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ASYMPTOTICS OF SOME GENERALIZED MATHIEU SERIES

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(Dedicated to Prof. Tibor Pogány on the occasion of his 65th birthday)

Abstract

We establish asymptotic estimates of Mathieu-type series defined by sequences with power-logarithmic or factorial behavior. By taking the Mellin transform, the problem is mapped to the singular behavior of certain Dirichlet series, which is then translated into asymptotics for the original series. In the case of power-logarithmic sequences, we obtain precise first order asymptotics. For factorial sequences, a natural boundary of the Mellin transform makes the problem more challenging, but a direct elementary estimate gives reasonably precise asymptotics. As a byproduct, we prove an expansion of the functional inverse of the gamma function at infinity.

1. Introduction and main results

Define, for $\mu \geq 0$, $r > 0$, and sequences $\mathbf{a} = (a_n)_{n\geq 0}$, $\mathbf{b} = (b_n)_{n\geq 0}$, the generalized Mathieu series

$$S_{\mathbf{a},\mathbf{b},\mu}(r) := \sum_{n=0}^{\infty} \frac{a_n}{(b_n + r^2)^{\mu+1}}. \tag{1.1}$$

The parametrization (i.e., r^2 and not r, $\mu + 1$ and not μ) is along the lines of [28]. Assumptions on the sequences $\mathbf{a}$ and $\mathbf{b}$ will be specified below. The study of such series began with 19th century work of Mathieu on elasticity of solid bodies, and has produced a considerable amount of literature, much of which focuses on integral representations and inequalities. See, e.g., [18], [28], [29], [30] for historical remarks, recent results and many references. As a special case of (1.1), define, for $\alpha, \beta, r > 0$, $\mu \geq 0$, with $\alpha - \beta(\mu + 1) < -1$ and $\gamma, \delta \in \mathbb{R}$,

$$S_{\alpha,\beta,\gamma,\delta,\mu}(r) := \sum_{n=2}^{\infty} \frac{n^{\alpha}(\log n)^{\gamma}}{(n^{\beta}(\log n)^{\delta} + r^2)^{\mu+1}}. \tag{1.2}$$

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Note that the summation in (1.2) starts at 2 to make the summand always well-defined. The series (1.2) is closely related to a paper by Paris [23] (see also [33]), but the presence of logarithmic factors is new. Another special case of (1.1) is the series

$$S^{!}_{\alpha,\beta,\mu}(r) := \sum_{n=0}^{\infty} \frac{(n!)^{\alpha}}{((n!)^{\beta} + r^2)^{\mu+1}}, \tag{1.3}$$

defined for $\alpha, \mu \geq 0$, $\beta, r > 0$ with $\alpha - \beta(\mu + 1) < 0$. We are not aware of any asymptotic estimates for (1.3) in the literature. See [30] for integral representations for some series of this kind. The subject of the present paper is the asymptotic behavior of the Mathieu-type series (1.2) and (1.3) for $r \uparrow \infty$. For the classical Mathieu series, the asymptotic expansion

$$\sum_{n=1}^{\infty} \frac{n}{(n^2 + r^2)^2} \sim \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{2r^{2k+2}}, \quad r \uparrow \infty,$$

was found by Elbert [7], whereas Pogány et al. [26] showed the expansion

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(n^2 + r^2)^2} \sim \sum_{k=1}^{\infty} \frac{(-1)^k G_{2k}}{4r^{2k+2}}, \quad r \uparrow \infty, \tag{1.4}$$

for its alternating counterpart; the B_n and G_n are Bernoulli resp. Genocchi numbers. (As noted by Paris [23], the factor $(-1)^k$ on the right hand side of (1.4) is missing in [26].) We refer to [23] for further references on asymptotics of Mathieu-type series, to which we add §19 and §20 of [13]. To formulate our results on (1.2), for

$$\delta(\alpha + 1)/\beta - \gamma \notin \mathbb{N} = \{1, 2, \ldots\}, \tag{1.5}$$

we define the constant

$$C_{\alpha,\beta,\gamma,\delta,\mu} := \frac{\left(\frac{1}{2}\beta\right)^{\delta(\alpha+1)/\beta-\gamma-1} \Gamma\left(\frac{\delta}{\beta}(\alpha + 1) - \gamma + 1\right)}{2\Gamma(\mu + 1)\Gamma\left(-\frac{\delta}{\beta}(\alpha + 1) + \gamma + 1\right)} \times \Gamma\left(-\frac{\alpha + 1}{\beta} + \mu + 1\right)\Gamma\left(\frac{\alpha + 1}{\beta}\right).$$

If, on the other hand, $m := \delta(\alpha + 1)/\beta - \gamma \in \mathbb{N}$ is a positive integer, then we define

$$C_{\alpha,\beta,\gamma,\delta,\mu} := \frac{\beta^{m-1} \Gamma\left(-\frac{\alpha+1}{\beta} + \mu + 1\right)\Gamma\left(\frac{\alpha+1}{\beta}\right)}{2^m \Gamma(\mu + 1)}. \tag{1.6}$$

THEOREM 1.1. *Let* $\alpha, \beta > 0$, $\mu \geq 0$, *with* $\alpha - \beta(\mu+1) < -1$, *and* $\gamma, \delta \in \mathbb{R}$. *Then we have*

$$S_{\alpha,\beta,\gamma,\delta,\mu}(r) \sim C_{\alpha,\beta,\gamma,\delta,\mu}\, r^{2(\alpha+1)/\beta-2(\mu+1)}(\log r)^{-\delta(\alpha+1)/\beta+\gamma}, \quad r \uparrow \infty. \tag{1.7}$$

Of course, the exponent of r is negative:

$$2(\alpha+1)/\beta - 2(\mu+1) = \frac{2}{\beta}\big(\alpha + 1 - \beta(\mu+1)\big) < 0.$$

Also, we note that for $\gamma = \delta = 0$ (no logarithmic factors), condition (1.5) is always satisfied, and the asymptotic equivalence (1.7) agrees with a special case of Theorem 3 in [23]. A bit more generally than Theorem 1.1, we have:

THEOREM 1.2. *Let the parameters* $\alpha, \beta, \gamma, \delta, \mu$ *be as in Theorem 1.1. Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be positive sequences that satisfy*

$$a_n \sim n^{\alpha}(\log n)^{\gamma}, \quad b_n \sim n^{\beta}(\log n)^{\delta}, \quad n \uparrow \infty.$$

Then $S_{\mathbf{a},\mathbf{b},\mu}(r)$ *has the asymptotic behavior stated in Theorem 1.1, i.e.*

$$S_{\mathbf{a},\mathbf{b},\mu}(r) \sim C_{\alpha,\beta,\gamma,\delta,\mu}\, r^{2(\alpha+1)/\beta-2(\mu+1)}(\log r)^{-\delta(\alpha+1)/\beta+\gamma}, \quad r \uparrow \infty.$$

This result includes sequences of the form $(\log n!)^{\alpha}$, see Corollary 5.1. Also, it clearly implies that shifts such as $a_n = (n+a)^{\alpha}(\log(n+b))^{\beta}$ are not visible in the first order asymptotics. Theorems 1.1 and 1.2 are proved in Section 2. The series (1.3) is more difficult to analyze than (1.2) by Mellin transform (see Appendix A for details), but it turns out that it is asymptotically dominated by only two summands. This yields the following result, which is proved in Section 3. We write $\{x\}$ for the fractional part of a real number x, and Γ^{-1} for the functional inverse of the gamma function.

THEOREM 1.3. *Let* $\alpha, \beta > 0$, $\mu \geq 0$, *with* $\alpha - \beta(\mu+1) < 0$, *and* $0 < d_1 < d_2 < 1$. *Then*

$$S^{!}_{\alpha,\beta,\mu}(r) = r^{-2(\mu+1-\alpha/\beta)} \exp\big(-m(r)\log\log r + O(\log\log\log r)\big) \tag{1.8}$$

as $r \to \infty$ *in the set*

$$\mathscr{R} := \big\{r > 0 : d_1 \leq \{\Gamma^{-1}(r^{2/\beta})\} \leq d_2\big\}, \tag{1.9}$$

where the function $m(\cdot)$ *is defined by*

$$m(r) := \min\big\{\alpha\{\Gamma^{-1}(r^{2/\beta})\},\ \big(\beta(\mu+1) - \alpha\big)\big(1 - \{\Gamma^{-1}(r^{2/\beta})\}\big)\big\} > 0.$$

Thus, under the constraint (1.9), the series $S^!_{\alpha,\beta,\mu}(r)$ decays like $r^{-2(\mu+1-\alpha/\beta)}$, accompanied by a power of $\log r$, where the exponent of the latter depends on r and fluctuates in a finite interval of negative numbers. The expression inside the fractional part $\{\cdot\}$ grows roughly logarithmically (see Appendix B):

$$\Gamma^{-1}(r^{2/\beta}) \sim \frac{2\log r}{\beta \log\log r}, \quad r \uparrow \infty.$$

Clearly, the proportion $\lim_{r\uparrow\infty} r^{-1}\,\mathrm{meas}(\mathscr{R}\cap[0,r])$ of "good" values of r can be made arbitrarily close to 1 by choosing d_1 and $1-d_2$ sufficiently small. Without the Diophantine assumption (1.9), a more complicated asymptotic expression for $S^!_{\alpha,\beta,\mu}(r)$ is obtained by combining (3.2), (3.10), and (3.11) below. From this expression it is easy to see that, for any $\varepsilon > 0$, we have

$$r^{2\alpha/\beta-2(\mu+1)-\varepsilon} \ll S^!_{\alpha,\beta,\mu}(r) \ll r^{2\alpha/\beta-2(\mu+1)+\varepsilon}, \quad r\uparrow\infty, \tag{1.10}$$

as well as logarithmic asymptotics:

$$\log S^!_{\alpha,\beta,\mu}(r) = -2(\mu+1-\alpha/\beta)\log r + O(\log\log r), \quad r\uparrow\infty. \tag{1.11}$$

The following result contains an asymptotic upper bound; like (1.10) and (1.11), it is valid without restricting r to (1.9):

THEOREM 1.4. *Let $\alpha, \beta > 0$, $\mu \ge 0$, with $\alpha - \beta(\mu+1) < 0$. Then*

$$S^!_{\alpha,\beta,\mu}(r) \le r^{-2(\mu+1-\alpha/\beta)} \exp\bigl(o(\log\log r)\bigr), \quad r\to\infty. \tag{1.12}$$

Theorem 1.4 is proved in Section 3, too. The proofs of Theorems 1.3 and 1.4 use an asymptotic expansion for the inverse of the gamma function, which is established in Appendix B. In Appendix A, we use a different method to show the following bound. It gives a weaker estimate, but also holds for $\alpha = 0$.

THEOREM 1.5. *Let $\alpha, \mu \ge 0$, $\beta > 0$ with $\alpha - \beta(\mu+1) < 0$. Then*

$$S^!_{\alpha,\beta,\mu}(r) = O\left(r^{-2(\mu+1-\alpha/\beta)} \frac{\log r}{\log\log r}\right), \quad r\uparrow\infty.$$

The difficulties concerning the factorial Mathieu-type series stem from the fact that the Mellin transform of $S^!_{\alpha,\beta,\mu}(\cdot)$ has a natural boundary in the form of a vertical line, whereas that of $S_{\alpha,\beta,\gamma,\delta,\mu}(\cdot)$ is more regular, featuring an analytic continuation with a single branch cut. See Section 2 and Appendix A for details. We therefore prove Theorem 1.3 by a direct estimate; see Section 3. It will be clear from the proof that the error term in (1.8) can be refined, if desired. Also, d_1 and $1 - d_2$ may depend on r, as long as they tend to zero sufficiently slowly.

2. Power-logarithmic sequences

Since (1.2) is a series with positive terms, the discrete Laplace method seems to be a natural asymptotic tool; see [22] for a good introduction and further references. However, while the summands of (1.2) do have a peak around $n \approx r^{2/\beta}$, the local expansion of the summand does not fully capture the asymptotics, and the central part of the sum yields an incorrect constant factor. A similar phenomenon has been observed in [6], [14] for *integrals* that are not amenable to the Laplace method. As in [15], [23], [24], [25], we instead use a Mellin transform approach. Since the Mellin transform seems not to be explicitly available in our case, we invoke results from [16] on the analytic continuation of a certain Dirichlet series. Before beginning with the Mellin transform analysis, we show that Theorem 1.2 follows from Theorem 1.1. This is the content of the following lemma.

LEMMA 2.1. *Let* **a** *and* **b** *be as in Theorem 1.2. Then*

$$S_{\mathbf{a},\mathbf{b},\mu}(r) = S_{\alpha,\beta,\gamma,\delta,\mu}(r)\big(1 + o(1)\big) + O\big(r^{-2(\mu+1)}(\log r)^{2\alpha+1}\big), \quad r \uparrow \infty.$$

PROOF. First consider the summation range $0 \le n \le \lfloor \log r \rfloor$ for the series defining $S_{\mathbf{a},\mathbf{b},\mu}(r)$, where $\lfloor \cdot \rfloor$ is the floor function. We have the estimate

$$\begin{aligned} b_n + r^2 &= O\big(n^\beta (\log n)^\delta\big) + r^2 \\ &= r^2\big(1 + O(\log r)^{2\beta}/r^2\big) \\ &= r^2\big(1 + o(1)\big), \quad r \uparrow \infty, \end{aligned}$$

and thus

$$(b_n + r^2)^{-(\mu+1)} = r^{-2(\mu+1)}\big(1 + o(1)\big), \quad 0 \le n \le \lfloor \log r \rfloor.$$

We obtain

$$\begin{aligned} \sum_{n=0}^{\lfloor \log r \rfloor} \frac{a_n}{(b_n + r^2)^{\mu+1}} &\lesssim \sum_{n=0}^{\lfloor \log r \rfloor} \frac{n^{2\alpha}}{(b_n + r^2)^{\mu+1}} \\ &\sim r^{-2(\mu+1)} \sum_{n=0}^{\lfloor \log r \rfloor} n^{2\alpha} \\ &= O\big(r^{-2(\mu+1)}(\log r)^{2\alpha+1}\big). \end{aligned} \tag{2.1}$$

Now consider the range $\lfloor \log r \rfloor < n < \infty$, which yields the main contribution. As for the denominator, we have

$$\begin{aligned} b_n + r^2 &= n^\beta (\log n)^\delta + o(n^\beta (\log n)^\delta) + r^2 \\ &= \big(n^\beta (\log n)^\delta + r^2\big)\left(1 + \frac{o(n^\beta (\log n)^\delta)}{n^\beta (\log n)^\delta + r^2}\right) \\ &= \big(n^\beta (\log n)^\delta + r^2\big)\big(1 + o(1)\big). \end{aligned} \tag{2.2}$$

Note that the first two $o(\cdot)$ are meant for $n \uparrow \infty$, but then the term $\frac{o(n^\beta (\log n)^\delta)}{n^\beta (\log n)^\delta + r^2}$ is also uniformly $o(1)$ as $r \uparrow \infty$, because $r \uparrow \infty$ implies $n \uparrow \infty$ in the range $\lfloor \log r \rfloor < n < \infty$. Similarly, we have

$$a_n = n^\alpha (\log n)^\gamma \big(1 + o(1)\big), \quad r \uparrow \infty. \tag{2.3}$$

Therefore,

$$\begin{aligned} \sum_{n > \lfloor \log r \rfloor} \frac{a_n}{(b_n + r^2)^{\mu+1}} &\sim \sum_{n > \lfloor \log r \rfloor} \frac{n^\alpha (\log n)^\gamma}{(n^\beta (\log n)^\delta + r^2)^{\mu+1}} \\ &= S_{\alpha,\beta,\gamma,\delta,\mu}(r) + O\big(r^{-2(\mu+1)} (\log r)^{2\alpha+1}\big). \end{aligned} \tag{2.4}$$

Here, the asymptotic equivalence follows from (2.2) and (2.3), and the equality follows from (2.1). The lemma now follows by combining (2.1) and (2.4).

We now begin the proof of Theorem 1.1. As in [16], define the Dirichlet series

$$\zeta_{\eta,\theta}(s) := \sum_{n=2}^{\infty} \frac{(\log n)^\eta}{(n(\log n)^\theta)^s}, \quad \mathrm{Re}(s) > 1, \tag{2.5}$$

with real parameters η, θ. We will see below that the Mellin transform of (1.2) can be expressed using $\zeta_{\eta,\theta}(s)$. The first two statements of the following lemma are taken from [16].

LEMMA 2.2. *The Dirichlet series $\zeta_{\eta,\theta}$ has an analytic continuation to the whole complex plane except $(-\infty, 1]$. As $s \to 1$ in this domain, we have the asymptotics*

$$\zeta_{\eta,\theta}(s) \sim \begin{cases} \dfrac{(-1)^{m-1}}{(m-1)!}(s-1)^{m-1} \log \dfrac{1}{s-1} & \text{if } m = \theta - \eta \in \mathbb{N}, \\ \Gamma(\eta - \theta + 1)(s-1)^{\theta-\eta-1} & \text{otherwise.} \end{cases}$$

The analytic continuation grows at most polynomially as $|\mathrm{Im}(s)| \uparrow \infty$ while $\mathrm{Re}(s)$ is bounded and positive.

PROOF. The statements about analytic continuation and asymptotics are proved in [16]. We revisit this proof in order to prove the polynomial estimate, which is needed later to apply Mellin inversion. By the Euler-Maclaurin summation formula, we have

$$\zeta_{\eta,\theta}(s) = \int_2^\infty f(x)\,dx + \frac{f(2)}{2} - \frac{B_2}{2} f'(2) - \int_2^\infty \frac{B_2(x - \lfloor x \rfloor)}{2} f''(x)\,dx, \tag{2.6}$$

where

$$f(x) := \frac{(\log x)^\eta}{(x(\log x)^\theta)^s}$$

and the Bs are Bernoulli numbers resp. polynomials. The last integral in (2.6) is a holomorphic function of s for $\mathrm{Re}(s) > -1$, and applying the Euler-Maclaurin formula of arbitrary order yields the full analytic continuation, after analyzing the first integral in (2.6). To prove our lemma, it remains to estimate the growth of the terms in (2.6). The dominating factor of $f''(x)$ satisfies

$$\left| \frac{\partial^2}{\partial x^2} x^{-s} \right| = |s(s+1)| x^{-\mathrm{Re}(s)-2},$$

from which it is very easy to see that the last integral in (2.6) grows at most polynomially under the stated conditions on s. In the first integral in (2.6), we substitute

$$x = \exp\big(z/(s-1)\big) \tag{2.7}$$

(as in [16]) and obtain

$$\begin{aligned} \int_2^\infty f(x)\,dx &= (s-1)^{\theta s - \eta - 1} \int_{(s-1)\log 2}^\infty z^{\eta - \theta s} e^{-z}\,dz \\ &= (s-1)^{\theta s - \eta - 1} \left(\Gamma(\eta - \theta s + 1) - \int_0^{(s-1)\log 2} z^{\eta - \theta s} e^{-z}\,dz \right). \end{aligned} \tag{2.8}$$

From Stirling's formula (see [3, p. 224]), we have

$$\Gamma(t) = O\big(e^{-\pi |\mathrm{Im}(t)|/2} |t|^{\mathrm{Re}(t) - 1/2}\big), \quad |t| \uparrow \infty,$$

uniformly w.r.t. $\mathrm{Re}(t)$, as long as $\mathrm{Re}(t)$ stays bounded. Using this and

$$|(s-1)^{-\theta s}| = \exp\big(-\theta\ \mathrm{Re}(s) \log|s-1| + \theta\ \mathrm{Im}(s) \arg(s-1)\big),$$

we see that $|(s-1)^{\theta s - \eta - 1} \Gamma(\eta - \theta s + 1)|$ can be bounded by a polynomial in s. Finally, we have

$$\int_0^{(s-1)\log 2} z^{\eta - \theta s} e^{-z}\,dz = \big((s-1)\log 2\big)^{\eta - \theta s + 1} \int_0^1 u^{\eta - \theta s} e^{(1-s)u \log 2}\,du,$$

from which it is immediate that the term

$$(s-1)^{\theta s-\eta-1}\int_0^{(s-1)\log 2} z^{\eta-\theta s}e^{-z}\,dz$$

in (2.8) admits a polynomial estimate. This completes the proof of Lemma 2.2.

For any sufficiently regular function f, we denote the Mellin transform by f^*,

$$f^*(s) := \int_0^\infty f(r)r^{s-1}\,dr. \tag{2.9}$$

We now compute the Mellin transform of the function $S_{\alpha,\beta,\gamma,\delta,\mu}(r)$, writing $a_n = n^\alpha(\log n)^\gamma$ and $b_n = n^\beta(\log n)^\delta$.

$$\begin{aligned} S^*_{\alpha,\beta,\gamma,\delta,\mu}(s) &= \int_0^\infty S_{\alpha,\beta,\gamma,\delta,\mu}(r)r^{s-1}\,dr \\ &= \sum_{n=2}^\infty a_n \int_0^\infty \frac{r^{s-1}}{(b_n+r^2)^{\mu+1}}\,dr \\ &= \frac{1}{2}\sum_{n=2}^\infty a_n b_n^{s/2-(\mu+1)} \int_0^\infty \frac{u^{s/2-1}}{(1+u)^{\mu+1}}\,du \\ &= \frac{D(s)\Gamma(\mu+1-s/2)\Gamma(s/2)}{2\Gamma(\mu+1)}, \end{aligned} \tag{2.10}$$

where we substituted $u = r^2/b_n$, and

$$\begin{aligned} D(s) &:= \sum_{n=2}^\infty n^\alpha(\log n)^\gamma \left(n^\beta(\log n)^\delta\right)^{s/2-(\mu+1)} \\ &= \sum_{n=2}^\infty (\log n)^{\delta s/2+\gamma-\delta(\mu+1)} n^{\beta s/2+\alpha-\beta(\mu+1)}. \end{aligned}$$

The Dirichlet series D can be expressed in terms of $\zeta_{\eta,\theta}$ from (2.5):

$$D(s) = \zeta_{\eta,\theta}\left(1+\tfrac{1}{2}\beta(\hat{s}-s)\right)\Big|_{\eta=\gamma-\alpha\delta/\beta,\ \theta=\delta/\beta} \tag{2.11}$$

with

$$\hat{s} := -2(\alpha+1)/\beta + 2(\mu+1) < 2\mu+2. \tag{2.12}$$

Formula (2.10) is valid for $\mathrm{Re}(s) \in (0,\hat{s})$. The function $\Gamma(\mu+1-s/2)$ has poles at $2\mu+2, 2\mu+4, \ldots$, and those of $\Gamma(s/2)$ are $0, -2, -4, \ldots$ All those poles are outside the strip $\{s \in \mathbb{C} : \mathrm{Re}(s) \in (0,\hat{s})\}$. The singular expansion of (2.10) at the dominating singularity $\hat{s}$ can be translated, via the Mellin inversion formula, into the asymptotic behavior of $S_{\alpha,\beta,\gamma,\delta,\mu}(r)$. See [10] for a standard introduction to this method; in fact, our generalized Mathieu series (1.1) is a harmonic sum in the terminology of [10]. By Mellin inversion, we have

$$\begin{aligned} S_{\alpha,\beta,\gamma,\delta,\mu}(r) &= \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} r^{-s} S^{*}_{\alpha,\beta,\gamma,\delta,\mu}(s)\,ds \\ &= \frac{1}{2\Gamma(\mu+1)}\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} r^{-s} D(s)\Gamma(\mu+1-s/2)\Gamma(s/2)\,ds, \end{aligned} \tag{2.13}$$

where $\kappa \in (0,\hat{s})$. Note that integrability of $S^{*}_{\alpha,\beta,\gamma,\delta,\mu}(s)$ follows from the polynomial estimate in Lemma 2.2 and Stirling's formula, as the latter implies

$$\Gamma(t) = O\big(\exp\big(-\big(\tfrac{1}{2}\pi - \varepsilon\big)|\operatorname{Im}(t)|\big)\big) \tag{2.14}$$

for bounded $\operatorname{Re}(t)$. Suppose first that

$$\theta - \eta = \delta(\alpha+1)/\beta - \gamma \notin \mathbb{N}.$$

Then, from Lemma 2.2 and (2.11), we have

$$\begin{aligned} D(s) &\sim \Gamma\big(\delta(\alpha+1)/\beta - \gamma + 1\big)\big(\tfrac{1}{2}\beta(\hat{s}-s)\big)^{\delta(\alpha+1)/\beta-\gamma-1} \\ &= c_1(\hat{s}-s)^{-c_2}, \quad s \to \hat{s}, \end{aligned} \tag{2.15}$$

with

$$\begin{aligned} c_1 &:= \Gamma\big(\delta(\alpha+1)/\beta - \gamma + 1\big)\big(\tfrac{1}{2}\beta\big)^{\delta(\alpha+1)/\beta-\gamma-1}, \\ c_2 &:= -\delta(\alpha+1)/\beta + \gamma + 1. \end{aligned} \tag{2.16}$$

Combining (2.10) and (2.15) yields

$$S^{*}_{\alpha,\beta,\gamma,\delta,\mu}(s) \sim c_3(\hat{s}-s)^{-c_2}, \quad s \to \hat{s}, \tag{2.17}$$

where

$$c_3 := \frac{c_1\Gamma(\mu+1-\hat{s}/2)\Gamma(\hat{s}/2)}{2\Gamma(\mu+1)}. \tag{2.18}$$

By a standard procedure, we can now extract asymptotics of the Mathieu-type series $S_{\alpha,\beta,\gamma,\delta,\mu}(r)$ from (2.13). The integration contour in (2.13) is pushed to the right, which is allowed by Lemma 2.2. The real part of the new contour is

$$\kappa_r := \hat{s} + \frac{\log\log r}{\log r},$$

where the singularity at $s = \hat{s}$ is avoided by a small C-shaped notch. In (2.19) below, this notch is the integration contour. The contour is then transformed to a Hankel contour $\mathscr{H}$ by the substitution $s = \hat{s} - w/\log r$. The contour $\mathscr{H}$ starts at $-\infty$, circles the origin counterclockwise and continues back to $-\infty$. Using (2.17), we thus obtain

$$\begin{aligned} S_{\alpha,\beta,\gamma,\delta,\mu}(r) &= \frac{1}{2\pi i}\int_{\kappa_r - i\infty}^{\kappa_r + i\infty} r^{-s} S^*_{\alpha,\beta,\gamma,\delta}(s)\, ds \\ &\sim \frac{c_3}{2\pi i}\int r^{-s}(\hat{s} - s)^{-c_2}\, ds \\ &\sim c_3 r^{-\hat{s}}(\log r)^{c_2 - 1}\frac{1}{2\pi i}\int_{\mathscr{H}} e^w w^{-c_2}\, dw \\ &= \frac{c_3}{\Gamma(c_2)} r^{-\hat{s}}(\log r)^{c_2-1}. \end{aligned} \tag{2.19}$$

See [10], [11], [14], [16] for details of this asymptotic transfer. This completes the proof of (1.7) in the case $\delta(\alpha+1)/\beta - \gamma \notin \mathbb{N}$. Recall the definitions of the constants $\hat{s}$, c_2, c_3 in (2.12), (2.16), and (2.18).

Now suppose that

$$m := \theta - \eta = \delta(\alpha+1)/\beta - \gamma \in \mathbb{N}. \tag{2.20}$$

We need to show that (1.7) still holds, but with the constant factor now given by (1.6). By Lemma 2.2 and (2.11), we have

$$\begin{aligned} D(s) &\sim \frac{(-1)^{m-1}}{(m-1)!}\left(\tfrac{1}{2}\beta\right)^{m-1}(\hat{s} - s)^{m-1}\log\frac{1}{\hat{s} - s} \\ &= c_4(\hat{s} - s)^{m-1}\log\frac{1}{\hat{s} - s}, \quad s \to \hat{s}, \end{aligned} \tag{2.21}$$

where

$$c_4 := \frac{(-1)^{m-1}}{(m-1)!}\left(\tfrac{1}{2}\beta\right)^{m-1}.$$

Define

$$c_5 := \frac{c_4\Gamma(\mu + 1 - \hat{s}/2)\Gamma(\hat{s}/2)}{2\Gamma(\mu+1)}. \tag{2.22}$$

Then, using (2.10) and (2.21),

$$S^*_{\alpha,\beta,\gamma,\delta,\mu}(s) \sim c_5(\hat{s} - s)^{m-1}\log\frac{1}{\hat{s} - s}, \quad s \to \hat{s}.$$

We proceed similarly as above (see again [10], [11], [16]) and find

$$\begin{aligned} S_{\alpha,\beta,\gamma,\delta,\mu}(r) &\sim c_5 r^{-\hat{s}} (\log r)^{-m} \frac{1}{2\pi i} \int_{\mathscr{H}} e^w w^{m-1} \left(\log \frac{e^r}{w} \right) dw \\ &\sim c_5 r^{-\hat{s}} (\log r)^{-m} \frac{1}{2\pi i} \int_{\mathscr{H}} e^w w^{m-1} (-\log w)\, dw \\ &= c_5 \left(\frac{1}{\Gamma} \right)' (1-m) \times r^{-\hat{s}} (\log r)^{-m}. \end{aligned} \tag{2.23}$$

As for the second $\sim$, note that

$$\frac{1}{2\pi i} \int_{\mathscr{H}} e^w w^{m-1}\, dw = \frac{1}{\Gamma(1-m)} = 0.$$

From the well-known residues of Γ and ψ at the non-positive integers (see, e.g., [32, p. 241]), we obtain

$$\left(\frac{1}{\Gamma} \right)' (1-m) = -\left(\frac{\psi}{\Gamma} \right) (1-m) = (-1)^{m-1} (m-1)!, \quad m \in \mathbb{N}.$$

Formula (1.7) is established, and Theorem 1.1 is proved. As for the constants in (2.23), recall the definitions in (2.12), (2.20), and (2.22). As mentioned above, Theorem 1.2 follows from Theorem 1.1 and Lemma 2.1.

3. Factorial sequences

This section contains the proofs of Theorems 1.3 and 1.4. Our estimates can be viewed as a somewhat degenerate instance of the Laplace method, where the central part of the sum consists of just two summands. We denote by A_n the summands of (1.3):

$$S^!_{\alpha,\beta,\mu}(r) = \sum_{n=0}^{\infty} A_n, \quad A_n := \frac{(n!)^\alpha}{\left((n!)^\beta + r^2\right)^{\mu+1}}.$$

Define $n_0 = n_0(r)$ by $n_0(r) := \lfloor \Gamma^{-1}(r^{2/\beta}) \rfloor - 1$, i.e.,

$$(n_0!)^\beta \le r^2 < (n_0+1)!^\beta. \tag{3.1}$$

We first show that $S^!_{\alpha,\beta,\mu}(r)$ is dominated by A_{n_0} and A_{n_0+1}. For brevity, we omit writing the dependence of A_n and n_0 on r.

LEMMA 3.1. *Let $\alpha, \beta > 0$, $\mu \ge 0$, with $\alpha - \beta(\mu+1) < 0$. Then*

$$S^!_{\alpha,\beta,\mu}(r) \sim A_{n_0} + A_{n_0+1}, \quad r \uparrow \infty. \tag{3.2}$$

PROOF. For $k \geq 2$, we estimate, using (3.1),

$$\begin{aligned} A_{n_0+k}/A_{n_0+1} &= \big((n_0+2)\dots(n_0+k)\big)^{\alpha}\left(\frac{(n_0+1)!^{\beta}+r^2}{(n_0+k)!^{\beta}+r^2}\right)^{\mu+1} \\ &\leq \big((n_0+2)\dots(n_0+k)\big)^{\alpha}\left(\frac{2(n_0+1)!^{\beta}}{(n_0+k)!^{\beta}}\right)^{\mu+1} \\ &= 2^{\mu+1}\big((n_0+2)\dots(n_0+k)\big)^{\alpha-\beta(\mu+1)}. \end{aligned}$$

Therefore,

$$\begin{aligned} A_{n_0+1}^{-1}\sum_{k=2}^{\infty} A_{n_0+k} &\leq 2^{\mu+1}\sum_{k=2}^{\infty}\big((n_0+2)\dots(n_0+k)\big)^{\alpha-\beta(\mu+1)} \\ &\leq 2^{\mu+1}\sum_{k=2}^{\infty} n_0^{(k-1)(\alpha-\beta(\mu+1))} \\ &\sim 2^{\mu+1} n_0^{\alpha-\beta(\mu+1)} = o(1). \end{aligned}$$

This shows that

$$\sum_{k=2}^{\infty} A_{n_0+k} \ll A_{n_0+1}.$$

For the initial segment $\sum_{k=1}^{n_0-1} A_{n_0-k}$ of the series, we use the following estimate for $k \geq 1$:

$$\begin{aligned} A_{n_0-k}/A_{n_0} &= \big(n_0(n_0-1)\dots(n_0-k+1)\big)^{-\alpha}\left(\frac{(n_0!)^{\beta}+r^2}{(n_0-k)!^{\beta}+r^2}\right)^{\mu+1} \\ &\leq \big(n_0(n_0-1)\dots(n_0-k+1)\big)^{-\alpha}\left(\frac{2r^2}{r^2}\right)^{\mu+1} \\ &= 2^{\mu+1}\big(n_0(n_0-1)\dots(n_0-k+1)\big)^{-\alpha} =: 2^{\mu+1}B_k. \end{aligned}$$

Pick an integer q with $q > 1/\alpha$. Then

$$\sum_{k=1}^{n_0} B_k = \sum_{k=1}^{q} B_k + \sum_{k=q+1}^{n_0} B_k. \tag{3.3}$$

Now $\sum_{k=1}^{q} B_k$ has a fixed number of summands, all $o(1)$, and is thus $o(1)$ as $r \uparrow \infty$. In the second sum, we pull out the factor $n_0^{-\alpha}$, estimate q of the remaining factors by $n_0 - k + 1$, and the other factors by 1:

$$\sum_{k=q+1}^{n_0} B_k \le n_0^{-\alpha} \sum_{k=q+1}^{n_0} (n_0 - k + 1)^{-\alpha q} \le n_0^{-\alpha} \sum_{k=1}^{n_0} k^{-\alpha q} = O(n_0^{-\alpha}).$$

The last equality follows from $q > 1/\alpha$. We conclude that (3.3) is $o(1)$, and thus

$$\sum_{k=1}^{n_0-1} A_{n_0-k} \ll A_{n_0}, \tag{3.4}$$

which finishes the proof.

We now evaluate A_{n_0} and A_{n_0+1} asymptotically. For this, we employ an asymptotic expansion for the functional inverse of the gamma function, which is established in Appendix B. We use the following notation:

$$\begin{aligned} x &:= r^{2/\beta}, \quad v = x/\sqrt{2\pi}, \\ g &:= \Gamma^{-1}(x), \\ n_0 &= \lfloor g \rfloor - 1 = g - \{g\} - 1, \\ w &= W\big((\log v)/e\big), \\ u &= (\log v)/w. \end{aligned} \tag{3.5}$$

Recall that $\{\cdot\}$ denotes the fractional part, and note that, according to (B.5), we have

$$u \log u - u = \log v. \tag{3.6}$$

PROOF OF THEOREM 1.3. By Stirling's formula and (3.5), we have

$$\begin{aligned} \log n_0! &= n_0 \log n_0 - n_0 + \tfrac{1}{2} \log n_0 + O(1) \\ &= (g - \{g\} - 1)\big(\log g + O(1/g)\big) - g + \tfrac{1}{2} \log g + O(1) \\ &= g \log g - g - \big(\tfrac{1}{2} + \{g\}\big) \log g + O(1). \end{aligned} \tag{3.7}$$

From Lemma B.1, we have the expansion

$$\Gamma^{-1}(x) = u + \frac{1}{2} + O\left(\frac{1}{uw}\right) \tag{3.8}$$

of the inverse gamma function. From (3.7) and (3.8), we obtain

$$\begin{aligned} \log n_0! &= u \log u - u + \tfrac{1}{2} \log u - \big(\tfrac{1}{2} + \{g\}\big) \log u + O(1) \\ &= u \log u - u - \{g\} \log u + O(1). \end{aligned}$$

Together with (3.6), this yields

$$n_0! = v\exp\bigl(-\{g\}\log u + O(1)\bigr) = r^{2/\beta}\exp\bigl(-\{g\}\log u + O(1)\bigr). \quad (3.9)$$

Equation (3.9) is crucial for determining the asymptotics of the right hand side of (3.2). Since

$$\exp\bigl(-\beta\{g\}\log u + O(1)\bigr) + 1 = e^{O(1)},$$

we can use (3.9) to evaluate the summand A_{n_0} as

$$\begin{aligned} A_{n_0} &= \frac{(n_0!)^\alpha}{\bigl((n_0!)^\beta + r^2\bigr)^{\mu+1}} \\ &= r^{2\alpha/\beta - 2(\mu+1)}\exp\bigl(-\alpha\{g\}\log u + O(1)\bigr) \\ &= r^{2\alpha/\beta - 2(\mu+1)}\exp\bigl(-\alpha\{g\}\log\log r + O(\log\log\log r)\bigr), \end{aligned} \quad (3.10)$$

where the last line follows from (B.2). By definition, we have

$$n_0 = g + O(1) = u + O(1),$$

and thus

$$n_0 \sim \frac{\log x}{\log\log x} \sim \frac{2\log r}{\beta\log\log r}, \quad r \uparrow \infty.$$

As for the summand A_{n_0+1}, we thus have (writing $\log^3 = \log\log\log$)

$$\begin{aligned} A_{n_0+1} &= \frac{(n_0+1)!^\alpha}{\bigl((n_0+1)!^\beta + r^2\bigr)^{\mu+1}} \\ &= \frac{(\log r)^\alpha e^{O(\log^3 r)}(n_0!)^\alpha}{\bigl((\log r/\log\log r)^\beta e^{O(1)}(n_0!)^\beta + r^2\bigr)^{\mu+1}} \\ &= (\log r)^\alpha r^{2\alpha/\beta}\exp\bigl(-\alpha\{g\}\log u + O(\log^3 r)\bigr) \\ &\quad \times \left(\left(\frac{\log r}{\log\log r}\right)^\beta r^2\exp\bigl(-\beta\{g\}\log u + O(1)\bigr) + r^2\right)^{-(\mu+1)} \\ &= r^{2\alpha/\beta - 2(\mu+1)}\exp\bigl(\alpha(1-\{g\})\log\log r + O(\log^3 r)\bigr) \\ &\quad \times \left(\left(\frac{\log r}{\log\log r}\right)^\beta \exp\bigl(-\beta\{g\}\log u + O(1)\bigr) + 1\right)^{-(\mu+1)}. \end{aligned} \quad (3.11)$$

This holds as $r \to \infty$, without any constraints on r. If $\{g\} \le d_2 < 1$, as assumed in Theorem 1.3, then the term inside the big parentheses in (3.11) tends to infinity; note that $\log u \sim \log\log r$ by (B.2). We then have

$$A_{n_0+1} = r^{2\alpha/\beta-2(\mu+1)} \exp\big((\alpha - \beta(\mu+1))(1-\{g\})\log\log r \\ + O(\log\log\log r)\big), \quad \{g\} \le d_2 < 1. \quad (3.12)$$

Define

$$\mathscr{R}_0 := \big\{r \in \mathscr{R} : -\alpha\{g\} \ge (\alpha - \beta(\mu+1))(1-\{g\})\big\}$$

and

$$\mathscr{R}_1 := \mathscr{R} \setminus \mathscr{R}_0.$$

Then, by Lemma 3.1, (3.10), and (3.12), we obtain

$$S^{!}_{\alpha,\beta,\mu}(r) \sim A_{n_0}, \quad r \in \mathscr{R}_0, \quad (3.13)$$

$$S^{!}_{\alpha,\beta,\mu}(r) \sim A_{n_0+1}, \quad r \in \mathscr{R}_1. \quad (3.14)$$

Theorem 1.3 now follows from this, (3.10), and (3.12). Note that the assumption $0 < d_1 \le \{g\} \le d_2 < 1$ of Theorem 1.3 ensures that the term $(\ldots)\log\log r$ in (3.10) and (3.12) asymptotically dominates the error term. Moreover, the asymptotic equivalence in (3.13) and (3.14) can be replaced by an equality, because the error factor $1 + o(1)$ is absorbed into the $O(\log^3 r)$ in the exponent.

Proof of Theorem 1.4. By (3.10), we have

$$A_{n_0} \le r^{2\alpha/\beta-2(\mu+1)} \exp\big(O(\log\log\log r)\big),$$

and so, by Lemma 3.1, it suffices to estimate A_{n_0+1}. Fix an arbitrary $\varepsilon > 0$. Recall the notation introduced around (3.5). If r is such that $\alpha(1-\{g\}) \le \varepsilon$, then we simply estimate the term in big parentheses in (3.11) by 1, and obtain

$$A_{n_0+1} \le r^{2\alpha/\beta-2(\mu+1)} \exp\big(\varepsilon \log\log r + O(\log\log\log r)\big).$$

If, on the other hand, $\{g\} < 1 - \varepsilon/\alpha$, then (3.12) holds, which implies

$$A_{n_0+1} \le r^{2\alpha/\beta-2(\mu+1)} \exp\big(O(\log\log\log r)\big),$$

because the quantity in front of $\log\log r$ in (3.12) is negative. We have thus shown that, for any $\varepsilon > 0$,

$$S^{!}_{\alpha,\beta,\mu}(r) \le r^{-2(\mu+1-\alpha/\beta)} \exp\big(\varepsilon \log\log r + O(\log\log\log r)\big). \quad (3.15)$$

From this, Theorem 1.4 easily follows. Indeed, were it not true, then there would be $\varepsilon' > 0$ and a sequence $r_n \uparrow \infty$ such that

$$\log\bigl(r_n^{2(\mu+1-\alpha/\beta)} S^!_{\alpha,\beta,\mu}(r_n)\bigr) \geq 2\varepsilon' \log\log r_n,$$

contradicting (3.15).

4. Power sequences: full expansion in a special case

In [28], an integral representation of the generalized Mathieu series

$$S_\mu(r) := \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^{\mu+1}}, \quad \mu > \tfrac{3}{2}, r > 0,$$

was derived. In our notation (1.2), this series is

$$S_\mu(r) = 2S_{1,2,0,0,\mu}(r) + \frac{2}{(1+r^2)^{\mu+1}}.$$

We use said integral representation and Watson's lemma to find a full expansion of $S_\mu(r)$ as $r \to \infty$. This expansion is not new (see Theorem 1 in [23]), and so we do not give full details. Still, our approach provides an independent check for (a special case of) Theorem 1 in [23], and it might be useful for other Mathieu-type series admitting a representation as a Laplace transform. The integral representation in Theorem 4 of [28] is

$$S_\mu(r) = c_\mu \int_0^\infty e^{-rt} t^{\mu+1/2} g_\mu(t)\, dt, \tag{4.1}$$

where

$$c_\mu := \frac{\sqrt{\pi}}{2^{\mu-1/2}\Gamma(\mu+1)},$$

and g_μ is the Schlömilch series

$$g_\mu(t) := \sum_{n=1}^{\infty} n^{1/2-\mu} J_{\mu+1/2}(nt).$$

For $\mathrm{Re}(s) > \frac{3}{2} - \mu$, the Mellin transform of g_μ is

$$\begin{aligned} g_\mu^*(s) &= \sum_{n=1}^{\infty} n^{1/2-\mu-s} 2^{s-1} \frac{\Gamma(\mu/2+1/4+s/2)}{\Gamma(\mu/2+5/4-s/2)} \\ &= \frac{2^{s-1}\zeta(s+\mu-1/2)\Gamma(\mu/2+1/4+s/2)}{\Gamma(\mu/2+5/4-s/2)}. \end{aligned}$$

The factor $\zeta(s+\mu-1/2)$ has a pole at $\check{s} := \frac{3}{2}-\mu$, and $\Gamma(\mu/2+1/4+s/2)$ has poles at $s_k := -2k-\mu-\frac{1}{2}$, $k \in \mathbb{N}_0$. By using Mellin inversion and collecting residues, we find that the expansion of $g_\mu(t)$ as $t \downarrow 0$ is

$$\begin{aligned} g_\mu(t) & \\ &\sim \frac{2^{\check{s}-1}\Gamma(\mu/2+1/4+\check{s}/2)}{\Gamma(\mu/2+5/4-\check{s}/2)} t^{-\check{s}} + \sum_{k=0}^{\infty} \frac{(-1)^k 2^{s_k} \zeta(s_k+\mu-1/2)}{k!\,\Gamma(\mu/2+5/4-\check{s}/2)} t^{-s_k} \\ &= \frac{2^{1/2-\mu}}{\Gamma(\mu+1/2)} t^{\mu-3/2} + \sum_{k=0}^{\infty} \frac{(-1)^k 2^{-2k-\mu-1/2} \zeta(-2k-1)}{k!\,\Gamma(k+\mu+3/2)} t^{2k+\mu+1/2}. \end{aligned}$$

Now we multiply this expansion by $t^{\mu+1/2}$ and use Watson's lemma [20, p. 71] in (4.1). In the notation of [20, p. 71] the parameters μ and λ are $\frac{1}{2}$ and our μ, respectively. Simplifying the resulting expansion using Legendre's duplication formula,

$$\Gamma(2k+2\mu+2) = \pi^{-1/2} 2^{2k+2\mu+1} \Gamma(k+\mu+1)\Gamma(k+\mu+3/2),$$

yields the expansion

$$S_\mu(r) \sim \frac{1}{\mu} r^{-2\mu} + \sum_{k=0}^{\infty} \frac{2(-1)^k \zeta(-2k-1)\Gamma(k+\mu+1)}{\Gamma(\mu+1)k!} r^{-2k-2\mu-2} \tag{4.2}$$

as $r \to \infty$. Recall that the values of the zeta function at negative odd integers can be represented by Bernoulli numbers:

$$\zeta(-2k-1) = -\frac{B_{2k+2}}{2k+2}, \quad k \in \mathbb{N}_0.$$

The expansion (4.2) indeed agrees with Theorem 1 in [23], and the first term agrees with our Theorem 1.1 (with $\alpha = 1$, $\beta = 2$, $\gamma = \delta = 0$). The divergent series in (4.2) looks very similar to formula (3.2) in [27], but there the argument of $\zeta(\cdot)$ in the summation is eventually positive instead of negative.

Finally, we give an amusing non-rigorous derivation of the asymptotic series on the right-hand side of (4.2), by using the binomial theorem, the "formula"

$\zeta(-2k-1) = \sum_{n=1}^{\infty} n^{2k+1}$, and interchanging summation:

$$\begin{aligned} S_\mu(r) &= 2r^{-2(\mu+1)} \sum_{n=1}^{\infty} n \left(1 + \frac{n^2}{r^2}\right)^{-(\mu+1)} \\ &= 2r^{-2(\mu+1)} \sum_{n=1}^{\infty} n \sum_{k=0}^{\infty} (-1)^k \binom{k+\mu}{k} \left(\frac{n}{r}\right)^{2k} \\ \text{“="} &\ 2r^{-2(\mu+1)} \sum_{k=0}^{\infty} (-1)^k \binom{k+\mu}{k} r^{-2k} \zeta(-2k-1) \\ &= \sum_{k=0}^{\infty} \frac{2(-1)^k \zeta(-2k-1)\Gamma(k+\mu+1)}{\Gamma(\mu+1)k!} r^{-2k-2\mu-2}. \end{aligned}$$

Note that the dominating term, of order $r^{-2\mu}$, is not found by this heuristic.

5. Miscellaneous

We now apply Theorem 1.2 (on power-logarithmic sequences) to an example taken from [30]. There, integral representations for some Mathieu-type series were deduced, and we state asymptotics for one of them.

Corollary 5.1. *Let* $\alpha, \beta > 0$, $\mu \geq 0$, *with* $\alpha - \beta(\mu+1) < -1$. *Then*

$$\sum_{n=2}^{\infty} \frac{(\log n!)^\alpha}{\left((\log n!)^\beta + r^2\right)^{\mu+1}} \sim C\, r^{2(\alpha+1)/\beta - 2(\mu+1)} / \log r, \quad r \uparrow \infty,$$

with

$$C = \frac{\Gamma\left(-\frac{\alpha+1}{\beta} + \mu + 1\right)\Gamma\left(\frac{\alpha+1}{\beta}\right)}{2\Gamma(\mu+1)}.$$

Proof. By Stirling's formula, we have $(\log n!)^\alpha \sim (n \log n)^\alpha$. The statement thus follows from Theorem 1.2, with $\gamma = \alpha$, $\delta = \beta$, and $m = \delta(\alpha + 1)/\beta - \gamma = 1 \in \mathbb{N}$.

A natural generalization of our main results on power-logarithmic sequences (Theorems 1.1 and 1.2) would be to replace log by arbitrary slowly varying functions: $a_n = n^\alpha \ell_1(n)$, $b_n = n^\beta \ell_2(n)$. Then the Dirichlet series (2.11) becomes

$$D(s) = \sum_{n=2}^{\infty} a_n b_n^{s/2-(\mu+1)} = \sum_{n=2}^{\infty} n^{\beta s/2 + \alpha - \beta(\mu+1)} \ell_1(n) \ell_2(n)^{s/2-(\mu+1)}.$$

The dominating singularity is still $\hat{s}$ defined in (2.12), as follows from [1, Proposition 1.3.6], but it seems not easy to determine the singular behavior of D at $\hat{s}$ for generic ℓ_1, ℓ_2. Still, for specific examples such as $(\log\log n)^\gamma$ or $\exp(\sqrt{\log n})$, this should be doable. Note that our second step, i.e. the asymptotic transfer from the Mellin transform to the original function, works for slowly varying functions under mild conditions; see [11].

Finally, we note that including a geometrically decaying factor x^n in the series (1.1) leads to a Mathieu-type *power series*. According to the following proposition, its asymptotics can be found in an elementary way, for rather general sequences **a**, **b**. We refer to [31] for integral representations and further references on certain Mathieu-type power series.

PROPOSITION 5.2. *Let $x \in \mathbb{C}$ with $|x| < 1$, $a_n \in \mathbb{C}$, $b_n \geq 0$, and $\mu \geq 0$. If $\sum_{n=0}^\infty a_n x^n$ is absolutely convergent and $b_n \uparrow \infty$, then*

$$\sum_{n=0}^\infty \frac{a_n}{(b_n + r^2)^{\mu+1}} x^n = r^{-2(\mu+1)} \sum_{n=0}^\infty a_n x^n + o(r^{-2(\mu+1)}), \quad r \uparrow \infty.$$

PROOF. We have

$$\left| \sum_{n\colon b_n > r} \frac{a_n}{(b_n + r^2)^{\mu+1}} x^n \right| \leq \sum_{n\colon b_n > r} \frac{|a_n|}{(b_n + r^2)^{\mu+1}} |x|^n \leq \sum_{n\colon b_n > r} \frac{|a_n|}{r^{2(\mu+1)}} |x|^n.$$

As $\sum_{n\colon b_n > r} |a_n||x|^n$ tends to zero, this is $o(r^{-2(\mu+1)})$. For the dominating part of the series, we find

$$\begin{aligned}\sum_{n\colon b_n \leq r} \frac{a_n}{(b_n + r^2)^{\mu+1}} x^n &= r^{-2(\mu+1)} \sum_{n\colon b_n \leq r} \frac{a_n}{(b_n/r^2 + 1)^{\mu+1}} x^n \\ &= r^{-2(\mu+1)} \Bigg(\sum_{n\colon b_n \leq r} a_n x^n + O(1/r) \Bigg) \\ &= r^{-2(\mu+1)} \Bigg(\sum_{n=0}^\infty a_n x^n + o(1) \Bigg).\end{aligned}$$

In the last equality, we used that $\sum_{n\colon b_n > r} a_n x^n = o(1)$, because $b_n \uparrow \infty$.

The next example immediately follows from this proposition.

EXAMPLE 5.3. In the spirit of (1.2), we can consider the Mathieu-type power series

$$\sum_{n=2}^\infty \frac{n^\alpha (\log n)^\gamma x^n}{\big(n^\beta (\log n)^\delta + r^2\big)^{\mu+1}} \sim r^{-2(\mu+1)} \sum_{n=2}^\infty n^\alpha (\log n)^\gamma x^n, \quad r \uparrow \infty,$$

where $|x| < 1$, $\mu \geq 0$, and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. The power series on the right hand side is a generalized polylogarithm [8].

In Proposition 5.2, we assumed $|x| < 1$. Our main results (Theorems 1.1–1.3) are concerned with the case $x = 1$, for some special sequences $\mathbf{a}, \mathbf{b}$. An alternating factor $(-1)^n$, on the other hand, induces cancellations that are difficult to handle, and may require the availability of an explicit Mellin transform, as in [23]. The special case $a_n = n$, $b_n = n^2$, $|x| \leq 1$ was recently settled in [15], where a full expansion was obtained. First order asymptotics for this example follow from Proposition 5.2.

EXAMPLE 5.4. Let $|x| < 1$ and $\mu \geq 0$. Then

$$\sum_{n=1}^{\infty} \frac{2nx^n}{(n^2 + r^2)^{\mu+1}} \sim r^{-2(\mu+1)} \frac{2x}{(1-x)^2}, \quad r \uparrow \infty.$$

In [31], several integral representations for this Mathieu-type power series were established.

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Appendix A. Factorial sequences: the associated Dirichlet series

In the Mellin transform of (1.3), the following Dirichlet series occurs:

$$\tau(s) := \sum_{n=0}^{\infty} (n!)^{-s}, \quad \mathrm{Re}(s) > 0. \tag{A.1}$$

As we will see in Lemma A.1, this function does not have an analytic continuation beyond the right half-plane. It is well known that the presence of a natural boundary is a severe obstacle when doing asymptotic transfers; see [9] and the references cited there. Therefore, our proof of Theorem 1.3 in Section 3 did *not* use Mellin transform asymptotics. Still, some analytic properties of (A.1) seem to be interesting in their own right, and will be discussed in the present appendix. We note that the arguments at the beginning of the proof of Lemma A.1 (analyticity, natural boundary) suffice to identify the *location* of the singularity of the Mellin transform of $S^{!}_{\alpha,\beta,\mu}(\cdot)$ (see (A.6) below), and thus yield the logarithmic asymptotics in (1.11) with the weaker error term $o(\log r)$. Moreover, in this appendix we will prove Theorem 1.5; see (A.11) below.

LEMMA A.1. *The function τ is analytic in the right half-plane, and the imaginary axis $i\mathbb{R}$ is a natural boundary. At the origin, we have the asymptotics*

$$\tau(s) \sim \frac{1}{s\log(1/s)}, \quad s \downarrow 0,\ s \in \mathbb{R}. \tag{A.2}$$

PROOF. Analyticity follows from a standard result on Dirichlet series, see e.g. [17, p. 5]. As $n/\log n! = o(1)$, the lacunary series $\sum_{n=0}^{\infty} z^{\log n!}$ has the unit circle as a natural boundary. We refer to the introduction of [5] for details. This implies that $i\mathbb{R}$ is the natural boundary of

$$\tau(s) = \sum_{n=0}^{\infty} z^{\log n!}\Big|_{z=e^{-s}}.$$

It remains to prove (A.2). We begin by showing that the Dirichlet series

$$\sum_{n=2}^{\infty} (\log n!)^{-s}, \quad \mathrm{Re}(s) > 1, \tag{A.3}$$

has an analytic continuation to $\mathrm{Re}(s) > 0$, with branch cut $(0, 1]$. The main idea is that replacing $\log n!$ by $n \log n$ leads to the series from Lemma 2.2, and the properties of (A.3) that we need are the same as those stated there. We just do not care about continuation further left than $\mathrm{Re}(s) > 0$, because we do not require it. The continuation of (A.3) is based on writing

$$\sum_{n=3}^{\infty} (\log n!)^{-s} = \sum_{n=3}^{\infty} \big((\log n!)^{-s} - (n\log n - n)^{-s}\big) + \sum_{n=3}^{\infty} (n \log n - n)^{-s}. \tag{A.4}$$

By Stirling's formula, we have

$$\begin{aligned}(\log n!)^{-s} &= (n\log n - n)^{-s}\big(1 + O(1/n)\big)^{-s} \\ &= (n\log n - n)^{-s}\big(1 + O(1/n)\big),\end{aligned}$$

locally uniformly w.r.t. s in the right half-plane. From this it follows that

$$\sum_{n=3}^{\infty} \big((\log n!)^{-s} - (n\log n - n)^{-s}\big)$$

defines an analytic function of s for $\mathrm{Re}(s) > 0$. Moreover, the last series in (A.4) has an analytic continuation to a slit plane. This is proved by the same argument as in Lemma 2.2, using the Euler-Maclaurin formula and (2.7).

Moreover, the polynomial estimate from that lemma easily extends to the continuation of (A.3) for $\mathrm{Re}(s) > 0$, $s \notin (0, 1]$. After these preparations we can prove (A.2) by Mellin transform asymptotics. Recall the definition of the Mellin transform in (2.9). We compute, using the definition of $\zeta_{\eta,\theta}$ in (2.5) and its asymptotics from Lemma 2.2,

$$\begin{aligned}\Big(s\sum_{n=2}^{\infty}(n!)^{-s}\Big)^{*}(t) &= \sum_{n=2}^{\infty}\int_0^{\infty}(n!)^{-s}s^t\,ds\\ &= \Gamma(t+1)\sum_{n=2}^{\infty}(\log n!)^{-t-1}\\ &\sim \Gamma(t+1)\sum_{n=2}^{\infty}(n\log n)^{-t-1}\\ &= \Gamma(t+1)\,\zeta_{0,1}(t+1)\\ &\sim \log\frac{1}{t},\qquad t\to 0.\end{aligned}\tag{A.5}$$

We have shown above that the Dirichlet series in (A.5) has an integrable analytic continuation to $\mathrm{Re}(t) > -1$, $t \notin (-1, 0]$, and so Lemma 2 in [16] is applicable (asymptotic transfer, with $a = 0$, $b = 1$ in the notation of [16]). We conclude

$$s\sum_{n=2}^{\infty}(n!)^{-s} \sim \Big(\log\frac{1}{s}\Big)^{-1},\qquad s\downarrow 0,\ s\in\mathbb{R},$$

and hence

$$\tau(s) \sim \frac{1}{s\log(1/s)},\qquad s\downarrow 0,\ s\in\mathbb{R}.$$

The lemma is proved.

Analogously to (2.10), we find the Mellin transform of (1.3):

$$\begin{aligned}S^{!\,*}_{\alpha,\beta,\mu}(s) &= \int_0^{\infty} S^{!}_{\alpha,\beta,\mu}(r)r^{s-1}\,dr\\ &= \frac{\Gamma(\mu+1-s/2)\Gamma(s/2)}{2\Gamma(\mu+1)}\sum_{n=0}^{\infty}(n!)^{\alpha}(n!)^{\beta(s/2-(\mu+1))}\\ &= \frac{\Gamma(\mu+1-s/2)\Gamma(s/2)}{2\Gamma(\mu+1)}\tau\big(\tfrac{1}{2}\beta(\tilde{s}-s)\big),\end{aligned}\tag{A.6}$$

where

$$\tilde{s} := 2(\mu+1-\alpha/\beta) > 0.$$

By the Mellin inversion formula, we have

$$S^{!}_{\alpha,\beta,\mu}(r) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} r^{-s} S^{!*}_{\alpha,\beta,\mu}(s)\,ds, \quad 0 < \sigma < \tilde{s}. \tag{A.7}$$

Note that integrability of the Mellin transform $S^{!*}_{\alpha,\beta,\mu}$ follows from (2.14) and the obvious estimate

$$|\tau(s)| \le \tau\bigl(\mathrm{Re}(s)\bigr), \quad \mathrm{Re}(s) > 0. \tag{A.8}$$

By (A.6) and Lemma A.1, the integrand in (A.7) has a singularity at $s = \tilde{s}$, with singular expansion

$$\log\bigl(r^{-s} S^{!*}_{\alpha,\beta,\mu}(s)\bigr) = -s\log r + \log\frac{1}{\tilde{s}-s} - \log\log\frac{1}{\tilde{s}-s} + O(1). \tag{A.9}$$

It is well known that this kind of singularity (polynomial growth of the transform) is *not* amenable to the saddle point method, as regards precise asymptotics. Still, a *saddle point bound* can be readily found. For an introduction to saddle point bounds and the saddle point method, we recommend Chapter VIII in [12]. Retaining only the first two terms on the right-hand side of (A.9) and taking the derivative w.r.t. s yields the saddle point equation

$$\log r = \frac{1}{\tilde{s}-s},$$

with solution

$$\sigma_r := \tilde{s} - \frac{1}{\log r}. \tag{A.10}$$

We take this as real part of the integration path in (A.7) and obtain, using (A.8),

$$\begin{aligned} |S^{!}_{\alpha,\beta,\mu}(r)| &\le r^{-\sigma_r}\tau\bigl(\tfrac{1}{2}\beta(\tilde{s}-\sigma_r)\bigr)\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\Gamma(\mu+1-s/2)\Gamma(s/2)|}{2\Gamma(\mu+1)}\bigg|_{s=\sigma_r+iy} dy \\ &= O\bigl(r^{-\sigma_r}\tau\bigl(\tfrac{1}{2}\beta(\tilde{s}-\sigma_r)\bigr)\bigr). \end{aligned}$$

The fact that the integral is $O(1)$ as $r \uparrow \infty$ follows from (2.14). From (A.10), we have

$$r^{-\sigma_r} = e r^{-\tilde{s}}.$$

Lemma A.1 implies

$$\tau\bigl(\tfrac{1}{2}\beta(\tilde{s}-\sigma_r)\bigr) = \tau\left(\frac{\beta}{2\log r}\right) \sim \frac{2\log r}{\beta\log\log r},$$

which results in the saddle point bound

$$S^{!}_{\alpha,\beta,\mu}(r) = O\left(r^{-2(\mu+1-\alpha/\beta)}\frac{\log r}{\log\log r}\right), \quad r \uparrow \infty, \tag{A.11}$$

which proves Theorem 1.5. Note that this bound is weaker than (1.12), but also holds for $\alpha = 0$. This case is excluded in Theorems 1.3 and 1.4, because our proof of (3.4) requires $\alpha > 0$.

Appendix B. Asymptotic inversion of the gamma function

The proofs of Theorems 1.3 and 1.4 rely on an expansion of Γ^{-1} at infinity. We use the following notation, in line with p. 417f. of [2], where the required expansion is stated without proof. We write $W(\cdot)$ for the Lambert W function, which satisfies $W(z)\exp(W(z)) = z$. The goal is to approximately solve the equation $\Gamma(y) = x$ for $y = y(x)$ as $x \uparrow \infty$. Put

$$v := x/\sqrt{2\pi}, \quad w := W\big((\log v)/e\big), \quad u := (\log v)/w. \tag{B.1}$$

It is well known (see [19, (4.13.10)], [21], or [4]) that

$$W(z) = \log z + O(\log\log z), \quad z \uparrow \infty,$$

and so

$$w = \log\log x + O(\log\log\log x)$$

and

$$u \sim \frac{\log x}{\log\log x}. \tag{B.2}$$

By Stirling's formula, we have

$$\log\Gamma(z) = z\log z - z - \frac{1}{2}\log z + \log\sqrt{2\pi} + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} + O\left(\frac{1}{z^7}\right), \quad z \uparrow \infty. \tag{B.3}$$

For fixed A, define $y_A = y_A(x)$ by

$$y_A = u + \frac{1}{2} + \frac{1}{24u\log u} - \frac{7}{2880u^3\log u} - \frac{1}{576u^3(\log u)^2} - \frac{1}{1152u^3(\log u)^3} + \frac{A}{u^5\log u}. \tag{B.4}$$

Now compose (B.3) and (B.4) – preferably using a computer algebra system – to obtain

$$\log\Gamma(y_A) = u\log u - u + \log\sqrt{2\pi} + \left(A - \frac{31}{40320}\right)\frac{1}{u^5} + O\left(\frac{1}{u^5\log u}\right).$$

It is easy to check, using the defining property of Lambert W, that

$$u\log u - u + \log\sqrt{2\pi} = \log x. \tag{B.5}$$

In fact, this is equation (63) in [2]. Therefore,

$$\Gamma(y_A(x)) = x\exp\left(\left(A - \frac{31}{40320}\right)\frac{1}{u^5} + O\left(\frac{1}{u^5\log u}\right)\right). \tag{B.6}$$

Fix two numbers A_1, A_2 satisfying

$$A_1 < \frac{31}{40320} < A_2.$$

By (B.6), we have

$$y_{A_1}(x) \le y(x) \le y_{A_2}(x), \quad x \text{ large},$$

which proves the following result:

LEMMA B.1. *As $x \uparrow \infty$, the functional inverse of the gamma function has the expansion*

$$\Gamma^{-1}(x) = u + \frac{1}{2} + \frac{1}{24u\log u} - \frac{7}{2880u^3\log u} - \frac{1}{576u^3(\log u)^2} - \frac{1}{1152u^3(\log u)^3} + O\left(\frac{1}{u^5\log u}\right),$$

where $u = u(x)$ is defined in (B.1).

This agrees with equation (70) in [2]; note that $\log u = 1 + w$.

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