know that $J(x,\infty) \nearrow \infty$ as $x \nearrow \infty$, so we get (3.2). □

We will next reason as in [12], p. 1121, using the LDP for $(\sqrt{\varepsilon}W_T, \sqrt{\varepsilon}B, \hat{V}^\varepsilon)$ in Corollary 2.6. Analogously to [12], we define the functional Φ on the space $M = \mathbb{R} \times C[0,T]^2$ by

$$\Phi(y, f, g) = \bar{\rho}\Big(\int_0^T \sigma(g(s))^2\, ds\Big)^{1/2} y + \rho \int_0^T \sigma(g(s)) \dot{f}(s)\, ds, \tag{3.4}$$

if $(f, g) = (f, \check{f})$ with $f \in H_0^1[0,T]$, and $\Phi(y, f, g) = 0$ otherwise (recall the definition (1.15)). Further, for any integer $m \ge 1$, define a functional on M by

$$\Phi_m(y, h, l) = \bar{\rho}\Big(\int_0^T \sigma(l(s))^2\, ds\Big)^{1/2} y + \rho \sum_{k=0}^{m-1} \sigma(l(t_k))\big(h(t_{k+1}) - h(t_k)\big), \tag{3.5}$$

where $t_k := \frac{kT}{m}$ for $k \in \{0, \dots, m\}$. The following approximation property is the key to applying the extended contraction principle (see (4.2.24) in [6]).

Lemma 3.2. *For every $\alpha > 0$,*

$$\limsup_{m\to\infty} \sup_{\{f \in H_0^1[0,T]:\, \frac{T}{2}y^2 + \frac{1}{2}\int_0^T \dot{f}(s)^2\, ds \le \alpha\}} |\Phi(y, f, \check{f}) - \Phi_m(y, f, \check{f})| = 0. \tag{3.6}$$

Proof. The proof is similar to that of Lemma 21 in [12]. We need to change the range of the integrals and suprema to $[0, T]$ instead of $[0, 1]$. Hence, the grid points for h_m are $t_k := \frac{Tk}{m}$ for $k \in \{0, \dots, m\}$, like in (3.5). We use a different integral operator than [12], and so we have to show that the set $E_\beta = \{\check{f} : f \in D_\beta\}$ is precompact in $C[0, T]$ for $D_\beta = \{f \in H_0^1[0, T] : \int_0^T \dot{f}(s)^2\, ds < \beta\}$. For $f \in D_\beta$, we have $\dot{f} \in L^2[0, T]$ and therefore can use Eq. (16) of [5] to estimate the solution of the ODE

$$v = v_0 + \int_0^{\cdot} \bar{b}(v(s))\, ds + \int_0^{\cdot} \bar{\sigma}(v(s))\dot{f}(s)\, ds$$

as follows:

$$\sup_{0 \le s \le T} |v(s)|^2 \le \big(3|v_0|^2 + 6\mu^2 T^2 + 6\mu^2 T \|\dot{f}\|_2^2\big) e^{6\mu^2 T (T + \|\dot{f}\|_2^2)} =: C_\beta^2.$$

Here, μ comes from the sub-linear growth condition for the coefficient functions of the diffusion equation for V in Assumption 1.4. Since the continuous function U is bounded on the interval $[-C_\beta, C_\beta]$,

$$\{U \circ v : f \in D_\beta,\ \dot{v} = \bar{b}(v) + \bar{\sigma}(v)\dot{f}\} \tag{3.7}$$

is a bounded subset of $C[0, T]$. The compact operator $\mathcal{K}$, as defined in (1.6), maps the set in (3.7) to a precompact set in $C[0, T]$. So we can conclude that E_β is precompact. After that, the proof continues like in [12]. □

Definition 3.3. Let $t \in [0, T]$ be fixed. Consider the grid $t_k := T\frac{k}{m}$ for $k \in \{0, \dots, m\}$. There is a k such that $t \in [t_k, t_{k+1})$. Denote by $\Xi(t)$ the left end of the previous interval. Explicitly, we put

$$\Xi(t) := \frac{T}{m}[\frac{mt}{T}], \tag{3.8}$$

where $[a]$ stands for the integer part of the number $a \in \mathbb{R}$. For $T = 1$, this reduces to $\Xi(t) = \frac{[mt]}{m}$.

We will next prove that $\Phi_m(\sqrt{\varepsilon}W_T, \sqrt{\varepsilon}B, \hat{V}^\varepsilon)$ is an exponentially good approximation as $m \nearrow \infty$ to $(\sqrt{\varepsilon}W_T, \sqrt{\varepsilon}B, \hat{V}^\varepsilon)$. We start with an auxiliary result.

Lemma 3.4. *For every $y > 0$,*

$$\limsup_{m\to\infty} \limsup_{\varepsilon \searrow 0} \varepsilon \log P\Big(\sup_{t\in[0,T]} |\hat{V}_t^\varepsilon - \hat{V}_{\Xi(t)}^\varepsilon| > y\Big) = -\infty. \tag{3.9}$$

Proof. This corresponds to Lemma 23 in [12], but we need to adjust some estimates in the proof, since we do *not* have Gaussianity in our setting. As in [12] we use

$$P\Big(\sup_{t\in[0,T]} |\hat{V}_t^\varepsilon - \hat{V}_{\Xi(t)}^\varepsilon| > y\Big) \le P\Bigg(\sup_{\substack{t_1,t_2\in[0,T]\\ |t_2-t_1|\le T/m}} |\hat{V}_{t_2}^\varepsilon - \hat{V}_{t_1}^\varepsilon| > y\Bigg). \tag{3.10}$$

Then, for $|s-t| \leq T/m$, we have

$$\begin{aligned}|\hat{V}^\varepsilon_t - \hat{V}^\varepsilon_s| &= \Big|\int_0^T \big(K(t,v)-K(s,v)\big)U(V^\varepsilon_v)\,dv\Big| \\ &\leq \sqrt{M(\frac{T}{m})}\sup_{v\in[0,T]}|U(V^\varepsilon_v)| \\ &\leq \Big(\frac{cT}{m}\Big)^{r/2}\sup_{v\in[0,T]}|U(V^\varepsilon_v)|,\end{aligned}$$

where M is the modulus of continuity of the kernel function in Assumption 1.1. We know that V^ε satisfies an LDP, by Theorem 2.2. Using this, we can estimate

$$\begin{aligned}P\Big(\sup_{t\in[0,T]}|\hat{V}^\varepsilon_t - \hat{V}^\varepsilon_{\Xi(t)}| > y\Big) &\leq P\Big(\sup_{s\in[0,T]}|U(V^\varepsilon_s)| > yc^{-r/2}T^{-r/2}m^{r/2}\Big) \\ &\leq \exp\Big(-\frac{\varepsilon^{-1}}{2}\cdot J\big((y(\frac{m}{cT})^{\frac{r}{2}},\infty)\big)\Big),\end{aligned}$$

for ε small enough. Here, J is the good rate function corresponding to $\sup_{s\in[0,T]}|U(V^\varepsilon_s)|$, which satisfies an LDP, as seen from applying the contraction principle to the continuous mapping $f \mapsto \sup_{s\in[0,T]}|U(f(s))|$. From this, we can write

$$\limsup_{\varepsilon\searrow 0}\varepsilon\log P\Big(\sup_{t\in[0,T]}|\hat{V}^\varepsilon_t - \hat{V}^\varepsilon_{\Xi(t)}| > y\Big) \leq -\frac{1}{2}J\Big(\Big(y\big(\frac{m}{cT}\big)^{\frac{r}{2}},\infty\Big)\Big). \tag{3.11}$$

Since J has compact level sets, the term on the right-hand side explodes for $m \nearrow \infty$. □

Next, we show that the discretization functionals Φ_m yield an exponentially good approximation.

Lemma 3.5. *For every $\delta > 0$,*

$$\lim_{m\to\infty}\limsup_{\varepsilon\searrow 0}\varepsilon\log P\Big(\big|\Phi(\sqrt{\varepsilon}W_T,\sqrt{\varepsilon}B,\hat{V}^\varepsilon) - \Phi_m(\sqrt{\varepsilon}W_T,\sqrt{\varepsilon}B,\hat{V}^\varepsilon)\big| > \delta\Big) = -\infty. \tag{3.12}$$

Proof. This lemma corresponds to Lemma 22 in [12]. As in the proof of that lemma, it suffices to show

$$\lim_{m\to\infty}\limsup_{\varepsilon\searrow 0}\varepsilon\log P\Big(\sqrt{\varepsilon}|\rho|\sup_{t\in[0,T]}\Big|\int_0^t \sigma^{(m)}_s\,dB_s\Big| > \delta\Big) = -\infty, \tag{3.13}$$

where $\sigma^{(m)}_t = \sigma(\hat{V}^\varepsilon_t) - \sigma(\hat{V}^\varepsilon_{\Xi(t)})$. We have to redefine $\xi^{(m)}_\eta$ in order to take a general $T > 0$ into account:

$$\xi^{(m)}_\eta = \inf\Big\{t\in[0,T] : \frac{\eta}{q(\eta)}|\hat{V}^\varepsilon| + |\hat{V}^\varepsilon_t - \hat{V}^\varepsilon_{\Xi(t)}| > \eta\Big\} \wedge T.$$

Note that we use the convention $\inf\emptyset = \infty$ here. The equations (55)–(65) in [12] remain the same, except that we replace $\varepsilon^H\hat{B}$ by $\hat{V}^\varepsilon$ and use our redefined versions of $\sigma^{(m)}$ and $\xi^{(m)}_\eta$. Thus, formula (65) in [12] can be applied. The estimates (66) and (67) have to be replaced by

$$P\Big(\sqrt{\varepsilon}|\rho| \sup_{t\in[0,T]} \Big| \int_0^t \sigma_s^{(m)}\, dB_s \Big| > \delta\Big)$$
$$\leq P(\xi_\eta^{(m)} < T) + P\Big(\sqrt{\varepsilon}|\rho| \sup_{t\in[0,\xi_\eta^{(m)}]} \Big| \int_0^t \sigma_s^{(m)}\, dB_s \Big| > \delta\Big)$$

and

$$\begin{aligned} P(\xi_\eta^{(m)} < T) &\leq P\Big(\sup_{t\in[0,T]} \big(\frac{\eta}{q(\eta)} |\hat{V}_t^\varepsilon| + |\hat{V}_t^\varepsilon - \hat{V}_{\Xi(t)}^\varepsilon| \big) > \eta \Big) \\ &\leq P\Big(\sup_{t\in[0,T]} |\hat{V}_t^\varepsilon| > \frac{q(\eta)}{2} \Big) + P\Big(\sup_{t\in[0,T]} |\hat{V}_t^\varepsilon - \hat{V}_{\Xi(t)}^\varepsilon| > \frac{\eta}{2} \Big). \end{aligned} \tag{3.14}$$

Using Lemma 3.4, we can handle the second term, and so it remains to find an appropriate estimate for the first term. Here we need to adapt the reasoning in [12] because of the lack of Gaussianity. By the LDP for $\hat{V}^\varepsilon$ and the contraction principle applied to the mapping $f \mapsto \sup_{t\in[0,T]} |f(t)|$, we get

$$P\Big(\sup_{t\in[0,T]} |\hat{V}_t^\varepsilon| > \frac{q(\eta)}{2} \Big) \leq \exp\Big(-\frac{\varepsilon^{-1}}{2} \cdot I_{\sup}\big((\tfrac{1}{2} q(\eta), \infty) \big) \Big), \tag{3.15}$$

for $\varepsilon > 0$ small enough, where $I_{\sup}$ is the rate function of $\sup_{t\in[0,T]} |\hat{V}_t^\varepsilon|$. Note that $q(\eta) \nearrow \infty$ for $\eta \searrow 0$. So, we get

$$\limsup_{\eta\searrow 0} \limsup_{\varepsilon\searrow 0} \varepsilon \log P\Big(\sup_{t\in[0,T]} |\hat{V}_t^\varepsilon| > \frac{q(\eta)}{2} \Big) = -\infty. \tag{3.16}$$

Using (3.9) and (3.16), we get (73) and (74) of [12]. Finally, we can complete the proof as in [12]. □

Let us continue the proof of Theorem 1.6. Lemma 3.2 states that condition (4.2.24) in [6] is satisfied. Furthermore, due to Lemma 3.5, we know that $\Phi_m(\sqrt{\varepsilon} W_T, \sqrt{\varepsilon} B, \hat{V}^\varepsilon)$ is an exponentially good approximation of $\Phi(\sqrt{\varepsilon} W_T, \sqrt{\varepsilon} B, \hat{V}^\varepsilon)$ as $m \nearrow \infty$. Hence, we can use the extended contraction principle (Theorem 4.2.23 in [6]), and get that $\hat{X}_T^\varepsilon$ satisfies an LDP with good rate function I and speed ε^{-1}. We know from Lemma 3.1 that $\hat{X}_T^\varepsilon$ and X_T^ε are exponentially equivalent, and so we finally arrive at Theorem 1.6.

According to the extended contraction principle, we have

$$I_T(x) = \inf\big\{ \hat{I}(y, f, g) : x = \Phi(y, f, g) \big\}.$$

The rate function $\hat{I}$ is only finite for

$$\hat{I}\big(y, f, \mathcal{K}(U \circ \Gamma(f))\big) = \frac{T}{2} y^2 + \frac{1}{2} \langle \dot{f}, \dot{f} \rangle.$$

Recall that Γ is the one-dimensional solution map that takes f to the solution of the ODE $\dot{v} = \bar{b}(v) + \bar{\sigma}(v) \dot{f}$, $v(0) = v_0$, and that the function Φ can be written as

$$\Phi(y, f, g) = \bar{\rho} \sqrt{\langle \sigma(g)^2, 1 \rangle}\, y + \rho \langle \sigma(g), \dot{f} \rangle.$$

Hence, if $x = \Phi(y, f, g)$, then

$$y = \frac{x - \rho \langle \sigma(g), \dot{f} \rangle}{\bar{\rho} \sqrt{\langle \sigma(g)^2, 1 \rangle}}.$$

Inserting this into the rate function obtained through the contraction principle, we get

$$\begin{aligned} I_T(x) &= \inf\left\{\hat{I}(y, f, g) : x = \Phi(y, f, g),\ \ f \in H_0^1,\ \ g = \mathcal{K}(U \circ \Gamma(f))\right\} \\ &= \inf\left\{\frac{T}{2}y^2 + \frac{1}{2}\langle \dot{f}, \dot{f}\rangle : y = \frac{x - \rho\langle \sigma(\mathcal{K}(U \circ \Gamma(f))), \dot{f}\rangle}{\bar{\rho}\sqrt{\langle \sigma(\mathcal{K}(U \circ \Gamma(f)))^2, 1\rangle}}, f \in H_0^1\right\} \\ &= \inf_{f \in H_0^1}\left\{\frac{T}{2}\left(\frac{x - \rho\langle \sigma(\mathcal{K}(U \circ \Gamma(f))), \dot{f}\rangle}{\bar{\rho}\sqrt{\langle \sigma(\mathcal{K}(U \circ \Gamma(f)))^2, 1\rangle}}\right)^2 + \frac{1}{2}\langle \dot{f}, \dot{f}\rangle\right\}. \end{aligned} \tag{3.17}$$

3.2. Proof of Theorem 1.7 (a sample path LDP)

We adapt the arguments on pp. 3655–3658 in [13]. As in the preceding section, our starting point is that we already have an LDP for $(\sqrt{\varepsilon}W, \sqrt{\varepsilon}B, \hat{V}^{\varepsilon})$, see Corollary 2.6. We redefine the functions Φ and Φ_m so that they map $C[0, T]^3$ to $C[0, T]$. For $l \in H_0^1[0, T]$ and $(f, g) \in C[0, T]^2$ such that $f \in H_0^1[0, T]$ and $g = \check{f}$,

$$\Phi(l, f, g)(t) = \bar{\rho}\int_0^t \sigma(\check{f}(s))\dot{l}(s)\,ds + \rho\int_0^t \sigma(\check{f}(s))\dot{f}(s)\,ds, \quad 0 \le t \le T. \tag{3.18}$$

In addition, for all the remaining triples (l, f, g), we set $\Phi(l, f, g)(t) = 0$ for all $t \in [0, T]$. By the following lemma, we can remove the drift term.

Lemma 3.6. *The families of processes X^{ε} and $\hat{X}^{\varepsilon}$ are exponentially equivalent, i.e. for every $\delta > 0$, the following equality holds:*

$$\limsup_{\varepsilon \searrow 0} \varepsilon \log P(\|X^{\varepsilon} - \hat{X}^{\varepsilon}\|_{C[0,T]} > \delta) = -\infty. \tag{3.19}$$

Here, $\hat{X}^{\varepsilon}$ is defined in (3.1).

Proof. By taking into account the proof of Lemma 3.1, we see that just one additional estimate is needed, namely

$$\|X^{\varepsilon} - \hat{X}^{\varepsilon}\|_{C[0,T]} = \sup_{0 \le t \le T}|X_t^{\varepsilon} - \hat{X}_t^{\varepsilon}| \le \frac{1}{2}\varepsilon T\eta\Big(\sup_{0 \le t \le T}|\hat{V}_t^{\varepsilon}|\Big).$$

Then we directly get

$$P(\|X^{\varepsilon} - \hat{X}^{\varepsilon}\| > \delta) \le P\left(\frac{1}{2}\varepsilon T\eta\Big(\sup_{0 \le t \le T}|\hat{V}_t^{\varepsilon}|\Big) > \delta\right) = P\left(\sup_{0 \le t \le T}|\hat{V}_t^{\varepsilon}| > \eta^{-1}\Big(\frac{2\delta}{\varepsilon T}\Big)\right),$$

which is exactly the same expression as in the proof of (3.2). $\square$

The sequence of functionals $(\Phi_m)_{m \ge 1}$ from $C[0, T]^3$ to $C[0, T]$ is given for $(r, h, l) \in C[0, T]^3$ and $t \in [0, T]$ by

$$\begin{aligned} \Phi_m(r, h, l)(t) = \bar{\rho}&\left(\sum_{k=0}^{[\frac{mt}{T}-1]} \sigma(l(t_k))[r(t_{k+1}) - r(t_k)] + \sigma\big(l(\Xi(t))\big)\big[r(t) - r(\Xi(t))\big]\right) \\ &+\rho\left(\sum_{k=0}^{[\frac{mt}{T}-1]} \sigma(l(t_k))[h(t_{k+1}) - h(t_k)] + \sigma\big(l(\Xi(t))\big)\big[h(t) - h(\Xi(t))\big]\right). \end{aligned} \tag{3.20}$$

It is not hard to see that for every $m \ge 1$, the mapping Φ_m is continuous.

Lemma 3.7. *For every $\zeta > 0$ and $y > 0$,*

$$\limsup_{m\nearrow\infty} \sup_{\{(r,f)\in H_0^1[0,T]^2:\ \frac{1}{2}\int_0^T \dot{r}(s)\,ds+\frac{1}{2}\int_0^T \dot{f}(s)\,ds\le\zeta\}} \|\Phi(r,f,\check{f})-\Phi_m(r,f,\check{f})\|_{C[0,T]^2}=0. \quad (3.21)$$

Proof. Lemma 3.7 can be obtained from the proofs of Lemma 3.2, Lemma 21 in [12] and Lemma 2.13 in [13]. The only difference here is that the supremum is taken over two functions from $D_\eta = \{w \in H_0^1[0,T] : \int_0^T \dot{w}^2\,ds \le \eta\}$. By the uniform bound in the proof of Lemma 21 of [12], this is actually irrelevant. □

Next, we will show that the family $\Phi_m(\sqrt{\varepsilon}W, \sqrt{\varepsilon}B, \hat{V}^\varepsilon)$ is an exponentially good approximation for $\Phi(\sqrt{\varepsilon}W, \sqrt{\varepsilon}B, \hat{V}^\varepsilon)$, as $m \nearrow \infty$.

Lemma 3.8. *For every $\delta > 0$*

$$\lim_{m\to\infty}\limsup_{\varepsilon\searrow 0}\varepsilon\log P(\|\Phi(\sqrt{\varepsilon}W,\sqrt{\varepsilon}B,\hat{V}^\varepsilon)-\Phi_m(\sqrt{\varepsilon}W,\sqrt{\varepsilon}B,\hat{V}^\varepsilon)\|_{C[0,T]}>\delta)=-\infty. \quad (3.22)$$

Proof. In the proof of Lemma 3.5, the estimate (3.13) was formulated stronger than needed. We can directly use this to show (2.13) of [13]. We can also get (2.14) of [13] this way. The ingredients of (55)–(65) in [12] do in fact depend on the Brownian motion B via the process $\hat{V}^\varepsilon$. However, the reasoning for the estimate

$$P\left(\sup_{t\in[0,\xi_\eta^{(m)}]} \varepsilon^H \left|\int_0^t \sigma_s^{(m)}\,dB_s\right| > \delta\right) \le \exp\left(-\frac{\delta^2}{2\varepsilon^{2H}L(q(\eta))^2\omega(\eta)^2}\right) \quad (3.23)$$

in [12] stays the same if we replace the driving Brownian motion B by W. The rest of the proof from here on is essentially the same as in the proof of Theorem 2.9 in [13]. □

Just as in the preceding section, we combine Lemmas 3.6–3.8 to see that Theorem 1.7 follows from the extended contraction principle. We have

$$Q(g) = \inf\{\hat{I}(\psi_0,\psi_1,\psi_1) : g = \Phi(\psi_0,\psi_1,\psi_2)\}.$$

The rate function $\hat{I}$ is only finite for

$$\hat{I}(\psi_0,\psi_1,\psi_2) = \frac{1}{2}\langle\dot{\psi}_0,\dot{\psi}_0\rangle + \frac{1}{2}\langle\dot{f},\dot{f}\rangle,$$

where $\psi_1 = f$ and $\psi_2 = \mathcal{K}(U \circ \Gamma(f))$ for some $f \in H_0^1[0,T]$. Recall that the function Φ is given by

$$\Phi(l,f,g)(t) = \bar{\rho}\int_0^t \sigma(g(s))\dot{l}(s)\,ds + \rho\int_0^t \sigma(g(s))\dot{f}(s)\,ds,$$

hence we can write

$$\dot{l} = \frac{\partial_t(\Phi(l,f,g)) - \rho\sigma(g)\dot{f}}{\bar{\rho}\sigma(g)}.$$

Finally, we get the rate function as follows:

$$\begin{aligned} Q(g) &= \inf\{\hat{I}(\psi_0,\psi_1,\psi_2) : g = \Phi(\psi_0,\psi_1,\psi_2)\} \\ &= \inf\Big\{\frac{1}{2}\langle\dot\psi_0,\dot\psi_0\rangle + \frac{1}{2}\langle\dot f,\dot f\rangle : f\in H_0^1,\ \psi_1 = f,\ \psi_2 = \mathcal{K}(U\circ\Gamma(f)), \\ &\qquad \dot\psi_0 = \frac{\partial_t(\Phi(\psi_0,\psi_1,\psi_2)) - \rho\sigma(\psi_2)\dot\psi_1}{\bar\rho\sigma(\psi_2)},\ g = \Phi(\psi_0,\psi_1,\psi_2)\Big\} \\ &= \inf\Big\{\frac{1}{2}\langle\dot\psi_0,\dot\psi_0\rangle + \frac{1}{2}\langle\dot f,\dot f\rangle : f\in H_0^1,\ \dot\psi_0 = \frac{\dot g - \rho\sigma(\mathcal{K}(U\circ\Gamma(f)))\dot f}{\bar\rho\sigma(\mathcal{K}(U\circ\Gamma(f)))}\Big\} \\ &= \inf_{f\in H_0^1}\Big\{\frac{1}{2}\int_0^T\Big(\frac{\dot g(t) - \rho\sigma(\mathcal{K}(U\circ\Gamma(f))(t))\dot f(t)}{\bar\rho\sigma(\mathcal{K}(U\circ\Gamma(f))(t))}\Big)^2 dt + \frac{1}{2}\int_0^T |\dot f(t)|^2\,dt\Big\}. \end{aligned} \tag{3.24}$$

4. Fractional CIR stochastic volatility

We describe an example of a model that fits our assumptions, and has already been studied in the literature on fractional volatility modelling [1]. Let V be a CIR process with positive parameters κ, θ and σ_{CIR}, satisfying $2\kappa\theta > \sigma_{\mathrm{CIR}}^2$. In this case,

$$\bar b(x) = \kappa(\theta - x) \quad \text{and} \quad \bar\sigma(x) = \sigma_{\mathrm{CIR}}\sqrt{x},$$

and the dynamics of V are

$$dV_t = \kappa(\theta - V_t)dt + \sigma_{\mathrm{CIR}}\sqrt{V_t}dB_t.$$

We choose the fractional kernel $K(t,s) = \Gamma(\alpha)^{-1}(t-s)^{\alpha-1}$, $0 \le s \le t$, and $U = \mathrm{id}$, so that the process $\hat V$ defined in (1.1) is the Riemann–Liouville integral of order α of the process V. We assume $\alpha \in (\frac{1}{2},\frac{3}{2})$, which overlaps with the parameter range $\alpha \in (0,1)$ considered in Section 2 of [1], and implies our assumption (1.3). The definition of the model is completed by putting

$$\sigma(x) = \sqrt{\sigma_0^2 + x}, \quad x \ge 0,$$

where $\sigma_0 > 0$ is the initial value of the stochastic volatility process $\sigma(\hat V_t)$. Note a small difference in notation compared to [1]: We write $v_0 = V_0$ for the initial value of V, and not for the initial value of the variance process $\sigma(\hat V_t)^2$ of the stock, which we denote by σ_0^2. Unlike [1], which is a paper on portfolio optimization, we set the drift of the stock to zero, because the application we have in mind is approximate option pricing in the small-noise regime.

The advantages of using a fractional CIR process instead of the classical CIR process are described in [1], Section 2, and the references given there. The model captures volatility persistence, in particular, steep implied volatility smiles for long maturity options and the comovement between implied and realized volatility. The paper [1] also gives a formula that makes the long-range dependence of the variance process explicit.

The model we just described is also closely related to the fractional Heston model from [11]. The main difference, besides the zero correlation assumption imposed in [11], is the range of α. They assume $\alpha \in (-\frac{1}{2},\frac{1}{2})$, whereas we have $\alpha \in (\frac{1}{2},\frac{3}{2})$. Thus, the models we consider in this paper could be seen as a complement to the fractional Heston model of [11], with positive correlation and rather general functions $\bar b(\cdot)$, $\bar\sigma(\cdot)$ and $\sigma(\cdot)$, but at the price of losing roughness of the volatility paths.

Remark 4.1. The paths of the CIR process V are $(\frac{1}{2}-\delta)$-Hölder continuous for any $\delta \in (0,\frac{1}{2})$ (see Lemma 7.1 in [1]). If we choose the fractional kernel $K(t,s) = \Gamma(H+\frac{1}{2})^{-1}(t-s)^{H-1/2}$,

$H \in (0,1)$, in the model considered in the present section, then the paths of $\hat{V}$ are in the Hölder space $\mathcal{H}^{H+1-\delta}$. See Definition 1.1.6 (p. 6) and Corollary 1.3.1 (p. 56) in [19]. In particular, since $H+1-\delta > 1$ for small δ, the paths of $\hat{V}$ are C^1 on $(0,T)$. By modifying the model, using $U(x) = |x - V_0|^\kappa$ with $\kappa \in (0,1]$ instead of $U = \mathrm{id}$, the paths of $\hat{V}$ become less smooth, namely $(\frac{1}{2}\kappa + H + \frac{1}{2} - \delta)$-Hölder continuous. In addition, if $\sigma(x) = \sigma_0(1 + x^\beta)$, $\beta \in (0,1)$, then the volatility paths $t \mapsto \sigma_0(1 + (\hat{V}_t)^\beta)$ are $(\frac{1}{2}\kappa\beta + (H + \frac{1}{2})\beta - \delta)$-Hölder continuous on $[0,T]$, for any small enough $\delta > 0$. While this Hölder exponent can be smaller than $\frac{1}{2}$, the volatility process is *not* rough, because $\sigma(\cdot)$ is smooth away from zero, and so "roughness" occurs only at time zero. Note that in truly rough models, the volatility process is constructed using stochastic integrals $\int_0^t K(t,s)dW_s$ or related processes, which is not the case in our setup.

5. Second order Taylor expansion of the rate function

In order to compute the rate function, a certain variational problem needs to be solved numerically. It might be preferable to use the Taylor expansion of the rate function instead, if it can be computed in closed form. In principle, this can be done using the approach used in [2], but would involve rather cumbersome calculations. We therefore illustrate the method by the example $V = B$ (a Brownian motion; thus $\bar{b} \equiv 0$ and $\bar{\sigma} \equiv 1$), $U(x) = x^2$, $v_0 = 0$. It is very easy to see that our main results hold for this example. Indeed, the required results from [5], for which we made our assumptions on the SDE for V, trivially hold here. The control ODE is degenerate, and its solution mapping Γ is just the identity map. The statement of Theorem 2.5 follows from Schilder's theorem and the contraction principle, and the transfer to the log-price is a simplified version of the arguments in Section 3.

Proposition 5.1. *Let $U(x) = x^2$ and $V = B$. Furthermore, assume that σ is smooth (at least locally around 0). Suppose that the rate function I is also smooth locally around 0. Then, with $\sigma_0 = \sigma(0)$, its Taylor expansion is*

$$
\begin{aligned}
I(x) &= I(0) + I'(0)x + I''(0)x^2 + O(x^3) \\
&= I''(0)x^2 + O(x^3) \\
&= \frac{1}{2\sigma_0^2}x^2 + O(x^3).
\end{aligned}
\tag{5.1}
$$

Remark 5.2. Formula (5.1) gives the second order Taylor expansion. However, the ideas in the proof of Proposition 5.1 can be used for higher orders. Clearly, the computations for the expansions get even more cumbersome in the latter case.

5.1. Proof of Proposition 5.1

The proof is very similar to the one of Theorem 3.1 in [2]. In the following, we will outline at which points adjustments are needed. Note that for the special we are treating we have $U(x) = x^2$ and $\Gamma \equiv \mathrm{id}$. To simplify computations in the proof, we put $T = 1$ and write $I = I_1$ for the rate function. In Proposition 5.1 of [2], there is a representation of the rate function that coincides with ours, except that different integral transforms are used. For our special case, we have

$$
I(x) = \inf_{f \in H_0^1} \Big[\frac{(x - \rho\tilde{G}(f))^2}{2\bar{\rho}^2\tilde{F}(f)} + \frac{1}{2}\tilde{E}(f) \Big] = \inf_{f \in H_0^1} \mathcal{I}_x(f), \tag{5.2}
$$

where

$$\tilde{G}(f) := \int_0^1 \sigma((\mathcal{K}(f^2))(s))\dot{f}(s)\,ds = \langle \sigma(\mathcal{K}(f^2)), \dot{f} \rangle, \tag{5.3}$$

$$\tilde{F}(f) := \int_0^1 \sigma((\mathcal{K}(f^2))(s))^2\,ds = \langle \sigma^2(\mathcal{K}(f^2)), 1 \rangle, \tag{5.4}$$

$$\tilde{E}(f) := \int_0^1 |\dot{f}(s)|^2\,ds = \langle \dot{f}, \dot{f} \rangle. \tag{5.5}$$

Recall that $\mathcal{K}f = \int_0^{\cdot} K(\cdot, s)f(s)\,ds$. In [2] the authors use the same integral transform as used in [12,13], i.e. $\mathcal{K}\dot{f}$. We have to adjust this to our case of $\mathcal{K}(f^2)$. Here, $\mathcal{I}_x$ denotes the functional that needs to be minimized to get the value of the rate function at x.

First, we need to get a representation for the minimizing configuration f^x of the functional $\mathcal{I}_x$. This is done like in Proposition 5.2 in [2]. The corresponding expansions of the ingredients of the rate function for our setting for $\delta > 0$ are

$$\tilde{E}(f + \delta g) \approx \tilde{E}(f) + 2\delta\langle \dot{f}, \dot{g} \rangle, \tag{5.6}$$

$$\tilde{F}(f + \delta g) \approx \tilde{F}(f) + 2\delta\langle (\sigma^2)'(\mathcal{K}(f^2)), \mathcal{K}(fg) \rangle, \tag{5.7}$$

$$\tilde{G}(f + \delta g) \approx \tilde{G}(f) + \delta(\langle \sigma(\mathcal{K}(f^2)), \dot{g} \rangle + 2\langle \sigma'(\mathcal{K}(f^2)), \dot{f}\mathcal{K}(fg) \rangle) \tag{5.8}$$

Note, that " $\approx$ " is defined in [2] as

$$A \approx B :\Leftrightarrow A = B + o(\delta), \quad \delta \searrow 0. \tag{5.9}$$

If $f = f^x$ is a minimizer then $\delta \mapsto \mathcal{I}_x(f + \delta g)$ has a minimum at $\delta = 0$ for all g. Using (5.6)–(5.8) we expand

$$\begin{aligned}
\mathcal{I}_x(f + \delta g) &= \frac{(x - \rho\tilde{G}(f + \delta g))^2}{2\bar{\rho}^2\tilde{F}(f + \delta g)} + \frac{1}{2}\tilde{E}(f + \delta g) \\
&\approx \frac{(x - \rho\tilde{G}(f))^2 - 2\delta\rho(x - \rho\tilde{G}(f))\big(\langle \sigma(\mathcal{K}(f^2)), \dot{g} \rangle + 2\langle \sigma'(\mathcal{K}(f^2)), \dot{f}\mathcal{K}(fg) \rangle\big)}{2\bar{\rho}^2\tilde{F}(f)\big(1 + \frac{2\delta}{\tilde{F}(f)}\langle (\sigma^2)'(\mathcal{K}(f^2)), K(fg) \rangle\big)} \\
&\quad + \frac{1}{2}\tilde{E}(f) + \delta\langle \dot{f}, \dot{g} \rangle \\
&\approx \frac{(x - \rho\tilde{G}(f))^2 - 2\delta\rho(x - \rho\tilde{G}(f))\big(\langle \sigma(\mathcal{K}(f^2)), \dot{g} \rangle + 2\langle \sigma'(\mathcal{K}(f^2)), \dot{f}\mathcal{K}(fg) \rangle\big)}{2\bar{\rho}^2\tilde{F}(f)} \\
&\quad - \frac{(x - \rho\tilde{G}(f))^2}{2\bar{\rho}^2\tilde{F}(f)}\frac{2\delta}{\tilde{F}(f)}\langle (\sigma^2)'(\mathcal{K}(f^2)), \mathcal{K}(fg) \rangle + \frac{1}{2}\tilde{E}(f) + \delta\langle \dot{f}, \dot{g} \rangle.
\end{aligned} \tag{5.10}$$

Now, as a consequence, for $f = f^x$ and every $g \in H_0^1[0, 1]$,

$$\begin{aligned}
0 = \partial_\delta(\mathcal{I}_x(f + \delta g))_{\delta=0} = &-\frac{2\rho(x - \rho\tilde{G}(f))\big(\langle \sigma(\mathcal{K}(f^2)), \dot{g} \rangle + 2\langle \sigma'(\mathcal{K}(f^2)), \dot{f}\mathcal{K}(fg) \rangle\big)}{2\bar{\rho}^2\tilde{F}(f)} \\
&- \frac{(x - \rho\tilde{G}(f))^2}{2\bar{\rho}^2\tilde{F}^2(f)}2\langle (\sigma^2)'(\mathcal{K}(f^2), \mathcal{K}(fg) \rangle + \langle \dot{f}, \dot{g} \rangle.
\end{aligned} \tag{5.11}$$

We have $f_0^x = 0$, for any x. We now test with $\dot{g} = \mathbb{1}_{[0,t]}$ for a fixed $t \in [0,1]$ and obtain

$$
\begin{aligned}
f_t^x = &\frac{\rho(x - \rho\tilde{G}(f^x))\big(\langle\sigma(\mathcal{K}((f^x)^2)), \mathbb{1}_{[0,t]}\rangle + 2\langle\sigma'(\mathcal{K}((f^x)^2)), \dot{f}^x\mathcal{K}(f^x\,\mathrm{id}_{\le t})\rangle\big)}{\bar{\rho}^2\tilde{F}(f^x)} \\
&+ \frac{(x-\rho\tilde{G}(f^x))^2}{2\bar{\rho}^2\tilde{F}^2(f^x)} 2\langle(\sigma^2)'(\mathcal{K}((f^x)^2)), \mathcal{K}(f^x\,\mathrm{id}_{\le t})\rangle,
\end{aligned}
\tag{5.12}
$$

where we write

$$
\mathrm{id}_{\le t}(s) = g(s) = \int_0^s \dot{g}(u)\,du = \int_0^s \mathbb{1}_{[0,t]}(u)\,du = \int_0^{s\wedge t} 1\,du = s\wedge t. \tag{5.13}
$$

Let us recall the ansatz in [2]. The authors of [2] choose for fixed x the optimizing function f^x for $\mathcal{I}_x$, i.e. $f^x = \mathrm{argmin}_{f\in H_0^1}\mathcal{I}_x(f)$. Therefore, the first order condition is $\mathcal{I}_x'(f^x) = 0$. The authors of [2] use the implicit function theorem to show that the minimizing configuration f^x is a smooth function in x (locally around $x = 0$). As $\mathcal{I}_x$ is a smooth function, too, this implies the smoothness of $x \mapsto \mathcal{I}_x(f^x) = I(x)$, at least in a neighbourhood of 0. Note that for (26) and Lemma 5.3 in [2], the embedding $\mathcal{K} : H_0^1 \to C$ works, because we have already established that $\mathcal{K}(U \circ f)$ is continuous (see Lemma 2.4).

In order to apply the implicit function theorem, the authors of [2] show that the ingredients of the rate function are Fréchet differentiable by computing their Gateaux derivative. This is more complicated in our case, because of the different integral transform we use. Therefore we *assume* that the rate function is locally smooth around 0 in Proposition 5.1, and, consequently, that Lemma 5.6 in [2] holds. After establishing that the implicit function theorem can be used, we can proceed as in [2] up to Theorem 5.12 there.

Next, we will imitate the computations in Theorem 5.12 of [2] in order to get the expansion of the minimizing configuration in our setting. In fact, if we just want to obtain the second order expansion of the rate function in our setting for Brownian motion squared, it suffices to find the first order expansion of f^x. Assuming the ansatz

$$
f_t^x = \alpha_t x + O(x^2), \tag{5.14}
$$

we get

$$
\begin{aligned}
f_t^x &= \alpha_t x + O(x^2),\\
\dot{f}_t^x &= \dot{\alpha}_t x + O(x^2),\\
\sigma(\mathcal{K}((f^x)^2)) &= \sigma_0 + O(x^2),\\
\sigma'(\mathcal{K}((f^x)^2)) &= \sigma_0' + O(x^2),\\
\tilde{F}(f^x) &= \sigma_0^2 + O(x^2),\\
\tilde{G}(f^x) &= \langle\sigma_0, \dot{\alpha}\rangle x + O(x^2).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\langle\sigma(\mathcal{K}((f^x)^2)), \mathbb{1}_{[0,t]}\rangle &= \sigma_0 t + O(x),\\
2\langle\sigma'(\mathcal{K}((f^x)^2)), \dot{f}^x\mathcal{K}(f^x\,\mathrm{id}_{\le t})\rangle &= O(x),\\
2\langle(\sigma^2)'(\mathcal{K}((f^x)^2)), \mathcal{K}(f^x\,\mathrm{id}_{\le t})\rangle &= O(x),\\
x - \rho\tilde{G}(f^x) &= (1 - \rho\sigma_0\alpha_1)x + O(x^2),\\
(x - \rho\tilde{G}(f^x))^2 &= O(x^2).
\end{aligned}
$$

We use the previous formulas in (5.12) to obtain

$$
\begin{aligned}
f_t^x &= \frac{\rho((1-\rho\sigma_0\alpha_1)x + O(x^2))(\sigma_0 t + O(x))}{\bar\rho^2(\sigma_0^2 + O(x^2))} + \frac{O(x^2)}{2\bar\rho^2(\sigma_0^4 + O(x^2))} O(x) \\
&= \frac{\rho(1-\rho\sigma_0\alpha_1)x\sigma_0 t}{\bar\rho^2\sigma_0^2} + O(x^2).
\end{aligned}
\tag{5.15}
$$

Comparing the coefficients, we get the same result as the authors of [2] for the first order expansion, i.e.

$$
\alpha_t = \frac{\rho(1-\rho\sigma_0\alpha_1)}{\bar\rho^2\sigma_0} t. \tag{5.16}
$$

Setting $t = 1$ and then computing α_1 leads to the formula

$$
\alpha_t = \frac{\rho}{\sigma_0} t. \tag{5.17}
$$

Note that the first order expansion of the minimizing configuration f^x is *exactly* the same as in [2]. The reason is that the expansions of the ingredients of (5.12) are relevant here, and these expansions coincide. For the second order expansion of the rate function, we need second order expansions of its ingredients. These are given in the following formulas, where id^2 denotes the quadratic function $s \mapsto s^2$:

$$
\begin{aligned}
\frac{1}{2}\tilde E(f^x) &= \frac{1}{2}\frac{\rho^2}{\sigma_0^2}x^2 + O(x^3), \\
(x - \rho\tilde G(f^x))^2 &= \bar\rho^4 x^2 + O(x^3) \\
\tilde F(f^x) &= \sigma_0^2 + (\sigma_0^2)'\langle \mathcal{K}(\alpha^2), 1\rangle x^2 + O(x^3) \\
&= \sigma_0^2 + (\sigma_0^2)'\frac{\rho^2}{\sigma_0^2}\langle \mathcal{K}(\mathrm{id}^2), 1\rangle x^2 + O(x^3).
\end{aligned}
$$

Finally, we get the Taylor expansion of the rate function by taking into account the reasoning above. We insert the expansion

$$
f_t^x = \alpha_t x + O(x^2) = \frac{\rho}{\sigma_0} t x + O(x^2) \tag{5.18}
$$

and the expansions above into Eq. (5.12) for the minimizing configuration. Then, we get

$$
\begin{aligned}
\mathcal{I}_x(f^x) &= \frac{(x - \rho\tilde G(f^x))^2}{2\bar\rho^2 \tilde F(f^x)} + \frac{1}{2}\tilde E(f^x) \\
&= \frac{\bar\rho^4 x^2 + O(x^3)}{2\bar\rho^2\left(\sigma_0^2 + (\sigma_0^2)'\frac{\rho^2}{\sigma_0^2}\langle \mathcal{K}(\mathrm{id}^2), 1\rangle x^2 + O(x^3)\right)} + \frac{1}{2}\frac{\rho^2}{\sigma_0^2}x^2 + O(x^3) \\
&= \frac{\bar\rho^2}{2\sigma_0^2}x^2 + O(x^3) + \frac{1}{2}\frac{\rho^2}{\sigma_0^2}x^2 + O(x^3) \\
&= \frac{1}{2\sigma_0^2}(\bar\rho^2 + \rho^2)x^2 + O(x^3) \\
&= \frac{1}{2\sigma_0^2}x^2 + O(x^3),
\end{aligned}
\tag{5.19}
$$

and hence the following expansion holds:

$$I(x) = \mathcal{I}_x(f^x) = \frac{1}{2\sigma_0^2}x^2 + O(x^3). \tag{5.20}$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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