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Asymptotic expansion of Mathieu power series and trigonometric Mathieu series [☆]

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ABSTRACT

We consider a generalized Mathieu series where the summands of the classical Mathieu series are multiplied by powers of a complex number. The Mellin transform of this series can be expressed by the polylogarithm or the Hurwitz zeta function. From this we derive a full asymptotic expansion, generalizing known expansions for alternating Mathieu series. Another asymptotic regime for trigonometric Mathieu series is also considered, to first order, by applying known results on the asymptotic behavior of trigonometric series.

1. Mathieu power series and the polylogarithm function

In [28], integral representations for the series

$$F_\mu(r,z) := \sum_{n=1}^{\infty} \frac{2nz^n}{(n^2+r^2)^{\mu+1}}, \tag{1.1}$$

where $r > 0$, $\mu > 0$, and $z \in \mathbb{C}$ with $|z| < 1$, have been established, in terms of the Bessel function of the first kind. The asymptotic behavior of (1.1) as $r \uparrow \infty$ has not been investigated so far, except for special values of z. For $z = 1$, this series becomes the generalized Mathieu series studied in [12,17], with positive summands and growth order $r^{-2\mu}$ as $r \uparrow \infty$. (Those papers also contain many further references on Mathieu series and their significance.) For any other number z on the complex unit circle, the oscillating character of the summands causes cancellations that make the sum decay faster, of order $r^{-2\mu-2}$. So far, this was only

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known for $z = -1$ (alternating Mathieu series); see [20] for $\mu = 1$ and [17,32] for general μ. For $|z| < 1$, the leading term is of order $r^{-2\mu-2}$, too. As in [12,17,18,32], we use a Mellin transform approach to expand (1.1) for $r \uparrow \infty$. The Mellin transform of $F_\mu(r, z)$ can be expressed by the polylogarithm function

$$\mathrm{Li}_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha}, \quad |z| < 1,\ \alpha \in \mathbb{C}. \tag{1.2}$$

It is well known that the polylogarithm has an analytic continuation, e.g. by the Lindelöf integral[1]

$$\mathrm{Li}_\alpha(z) = -\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{(-z)^u}{u^\alpha} \frac{\pi}{\sin \pi u} du. \tag{1.3}$$

From this representation and the definition (1.2), it is easy to see that $\mathrm{Li}_\alpha(z)$ is an entire function of α for any $z \in \mathbb{C} \setminus [1, \infty)$. See [7,10] and p. 409 in [8] for details. In our main result, we need an estimate for $\mathrm{Li}_\alpha(z)$ for fixed z and large $\mathrm{Im}(\alpha)$. This will be established in Section 3, using the well-known representation of $\mathrm{Li}_\alpha(z)$ by the Hurwitz zeta function. Moreover, we will require the following complex extension of Abel's convergence theorem (Stolz 1875); see p. 406 in [15].

Theorem 1.1. *Let $\sum_{n=0}^\infty a_n z^n$ be a complex power series with radius of convergence 1. If this series converges at a point z_0 of the unit circle, then*

$$\lim_{\substack{z \to z_0 \\ z \in \Delta}} \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z_0^n,$$

where Δ is any triangle in the unit disk with z_0 as one of its vertices.

This theorem implies consistency of (1.2) with the analytic continuation of the polylogarithm, i.e. that $\sum_{n=1}^\infty n^{-\alpha} e^{inx} = \mathrm{Li}_\alpha(e^{ix})$ for $\mathrm{Re}(\alpha) > 1$ and $x \in (0, 2\pi)$, which will be used below. (This actually holds for $\mathrm{Re}(\alpha) > 0$, see p. 401 in [15] for convergence of $\sum n^{-\alpha} e^{inx}$, but we do not need this fact.)

2. Main result

Theorem 2.1. *Let $\mu > 0$ and $1 \neq z \in \mathbb{C}$ with $|z| \le 1$. As $r \uparrow \infty$, we have the asymptotic expansion*

$$F_\mu(r, z) \sim \sum_{k=0}^{\infty} r^{-2k-2\mu-2} \frac{2(-1)^k\, \Gamma(k + \mu + 1)}{k!\, \Gamma(\mu + 1)} \mathrm{Li}_{-2k-1}(z) \tag{2.1}$$

$$= 2 \sum_{k=0}^{\infty} r^{-2k-2\mu-2} (-1)^k \binom{k+\mu}{k} \mathrm{Li}_{-2k-1}(z), \tag{2.2}$$

where $\mathrm{Li}_{-2k-1}(z)$ is defined by (1.2) or (1.3).

Proof. The Mellin transform of $F_\mu(r, z)$ with respect to r is (cf. [12,17])

$$(\mathcal{M}F_\mu)(u) = \int_0^\infty r^{u-1} F_\mu(r, z) dr$$

[1] The name Li does not originate from *Lindelöf integral*, but rather from *logarithmic integral* (see [8]).

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} 2nz^n \int_0^{\infty} \frac{r^{u-1}}{(n^2+r^2)^{\mu+1}} dr \\
&= \frac{\Gamma(\mu+1-u/2)\Gamma(u/2)}{\Gamma(\mu+1)} \sum_{n=1}^{\infty} n^{u-2\mu-1} z^n \\
&= \frac{\Gamma(\mu+1-u/2)\Gamma(u/2)}{2\Gamma(\mu+1)} \mathrm{Li}_{2\mu+1-u}(z), \quad 0 < \mathrm{Re}(u) < 2\mu. \qquad (2.3)
\end{aligned}
$$

For $|z| < 1$, the last equality is clear from (1.2). For $z \neq 1$ with $|z| = 1$, we have

$$
\mathrm{Li}_{2\mu+1-u}(z) = \lim_{\substack{w \to z \\ w \in \Delta}} \sum_{n=1}^{\infty} n^{u-2\mu-1} w^n = \sum_{n=1}^{\infty} n^{u-2\mu-1} z^n
$$

for $0 < \mathrm{Re}(u) < 2\mu$, where the first equality is clear from analytic continuation and the second one from Theorem 1.1, with Δ as in that theorem.

As mentioned above, $\mathrm{Li}_{2\mu+1-u}(z)$ is an entire function of u. Thus, $\mathcal{M}F_\mu$ is a meromorphic function. For the desired asymptotic expansion ($r \uparrow \infty$), the poles in the right half-plane are the relevant ones. They are those of the factor $\Gamma(\mu+1-u/2)$, located at $2k+2\mu+2$ for $k \in \mathbb{N}_0$. We can now use the standard procedure of expanding a function whose Mellin transform is meromorphic (see, e.g., [9] or Section 4.1.1 in [19]). To justify Mellin inversion, we have to argue that $(\mathcal{M}F_\mu)(\sigma+it)$ is an integrable function of t for fixed σ, where

$$
u = \sigma + it, \quad \sigma, t \in \mathbb{R}.
$$

By Stirling's formula (see p. 224 in [5]), we have

$$
|\Gamma(w)| \sim \sqrt{2\pi}\, |\mathrm{Im}(w)|^{\mathrm{Re}(w)-1/2} \exp(-\tfrac{1}{2}\pi|\mathrm{Im}(w)|) \qquad (2.4)
$$

for $\mathrm{Re}(w)$ bounded and $|\mathrm{Im}(w)| \uparrow \infty$. This implies

$$
\Gamma(\mu+1-u/2)\Gamma(u/2) = \mathrm{O}\Big(\exp\big(-\tfrac{1}{2}\pi|t|\big)\, |t|^{\mu}\Big)
$$

for σ bounded and $|t| \uparrow \infty$. Using this and Proposition 3.1 below, we see from (2.3) that $(\mathcal{M}F_\mu)(u)$ decays exponentially along vertical lines,

$$
(\mathcal{M}F_\mu)(u) = \mathrm{O}\big(\exp(-\varepsilon|t|)\big),
$$

and is thus integrable for $\sigma = \mathrm{Re}(u) > 0$, as long as the vertical contour avoids the poles of $\Gamma(\mu+1-u/2)$. The Mellin inversion formula then says that

$$
F_\mu(r,z) = \frac{1}{2\pi i} \int_{u_0-i\infty}^{u_0+i\infty} r^{-u} (\mathcal{M}F_\mu)(u) du, \quad 0 < u_0 < 2\mu+2.
$$

The above locally uniform estimate for $\mathcal{M}F_\mu$ allows to push the contour to the right, and the residue theorem yields the expansion

$$
F_\mu(r,z) \sim -\sum_{k=0}^{\infty} \mathrm{res}_{u=2k+2\mu+2}\; r^{-u} (\mathcal{M}F_\mu)(u)
$$

$$= -\sum_{k=0}^{\infty} r^{-2k-2\mu-2} \operatorname{res}_{u=2k+2\mu+2} (\mathcal{M}F_\mu)(u)$$
$$= -\sum_{k=0}^{\infty} r^{-2k-2\mu-2} \frac{\Gamma(k+\mu+1)}{\Gamma(\mu+1)} \operatorname{Li}_{-2k-1}(z) \operatorname{res}_{u=2k+2\mu+2} \Gamma(\mu+1-u/2).$$

It easily follows from $\operatorname{res}_{s=-k}\Gamma(s) = (-1)^k/k!$, $k \in \mathbb{N}_0$, that

$$\operatorname{res}_{u=2k+2\mu+2} \Gamma(\mu+1-u/2) = \frac{2(-1)^{k+1}}{k!}, \quad k \in \mathbb{N}_0.$$

This implies the result. $\square$

In Section 4 we will comment on the relation between Theorem 2.1 and some results from the literature on Mathieu series.

3. Estimates for the polylogarithm and the Hurwitz zeta function

The Hurwitz zeta function is defined by

$$\zeta(s,q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^s}, \quad \operatorname{Re}(s) > 1, \operatorname{Re}(q) > 0, \tag{3.1}$$

and can be extended to $s \in \mathbb{C}\setminus\{1\}$ by analytic continuation. It is related to the polylogarithm by Jonquière's formula [14]

$$\operatorname{Li}_\alpha(z) = \frac{\Gamma(1-\alpha)}{(2\pi)^{1-\alpha}} \left(i^{1-\alpha}\zeta\Big(1-\alpha, \frac{1}{2} + \frac{\log(-z)}{2\pi i}\Big) + i^{\alpha-1}\zeta\Big(1-\alpha, \frac{1}{2} - \frac{\log(-z)}{2\pi i}\Big)\right), \tag{3.2}$$

valid for $z \in \mathbb{C} \setminus [0,\infty)$ and $\alpha \in \mathbb{C} \setminus \mathbb{N}_0$. To ensure integrability of the Mellin transform in the proof of Theorem 2.1, we need a growth estimate for $\operatorname{Li}_\alpha(z)$, or equivalently for $\zeta(s,q)$, for large $\operatorname{Im}(\alpha)$ resp. $\operatorname{Im}(s)$. A related estimate for the polylogarithm occurs in Lemma 2 of [6].

Proposition 3.1. *Let* $\operatorname{Re}(q) > 0$ *and* $\theta_1, \theta_2 > 0$. *Then there is an* $\varepsilon > 0$ *such that*

$$\zeta(s,q) = \mathrm{O}\Big(\exp\big((\tfrac{1}{2}\pi - \varepsilon)|\operatorname{Im}(s)|\big)\Big) \tag{3.3}$$

as $|\operatorname{Im}(s)| \uparrow \infty$, *uniformly with respect to* $-\theta_1 \le \operatorname{Re}(s) \le \theta_2$. *Similarly, for* $z \in \mathbb{C} \setminus [1,\infty)$ *and* $\theta_1, \theta_2 > 0$ *there is* $\varepsilon > 0$ *such that*

$$\operatorname{Li}_\alpha(z) = \mathrm{O}\Big(\exp\big((\tfrac{1}{2}\pi - \varepsilon)|\operatorname{Im}(\alpha)|\big)\Big) \tag{3.4}$$

for $-\theta_1 \le \operatorname{Re}(\alpha) \le \theta_2$.

The proposition will be proved at the end of this section. For $q \in (0,1]$, (3.3) can be strengthened to a *polynomial* bound by §13.5 in [31], used in the Mathieu series context in [32]. This easily yields a polynomial bound instead of (3.4) under the additional assumption that $|z| = 1$ (see the proof of Proposition 3.1 below). We also mention that, for $q \in (0,1]$ and $\operatorname{Re}(s) \in [\frac{1}{2}, 1]$, rather tight polynomial bounds for $\zeta(s,q)$ have been obtained [11,16,22,30]. However, for non-real q, it is not obvious how to adapt §13.5 in [31], which uses Hurwitz' Fourier series for $\zeta(s,q)$. We thus use a different approach to prove (3.3), similar to p. 271 in [21].

Lemma 3.2. *Let* $\operatorname{Re}(q) > 0$ *and* $\theta_1, \theta_2 > 0$. *Then*

$$\zeta(s,q) = \sum_{n=0}^{\lfloor|\operatorname{Im}(s)|\rfloor} (n+q)^{-s} \quad + \mathrm{O}\big(|\operatorname{Im}(s)|^{\theta_1+1}\big) \tag{3.5}$$

as $|\operatorname{Im}(s)| \uparrow \infty$, *uniformly with respect to* $-\theta_1 \le \operatorname{Re}(s) \le \theta_2$.

Proof. Define $k := \lfloor\theta_1/2\rfloor + 1$ and $f(x) := (x+q)^{-s}$. By the Euler–Maclaurin formula, we have

$$\sum_{n=a}^{b}(n+q)^{-s} = \int_a^b f(x)dx + \frac{1}{2}\big(f(a)+f(b)\big) + \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!}\big(f^{(2j-1)}(b) - f^{(2j-1)}(a)\big) + \int_a^b \frac{B_{2k+1}(x-\lfloor x\rfloor)}{(2k+1)!} f^{(2k+1)}(x)dx$$

for $\operatorname{Re}(s) > 1$, where $a \le b \in \mathbb{N}_0$. As usual, the B_k denote Bernoulli numbers resp. polynomials. Since

$$f^{(m)}(x) = (-1)^m (s)_m (x+q)^{-s-m}, \quad m \ge 0,$$

where $(s)_m = s(s+1)\cdots(s+m-1)$ is the Pochhammer symbol, we obtain

$$\begin{aligned}\zeta(s,q) &= \sum_{n=0}^{a-1}(n+q)^{-s} + \frac{(a+q)^{1-s}}{s-1} + \frac{1}{2}(a+q)^{-s} \\ &+ \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!}(s)_{2j-1}(a+q)^{-s-2j+1} \\ &- \frac{(s)_{2k+1}}{(2k+1)!}\int_a^\infty \frac{B_{2k+1}(x-\lfloor x\rfloor)}{(x+q)^{s+2k+1}}dx\end{aligned} \tag{3.6}$$

for $\operatorname{Re}(s) > 1$. By analytic continuation, this equality extends to $s \ne 1$ with $\operatorname{Re}(s) > -2k$. We now put

$$a := \lfloor|\operatorname{Im}(s)|\rfloor$$

and consider $a \uparrow \infty$ with the specified restriction $-\theta_1 \le \operatorname{Re}(s) \le \theta_2$. Note that, by definition of k,

$$-2k = -2\lfloor\theta_1/2\rfloor - 2 < -2(\theta_1/2 - 1) - 2 = -\theta_1,$$

and so (3.6) holds in a sufficiently large half-plane. Since

$$\arg(a+q) = \arctan\frac{\operatorname{Im}(q)}{a+\operatorname{Re}(q)} = \mathrm{O}(1/a),$$

we have for any $m \in \{-2k-1, \dots, 0, 1\}$

$$|(a+q)^{-s+m}| = |a+q|^{m-\operatorname{Re}(s)} e^{\operatorname{Im}(s)\arg(a+q)} = \mathrm{O}(a^{m-\operatorname{Re}(s)}). \tag{3.7}$$

As $\operatorname{Re}(s)$ is bounded, we have $(s)_m = \mathrm{O}(a^m)$. We use this, (3.7), and the boundedness of $B_{2k+1}(x-\lfloor x\rfloor)$ in (3.6) to get

$$\zeta(s,q) = \sum_{n=0}^{a-1}(n+q)^{-s} + \mathrm{O}(a^{1-\mathrm{Re}(s)})$$
$$= \sum_{n=0}^{a-1}(n+q)^{-s} + \mathrm{O}(a^{\theta_1+1}). \quad \square$$

We now prove Proposition 3.1, using a crude estimate for the sum in (3.5), which suffices for our purposes.

Proof of Proposition 3.1. The sum in (3.5) can be estimated by

$$\sum_{0\le n\le |\mathrm{Im}(s)|} |(n+q)^{-s}| = \sum_{0\le n\le |\mathrm{Im}(s)|} |n+q|^{-\mathrm{Re}(s)} e^{\mathrm{Im}(s)\arg(n+q)}.$$

The factor

$$|n+q|^{-\mathrm{Re}(s)} \le \max\Big\{(|\mathrm{Im}(s)|+|q|)^{-\mathrm{Re}(s)},\ |q|^{-\mathrm{Re}(s)}\Big\}$$

is of at most polynomial growth. Note that any polynomial factor does not affect the validity of (3.3), by possibly shrinking ε. Since $\mathrm{Re}(q) > 0$, we have

$$|\arg(n+q)| \le |\arg(q)| < \tfrac{1}{2}\pi, \quad n \ge 0.$$

This proves (3.3). It remains to prove (3.4). For $z \in [0,1)$, we obviously have $|\mathrm{Li}_\alpha(z)| \le \zeta(\alpha)$. The Riemann zeta function $\zeta(\cdot) = \zeta(\cdot, 1)$ is of at most polynomial growth in any right half-plane (see p. 95 in [26]), and so we may from now on assume $z \in \mathbb{C} \setminus [0,\infty)$ and apply (3.2), with

$$\mathrm{Re}(q_\pm) = \mathrm{Re}\Big(\frac{1}{2} \pm \frac{\log(-z)}{2\pi i}\Big) > 0.$$

The factor $(2\pi)^{\alpha-1}$ in (3.2) is clearly $\mathrm{O}(1)$, and

$$|i^{\pm(1-\alpha)}| = \exp\big(\pm\tfrac{1}{2}\pi\mathrm{Im}(\alpha)\big).$$

By Stirling's formula (see (2.4)), we have

$$\Gamma(1-\alpha) = \mathrm{O}\Big(|\mathrm{Im}(\alpha)|^{1/2-\mathrm{Re}(\alpha)}\exp(-\tfrac{1}{2}\pi|\mathrm{Im}(\alpha)|)\Big).$$

Therefore, the exponential estimates contributed by $\Gamma(1-\alpha)$ and $i^{1-\alpha}$ in (3.2) cancel, and using (3.3) in (3.2) proves (3.4). $\square$

4. Trigonometric Mathieu series

When $z = e^{ix}$ lies on the unit circle, then the real resp. imaginary part of (1.1) become Mathieu cosine resp. sine series. These, and their partial sums, were considered in [23]. In particular, several inequalities for trigonometric Mathieu series were proved there. For asymptotics in the large r regime, the following result immediately follows from Theorem 2.1, by putting $z = e^{ix}$.

Corollary 4.1. *Let $\mu > 0$ and $x \in (0, 2\pi)$. As $r \uparrow \infty$, we have the asymptotic expansions*

$$\sum_{n=1}^{\infty} \frac{2n\cos(nx)}{(n^2+r^2)^{\mu+1}} \sim \sum_{k=0}^{\infty} r^{-2k-2\mu-2}\frac{2(-1)^k\,\Gamma(k+\mu+1)}{k!\,\Gamma(\mu+1)}\,\mathrm{Re}\big(\mathrm{Li}_{-2k-1}(e^{ix})\big) \tag{4.1}$$

and

$$\sum_{n=1}^{\infty} \frac{2n\sin(nx)}{(n^2+r^2)^{\mu+1}} \sim \sum_{k=0}^{\infty} r^{-2k-2\mu-2} \frac{2(-1)^k\,\Gamma(k+\mu+1)}{k!\,\Gamma(\mu+1)} \operatorname{Im}\bigl(\mathrm{Li}_{-2k-1}(e^{ix})\bigr). \tag{4.2}$$

In particular, setting $x = \pi$ in (4.1) yields the alternating Mathieu series treated in [17] and [32]. It can be easily checked that this special case of (4.1) is consistent with (2.7) in [17]. To see this, note that

$$\mathrm{Li}_{-2k-1}(-1) = (2^{2k+2}-1)\zeta(-2k-1),$$

which follows from the basic relation

$$\eta(s) = (1-2^{1-s})\zeta(s)$$

between the Riemann zeta function and the Dirichlet eta function $\eta(s) = \sum_{n=1}^{\infty}(-1)^{n+1}n^{-s}$. Moreover, the parameter μ from [17] is our $\mu+1$, and there is a typo in (2.7) of [17]: the summation should start at $k = 0$. (Besides, the last sum at the bottom of p. 6213 in [17] should be multiplied by -1.)

More generally, if x in (4.1) is a rational multiple of π, we can split the trigonometric Mathieu series into finitely many segments to which the following result from [32] can be applied.

Theorem 4.2. *For $a > 0$, $\gamma \in \mathbb{R}$, $\alpha > 0$, $\mu > \max\{(\gamma+1)/\alpha, 0\}$, and $-(\gamma+1)/\alpha \notin \mathbb{N}_0$, we have*

$$\sum_{\nu=0}^{\infty} \frac{(\nu+a)^{\gamma}}{(y(\nu+a)^{\alpha}+1)^{\mu}} \sim \frac{\Gamma\bigl(\frac{\gamma+1}{\alpha}\bigr)\Gamma\bigl(\mu-\frac{\gamma+1}{\alpha}\bigr)}{\alpha\Gamma(\mu)} y^{-\frac{\gamma+1}{\alpha}} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(\mu+k)}{\Gamma(\mu)} \zeta(-\alpha k-\gamma, a)\, y^k, \quad y \downarrow 0.$$

To see that Theorem 4.2 gives an alternative proof of (4.1) for x a rational multiple of π, let $x = p\pi/q$ with $p/q \in \mathbb{Q} \cap (0,2)$. Then, putting $y := (2q/r)^2$, we can write the left hand side of (4.1) as

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{2n\cos(nx)}{(n^2+r^2)^{\mu+1}} = \sum_{m=0}^{2q-1}\sum_{\nu=0}^{\infty} \frac{2(2\nu q+m)}{\bigl((2\nu q+m)^2+r^2\bigr)^{\mu+1}} \cos\Bigl(2\nu p\pi + \frac{mp\pi}{q}\Bigr) \\
&= 4qr^{-2\mu-2}\Biggl(\sum_{\nu=0}^{\infty} \frac{\nu+1}{\bigl(y(\nu+1)^2+1\bigr)^{\mu+1}} + \\
&\sum_{m=1}^{2q-1} \cos\Bigl(\frac{mp\pi}{q}\Bigr) \sum_{\nu=0}^{\infty} \frac{\nu+m/2q}{\bigl(y(\nu+m/2q)^2+1\bigr)^{\mu+1}}\Biggr) \\
&\sim 4qr^{-2\mu-2}\Biggl(\frac{\Gamma(\mu)}{2\Gamma(\mu+1)}y^{-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\frac{\Gamma(k+\mu+1)}{\Gamma(\mu+1)}\zeta(-2k-1)y^k + \\
&\sum_{m=1}^{2q-1} \cos\Bigl(\frac{mp\pi}{q}\Bigr)\Bigl(\frac{\Gamma(\mu)}{2\Gamma(\mu+1)}y^{-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\frac{\Gamma(k+\mu+1)}{\Gamma(\mu+1)}\zeta(-2k-1, \tfrac{m}{2q})y^k\Bigr)\Biggr) \\
&= 2^{2k+2}q^{2k+1}\sum_{k=0}^{\infty} r^{-2k-2\mu-2}\frac{(-1)^k}{k!}\frac{\Gamma(k+\mu+1)}{\Gamma(\mu+1)}\Bigl(\zeta(-2k-1) + \\
&\quad \sum_{m=1}^{2q-1} \cos\Bigl(\frac{mp\pi}{q}\Bigr)\zeta(-2k-1, \tfrac{m}{2q})\Bigr),
\end{aligned} \tag{4.3}$$

where the asymptotic expansion follows from Theorem 4.2, with μ replaced by $\mu+1$, $\gamma=1$, $\alpha=2$, and $a=1$ resp. $a=2m/q$, and

$$\sum_{m=0}^{2q-1}\cos\left(\frac{mp\pi}{q}\right)=0$$

was used in the last equality. Now (4.1) for $x=p\pi/q$ easily follows from (4.3). Note that

$$\mathrm{Li}_\alpha(e^{ip\pi/q})=(2q)^{-\alpha}\zeta(\alpha)+(2q)^{-\alpha}\sum_{m=1}^{2q-1}e^{imp\pi/q}\zeta(\alpha,m/2q),\quad \alpha\neq 1,$$

which we apply with $\alpha=-2k-1$. The latter (well-known) identity easily follows from (1.2) and (3.1) by analytic continuation. Clearly, the sine series (4.2) can be treated analogously.

We now comment on a different asymptotic regime for trigonometric Mathieu series, namely $x\downarrow 0$ for $r>0$ fixed. For the series in (4.1) and (4.2), such an expansion can be obtained using again the Mellin transform approach. However, this would require an analysis of the singularity structure of the Dirichlet series $\sum_{k=1}^\infty k^{-s}(k^2+r^2)^{-(\mu+1)}$, which is doable, but deferred to future work. First order asymptotics, however, follow from known results, even for the more general Mathieu-type sine series

$$\tilde{S}^{\gamma,\delta}_{\alpha,\beta,\mu}(r,x):=\sum_{n=2}^{\infty}\frac{n^\alpha(\log n)^\gamma\sin(nx)}{\left(n^\beta(\log n)^\delta+r^2\right)^{\mu+1}}. \tag{4.4}$$

The corresponding series without the factor $\sin(nx)$ was introduced in [12]. The coefficients of the series (4.4) behave roughly like $n^{\alpha-\beta(\mu+1)}$, and the asymptotic behavior is markedly different for $\alpha-\beta(\mu+1)<-2$ and $\alpha-\beta(\mu+1)>-2$.

Proposition 4.3. *Let $\mu,r\geq 0$ and $\alpha,\beta,\gamma,\delta\in\mathbb{R}$ such that*

$$0\leq\theta:=\beta(\mu+1)-\alpha<2.$$

If $\theta=0$, then we assume that $\gamma-\delta(\mu+1)<0$. For $x\downarrow 0$, we have

$$\tilde{S}^{\gamma,\delta}_{\alpha,\beta,\mu}(r,x)\sim\begin{cases}\dfrac{\pi}{2\Gamma(\theta)\sin(\pi\theta/2)}x^{\theta-1}\left(\log\dfrac{1}{x}\right)^{\gamma-\delta(\mu+1)} & 0<\theta<2,\\[2ex] \dfrac{1}{x}\left(\log\dfrac{1}{x}\right)^{\gamma-\delta(\mu+1)} & \theta=0.\end{cases}$$

Proof. As the coefficient sequence

$$a_n=\frac{n^\alpha(\log n)^\gamma}{\left(n^\beta(\log n)^\delta+r^2\right)^{\mu+1}} \tag{4.5}$$

eventually decreases, the series is convergent for $x\in(0,\pi)$ (see p. 3 in [33]). First assume $\theta=0$. Since powers of logarithms are slowly varying, and asymptotic equivalence preserves slow variation, the sequence

$$a_n\sim(\log n)^{\gamma-\delta(\mu+1)},\quad n\to\infty,$$

is slowly varying. See [2,3] for more information on slow variation. Moreover, it is easy to see that the second derivative of (4.5) does not change sign for n sufficiently large, and so a_n is eventually convex or eventually concave. We can thus apply the corollary on p. 48 of Telyakovskiĭ [25] (with a_n replaced by $-a_n$ in case of concavity) to conclude that

$$\tilde{S}^{\gamma,\delta}_{\alpha,\beta,\mu}(r,x) \sim a_m/x, \quad x \downarrow 0,\ x \in \Big(\frac{\pi}{m+1}, \frac{\pi}{m}\Big],$$

where we note that in [25] asymptotic equivalence is denoted by $\approx$ and not by $\sim$. Since $m = \lceil \pi/x \rceil = \pi/x + \mathrm{O}(1)$, the statement easily follows.

For $\theta > 0$, the sequence

$$a_n \sim n^{-\theta}(\log n)^{\gamma-\delta(\mu+1)}, \quad n \to \infty,$$

is *regularly* varying with index $-\theta$. Recall that this means that $\lim_{n\to\infty} a_{\lfloor \lambda n \rfloor}/a_n = \lambda^{-\theta}$ for $\lambda > 0$ (see [2,3]). Thus, our assertion is an immediate consequence of Theorem 1 in [1]. Note that it is easy to see that $n^\theta a_n$ is eventually decreasing, as assumed in that theorem. However, the required implication is true even without this condition; see [4] for this, and for further references. □

If $\theta = \beta(\mu+1)-\alpha$ is larger than 2, we can use a result of Hartman and Wintner [13]. Unlike Proposition 4.3, the parameter r now appears in the first asymptotic term, and there is no $\log x$ term. Essentially, the series now converges fast enough to justify exchanging summation and the asymptotic equivalence $\sin(nx) \sim nx$.

Proposition 4.4. *Let $\mu, r \geq 0$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\theta = \beta(\mu+1) - \alpha > 2$. Then we have*

$$\tilde{S}^{\gamma,\delta}_{\alpha,\beta,\mu}(r,x) \sim x \sum_{n=2}^{\infty} \frac{n^{\alpha+1}(\log n)^\gamma}{\big(n^\beta(\log n)^\delta + r^2\big)^{\mu+1}}, \quad x \downarrow 0.$$

Proof. According to [13], for a decreasing sequence $a_n \downarrow 0$ with $\sum n a_n < \infty$, we have

$$\sum_{n=1}^{\infty} a_n \sin(nx) \sim x \sum_{n=1}^{\infty} n a_n, \quad x \downarrow 0.$$

The sequence (4.5) decreases for large n, say for $n \geq n_0$. By considering the sequence $\tilde{a}_n = a_{n_0}\mathbf{1}_{n \leq n_0} + a_n \mathbf{1}_{n > n_0}$, it is very easy to see that "decreasing" can be replaced by "eventually decreasing" in the above statement. This implies the assertion. □

The series

$$\sum_{n=2}^{\infty} \frac{(\log n!)^\alpha}{\big((\log n!)^\beta + r^2\big)^{\mu+1}}$$

was considered in Corollary 7.1 of [12]. The corresponding sine series can be analyzed analogously to the preceding propositions. By Stirling's formula,

$$\frac{(\log n!)^\alpha}{\big((\log n!)^\beta + r^2\big)^{\mu+1}} \sim (n \log n)^{-\theta}, \quad n \to \infty,$$

and so the coefficients are regularly varying. In particular, for $\theta = \beta(\mu+1) - \alpha > 2$, we can proceed as in Proposition 4.4 to obtain

$$\sum_{n=2}^{\infty} \frac{(\log n!)^\alpha \sin(nx)}{\big((\log n!)^\beta + r^2\big)^{\mu+1}} \sim x \sum_{n=2}^{\infty} \frac{n(\log n!)^\alpha)}{\big((\log n!)^\beta + r^2\big)^{\mu+1}}, \quad x \downarrow 0.$$

The paper [24] contains several estimates that can be applied to Mathieu sine series. For instance, it was proved there (Corollary 2) that

$$\int_0^\pi \Big|\sum_{k=1}^n a_k \sin(kx)\Big| dx = \sum_{k=1}^n \frac{a_k}{k} + \mathrm{O}(a_1)$$

for coefficient sequences $a_n \downarrow 0$. This result can be readily applied to our Mathieu sine series, with obvious constraints on the parameters to ensure monotonicity of the coefficient sequence. Moreover, Ul'yanov [29] studied convergence in the L^p-quasinorm for $p \in (0,1)$, for both sine and cosine series, which yields the following result.

Proposition 4.5. *Let $\mu, r \geq 0$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\theta = \beta(\mu+1) - \alpha \geq 0$. If $\theta = 0$, then we assume that $\gamma - \delta(\mu+1) < 0$. We write $\tilde{S}^{\gamma,\delta}_{\alpha,\beta,\mu}(r,x;n)$ for the n-th partial sum of the series (4.4). Then*

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \big|\tilde{S}^{\gamma,\delta}_{\alpha,\beta,\mu}(r,x) - \tilde{S}^{\gamma,\delta}_{\alpha,\beta,\mu}(r,x;n)\big|^p dx = 0, \quad p \in (0,1).$$

The same result holds for the corresponding cosine series.

Proof. The coefficient sequence (4.5) eventually decreases. Therefore, it is of bounded variation, which by definition means that $\sum |\Delta a_n| < \infty$. Thus, both assertions are immediate from [29]. □

Finally, we mention that L^1-convergence of the Mathieu-type cosine series

$$S^{\gamma,\delta}_{\alpha,\beta,\mu}(r,x) := \sum_{n=2}^{\infty} \frac{n^\alpha (\log n)^\gamma \cos(nx)}{\big(n^\beta (\log n)^\delta + r^2\big)^{\mu+1}}, \tag{4.6}$$

follows from a result in [27].

Proposition 4.6. *Let $\mu, r \geq 0$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that either*

$$\alpha - \beta(\mu+1) < 0$$

or

$$\alpha = \beta = 0 \quad \text{and} \quad \gamma - \delta(\mu+1) < -1.$$

Then the series (4.6) converges in $L^1(0,\pi)$.

Proof. As noted above, the coefficient sequence (4.5) of the series $S^{\gamma,\delta}_{\alpha,\beta,\mu}(r,x)$ is regularly varying. According to [27], it then suffices to verify that $a_n \log n \to 0$. But this easily follows from our assumption on the parameters. □

As for the asymptotic behavior of the cosine series $S^{\gamma,\delta}_{\alpha,\beta,\mu}(r,x)$ for $x \downarrow 0$, we can use Theorem 2.1 of [4] (going back to Zygmund) in the case $0 < \theta = \beta(\mu+1) - \alpha < 1$. For θ outside of this interval, the other results for sine series we used in Propositions 4.3 and 4.4 seem not to be available for cosine series so far.

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