SMALL-MATURITY DIGITAL OPTIONS IN LÉVY MODELS: AN ANALYTIC APPROACH

STEFAN GERHOLD

ABSTRACT. We prove a small-time Tauberian theorem for transition probabilities of certain Lévy processes. The main assumption is a condition on the asymptotic behavior of the characteristic function. This gives an alternative derivation of some results on digital options and implied volatility slopes in Lévy models. In probabilistic terms, it gives a sufficient criterion for Spitzer's condition.

This note is concerned with evaluating $\lim_{T\to 0} \mathbb{P}[X_T \geq 0]$, where X is a Lévy process with $X_0 = 0$. We focus on Lévy processes commonly used in mathematical finance, in particular on the Normal Inverse Gaussian (NIG), CGMY, and Meixner process. While the literature on $\lim_{T\to 0} \mathbb{P}[X_T \geq x]$, where $x \neq 0$, is considerable (see, e.g., [2] and the references therein), results for $x = 0$ are scarcer.

For the Lévy processes that have been proposed in the literature as financial models, the characteristic function is known explicitly. Then, it is usually easy to identify scalings such that the normalized X_T converges in distribution as $T \to 0$, which in turn yields the value of $\lim_{T\to 0} \mathbb{P}[X_T \geq 0]$. Another approach proceeds from the results of [5], who discuss convergence at process level under certain assumptions on the Lévy triplet; see [3] for details.

We propose a third route. Our aim is not at new limit results for concrete Lévy models, but at a novel Tauberian transfer theorem with analytic assumptions on the characteristic function. The latter amount to a certain asymptotic behavior of the characteristic function at infinity. In future research, we intend to connect the latter with properties of the Lévy triplet.

It has been shown (see Chapter 7 of [1]) that the existence of the limit

$$\lim_{T\to 0} \mathbb{P}[X_T > 0] = \rho$$

is equivalent to Spitzer's condition, i.e.,

$$\lim_{T\to 0} \frac{1}{T} \int_0^T \mathbb{P}[X_t > 0]\, dt = \rho.$$

In general, it is an open problem to infer from the characteristics of a Lévy process whether Spitzer's condition holds. In Section 7.3 of [1], several sufficient conditions are stated. In particular, condition (iii) there covers all our examples (Corollaries 2–4), as these processes are in the domain of attraction of a strictly

Date: October 17, 2014.

I gratefully acknowledge financial support from the Austrian Science Fund (FWF) under grant P 24880-N25, and thank the anonymous referees for helpful remarks.

stable process, by the results of [5]. We leave as an open problem whether the latter is always the case under the assumptions of our Theorem 1, or it can produce new examples where Spitzer's condition holds.

In mathematical finance, the quantity $\mathbb{P}[X_T \geq 0]$ is the price of an at-the-money digital call option. We assume here that the underlying is an exponential Lévy martingale $S = \exp(X)$, normalized to $S_0 = 1$, and that interest rate and dividends are zero. As explained in [3], it is well known that digital prices are intimately related with the at-the-money slope (i.e., strike derivative) of implied volatility. If the Lévy process X has no Brownian component, the latter typically explodes for short maturity, more precisely:

$$\partial_K|_{K=1}\sigma_{\mathrm{imp}}(K,T) \sim \sqrt{\frac{2\pi}{T}}\left(\frac{1}{2} - \lim_{t\to 0}\mathbb{P}[X_t \geq 0]\right), \quad T \to 0. \tag{1}$$

If there *is* a Brownian component, we have asymptotic symmetry in the sense that $\lim_{t\to 0}\mathbb{P}[X_t \geq 0] = \frac{1}{2}$. Then the implied volatility slope tends to a finite limit, as for a pure diffusion model; this case is discussed in detail in [3]. The main practical application of such results is model calibration. In particular, identifying the parameter ranges where a model produces a positive resp. negative slope yields a useful constraint when fitting to the market smile.

Let (b, σ^2, ν) be the characteristic triplet of X. The moment generating function (mgf) of X_T is

$$M(s,T) = \mathbb{E}[e^{sX_T}] = \exp\left(T(bs + \tfrac{1}{2}\sigma^2 s^2 + \psi(s))\right), \tag{2}$$

where

$$\psi(s) := \int_{\mathbb{R}} (e^{sx} - 1 - sx\mathbf{1}_{\{|x|\leq 1\}})\nu(dx). \tag{3}$$

Since S is a martingale, we must have $b + \sigma^2/2 + \psi(1) = 0$. The mgf is analytic in a strip $s_- < \mathrm{Re}(s) < s_+$, given by the critical moments

$$s_+ = \sup\{s \in \mathbb{R} : \mathbb{E}[e^{sX_T}] < \infty\} \tag{4}$$

and

$$s_- = \inf\{s \in \mathbb{R} : \mathbb{E}[e^{sX_T}] < \infty\}. \tag{5}$$

We will obtain asymptotic information on the transition probabilities (i.e., digital call prices) from its Fourier representation [4]

$$\mathbb{P}[S_T \geq S_0] = \mathbb{P}[X_T \geq 0] = \frac{1}{2i\pi}\int_{a-i\infty}^{a+i\infty} \frac{M(s,T)}{s} ds, \tag{6}$$

where the real part of the vertical integration contour satisfies $0 < a < s_+$, and absolute convergence of the integral is assumed. Our notation is a bit different from that of [4]. In that paper, maturity T is fixed, and the underlying at maturity is denoted by X. With the notation from [4, Section 2.2], we are interested in the payoff function $G_3(x,k) = \mathbf{1}_{\{x>k\}}$, with $b_1 = 0$ and $b_0 = 1$, and log-strike $k = \log K$. The set A_X is the strip of analyticity of the mgf, i.e., $A_X = \{s \in \mathbb{C} : s_- < \mathrm{Re}(s) < s_+\}$. By [4, Theorem 4.2], the damped payoff $e^{-\alpha k}\mathbb{E}[G_3(X,k)]$ has Fourier transform $u \mapsto f(u - \alpha i)/(\alpha + ui)$, where $\alpha > 0$ is an arbitrary real and positive element of A_X, and f is the characteristic function of X. Then, by [4, Theorem 4.3],

$$\mathbb{P}[X > k] = \mathbb{E}[G_3(X,k)] = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-uki}\frac{f(u-\alpha i)}{\alpha+ui}du,$$

assuming integrability. Now it suffices to substitute $s = \alpha + ui$ to get (6), with $a = \alpha$; note that we are interested only in the at-the-money case, i.e., $k = 0$.

The case $\sigma > 0$ being handled by [3], we from now on assume that $\sigma = 0$, i.e., that there is no diffusion component. The following theorem is our main result; it is of Tauberian type, deducing the limit of (6) from asymptotic information on the transform $M(s,T)$. As usual, for functions $F, G : (0,\infty) \to \mathbb{C}$, we write

$$F(x) = O(G(x)) \quad \text{as } x \to 0$$

if there is a constant $c > 0$ such that $|F(x)| \le c|G(x)|$ for x sufficiently small, and

$$F(x) = O(G(x)) \quad \text{as } x \to \infty$$

if there is a constant $c > 0$ such that $|F(x)| \le c|G(x)|$ for x sufficiently large.

Theorem 1. *Assume that $\sigma = 0$, $b \neq 0$ and that there is a real number $a \in (0, s_+)$ such that ψ has an expansion of the form*

$$\psi(a+iy) = -c_1 y^{\eta} + \sum_{k=2}^{m} c_k y^{\eta_k} + c_{m+1} + O(y^{-\gamma}), \quad y \to \infty, \tag{7}$$

where $m \ge 2$, $c_1 > 0$, $c_2, \dots, c_{m+1} \in \mathbb{C}$, $\eta > 0$, $(\eta \wedge 1) > \eta_2 > \dots > \eta_m > 0$, and $\gamma > 0$. Then

$$\lim_{T\to 0} \mathbb{P}[X_T \ge 0] = \begin{cases} 0 & 0 < \eta < 1,\ b < 0, \\ 1 & 0 < \eta < 1,\ b > 0, \\ \frac{1}{2} + \frac{1}{\pi}\arctan\frac{b}{c_1} & \eta = 1, \\ \frac{1}{2} & \eta > 1. \end{cases} \tag{8}$$

The proof of Theorem 1 is given below. We will apply Theorem 1 in conjunction with (1) to determine digital asymptotics and the asymptotic implied volatility slope for the CGMY, NIG, and Meixner models.

Note that the "drift" b and the function ψ depend on the choice of the truncation function in (3). Suppose that the assumptions of Theorem 1 hold, and that we replace ψ by ψ_h, using some truncation function h other than $\mathbf{1}_{\{|x|\le 1\}}$. Then we get a new drift $b_h = b - c$, where $c = \int x(\mathbf{1}_{\{|x|\le 1\}} - h(x))\nu(dx)$, which would apparently change the limiting probability as given in (8). However, the assumption (7) would then no longer be valid, because of the term ciy in $\psi_h(a+iy) = \psi(a+iy) + ciy + O(1)$.

The exponent η in (7) seems to be related to the path regularity of the Lévy process X. In our examples, $\eta < 1$ holds for finite variation processes, and $\eta \ge 1$ for infinite variation. We are not aware of any results in this direction, though. For $\eta > 1$, Theorem 1 yields $1/2$ for the limiting probability, i.e., the same value as if there was a Brownian component.

In the CGMY model, the mgf is

$$M(s,T) = \exp\left(Tbs + TC\Gamma(-Y)((M-s)^Y - M^Y + (G+s)^Y - G^Y)\right), \tag{9}$$

where we assume $C > 0$, $G > 0$, $M > 1$, and $0 < Y < 1$. For $s_- < a < s_+$, $s = a + iy$, and $y \to \infty$, we have

$$(M-s)^Y = y^Y(\cos(\pi Y/2) - i\sin(\pi Y/2)) + O(y^{Y-1})$$

and

$$(G+s)^Y = y^Y(\cos(\pi Y/2) + i\sin(\pi Y/2)) + O(y^{Y-1}).$$

Therefore, we obtain

$$(M-s)^Y + (G+s)^Y = 2\cos(\pi Y/2)y^Y + O(y^{Y-1}),$$

and Theorem 1 (with $\eta = Y$), together with (1), shows the following:

Corollary 2. *In the CGMY model, with the above parameter ranges and $b \neq 0$, digital prices and implied volatility slope satisfy*

$$\lim_{T\to 0} \mathbb{P}[X_T \geq 0] = \begin{cases} 0 & \textit{if } b < 0 \\ 1 & \textit{if } b > 0 \end{cases}$$

respectively

$$\partial_K|_{K=1}\sigma_{\mathrm{imp}}(K,T) \sim -\sqrt{\pi/2}\,\mathrm{sgn}(b)T^{-1/2}, \quad T \to 0.$$

In the Normal Inverse Gaussian (NIG) model, the mgf of log-spot is

$$M(s,T) = \exp(Tbs + \delta T(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+s)^2}))$$

with $\delta > 0$, $\alpha > \max\{\beta + 1, -\beta\}$. Since S is a martingale, we must have

$$b = \sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2 - \beta^2}.$$

Corollary 3. *For $b \neq 0$, digital prices and implied volatility slope of the NIG model satisfy*

$$\lim_{T\to 0} \mathbb{P}[X_T \geq 0] = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{b}{\delta}$$

respectively

$$\partial_K|_{K=1}\sigma_{\mathrm{imp}}(K,T) \sim -\sqrt{2/\pi}\arctan(\frac{b}{\delta}) \cdot T^{-1/2}, \quad T \to 0.$$

Since, for $s_- < a < s_+$, $s = a + iy$, and $y \to \infty$,

$$\delta\sqrt{\alpha^2 - (\beta+s)^2} = \delta y - i\delta(a+\beta) + O(1/y),$$

Corollary 3 follows immediately from (1) and Theorem 1 (with $\eta = 1$ and $c_1 = \delta$).

Finally, we consider the Meixner model. The mgf is

$$M(s,T) = \exp\left(T\left(bs + 2\bar{d}\log\frac{\cos(\bar{b}/2)}{\cosh\frac{1}{2}(-\bar{a}is - i\bar{b})}\right)\right),$$

where $\bar{d} > 0$, $\bar{b} \in (-\pi, \pi)$, and $0 < \bar{a} < \pi - \bar{b}$. (We follow the notation of Schoutens [6], except that we denote the drift by b instead of m, and his parameters a, b, d by $\bar{a}, \bar{b}, \bar{d}$.) A straightforward calculation shows that

$$\psi(a + iy) = -\bar{a}\bar{d}y + C + O(e^{-\bar{a}y})$$

for $0 < a < s_+$ and $y \to \infty$. Formula (1) and Theorem 1 thus imply (with $\eta = 1$ and $c_1 = \bar{a}\bar{d}$):

Corollary 4. *For $b \neq 0$, the digital price and implied volatility slope of the Meixner model satisfy*

$$\lim_{T\to 0} \mathbb{P}[X_T \geq 0] = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{b}{\bar{a}\bar{d}}\right)$$

respectively

$$\partial_K \sigma_{\mathrm{imp}}(1, T) \sim -\sqrt{2/\pi} \arctan\left(\frac{b}{\bar{a}\bar{d}}\right) \cdot T^{-1/2}, \quad T \to 0.$$

We now give the proof of our main result.

Proof of Theorem 1. Our assumptions imply that the Fourier representation (6) of the transition probability holds:

$$\mathbb{P}[X_T \geq 0] = \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} \frac{1}{s} M(s, T) ds = \frac{1}{\pi} \operatorname{Im} \int_a^{a+i\infty} \frac{1}{s} M(s, T) ds.$$

Since $e^{c_{m+1}T} \sim 1$, we may w.l.o.g. assume $c_{m+1} = 0$. Define

$$f(y) := -c_1 y^\eta + \sum_{k=2}^m \operatorname{Re}(c_k) y^{\eta_k}$$

and

$$g(y) := by + \sum_{k=2}^m \operatorname{Im}(c_k) y^{\eta_k}$$

so that $\psi(a + iy) + b(a + iy) = ab + f(y) + ig(y) + O(y^{-\gamma})$. Now we use the assumption on the asymptotics of ψ, and note that $e^{Tab} = 1 + o(1)$:

(10)

$$\begin{aligned}
&(1 + o(1))\mathbb{P}[X_T \geq 0] \\
&\quad = \frac{1}{\pi} \operatorname{Im} \int_a^{a+i\infty} \frac{1}{s} \exp\Big(T\big(f(\operatorname{Im}(s)) + ig(\operatorname{Im}(s)) + O(\operatorname{Im}(s)^{-\gamma})\big)\Big) ds \\
&\quad = \frac{1}{\pi} \operatorname{Im} \int_a^{a+i\infty} \frac{1}{s} \exp\Big(T\big(f(\operatorname{Im}(s)) + ig(\operatorname{Im}(s))\big)\Big)\big(\exp(T \cdot O(\operatorname{Im}(s)^{-\gamma})) - 1\big) ds
\end{aligned}$$

(11)

$$+ \frac{1}{\pi} \operatorname{Im} \int_a^{a+i\infty} \frac{1}{s} \exp\big(T(f(\operatorname{Im}(s)) + ig(\operatorname{Im}(s)))\big) ds.$$

For $x > 0$ and $0 < T < 1$, we have

$$\begin{aligned}\frac{e^{Tx}-1}{T} &= \sum_{k=1}^{\infty} T^{k-1}x^k/k! \leq \sum_{k=1}^{\infty} x^k/k! \\ &= e^x - 1 = O(x), \quad x \to 0, \text{ uniformly in } T,\end{aligned}$$

and so

$$\frac{\exp(T \cdot O(\mathrm{Im}(s)^{-\gamma})) - 1}{T} = O(\mathrm{Im}(s)^{-\gamma}), \quad \mathrm{Im}(s) \to \infty,$$

uniformly for small T. By dominated convergence, this implies that the first integral in (11) is $O(T)$. We continue with the second integral in (11). It equals

$$\begin{aligned}&\frac{1}{\pi}\,\mathrm{Re}\int_0^\infty \frac{a-iy}{a^2+y^2}\exp(T(f(y)+ig(y))dy \\ &\quad = \frac{1}{\pi}\int_0^\infty \frac{a}{a^2+y^2}dy + o(1) + \frac{1}{\pi}\int_0^\infty \frac{y}{a^2+y^2}e^{Tf(y)}\sin(Tg(y))dy \\ &\quad = \frac{1}{2} + \frac{1}{\pi}\int_{C_0}^\infty \frac{1}{y}e^{Tf(y)}\sin(Tg(y))dy + o(1). \qquad (12)\end{aligned}$$

The last equality follows from $y/(a^2+y^2) = 1/y + O(y^{-3})$ and dominated convergence. Moreover, C_0 denotes a large constant (more precisely, big enough so that all "sufficiently large" statements in the following are applicable). From now on, we assume $b > 0$, and evaluate the limit of the integral in (12) as $T \to 0$. (If $b < 0$, we factor out -1 from the sine in (12), thus returning to the case $b > 0$.)

For large y, the function $g(y)$ is increasing, so that the substitution $y = g^{-1}(u/T)$ makes sense. The integral in (12) thus becomes

$$\frac{1}{T}\int_{Tg(C)}^\infty \frac{(g^{-1})'(u/T)}{g^{-1}(u/T)}e^{Tf(g^{-1}(u/T))}\sin u \; du. \qquad (13)$$

The expansion of g^{-1} at infinity is

$$g^{-1}(x) = x/b + Cx^{\eta_2} + \dots,$$

and so

$$\frac{1}{g^{-1}(x)} = \frac{b}{x}(1 + Cx^{\eta_2 - 1} + \dots),$$

from which we deduce, for large x, that

$$|1/g^{-1}(x) - b/x| \leq Cx^{\eta_2 - 2}.$$

For $x = u/T$, this estimate becomes $CT^{2-\eta_2}u^{\eta_2-2}$. Hence, by dominated convergence, we may replace the factor $1/g^{-1}(u/T)$ in (13) by bT/u:

$$\mathbb{P}[X_T \geq 0] = \frac{1}{2} + \frac{b}{\pi}\int_{Tg(C)}^\infty (g^{-1})'(u/T)e^{Tf(g^{-1}(u/T))}\frac{\sin u}{u}du + o(1). \qquad (14)$$

Note that, in general, asymptotic expansions may not be differentiated; for the special function g^{-1} it is easy to see, though, that this is valid, and so

$$(g^{-1})'(x) = 1/b + Cx^{\eta_2 - 1} + \dots,$$

hence

$$|(g^{-1})'(x) - 1/b| \le Cx^{\eta_2-1}$$

for large x. Similarly, we have a bound

$$|f(g^{-1}(x)) + c_1(x/b)^\eta| \le Cx^{\eta_2}.$$

To simplify the integrand further, note that

$$(g^{-1})'(u/T)e^{Tf(g^{-1}(u/T))} - \frac{1}{b}e^{-c_1T^{1-\eta}(u/b)^\eta} = \frac{1}{b}e^{-c_1T^{1-\eta}(u/b)^\eta} \cdot \Big(e^{c_1T^{1-\eta}(u/b)^\eta+Tf(g^{-1}(u/T))}\big(b(g^{-1})'(u/T)-1\big) + e^{c_1T^{1-\eta}(u/b)^\eta+Tf(g^{-1}(u/T))} - 1\Big). \tag{15}$$

We have

$$|(g^{-1})'(u/T) - 1/b| \le CT^{1-\eta_2}u^{\eta_2-1},$$

and

$$\begin{aligned}|c_1T^{1-\eta}(u/b)^\eta + Tf(g^{-1}(u/T))| &= T|c_1(u/bT)^\eta + f(g^{-1}(u/T))| \\ &\le CT^{1-\eta_2}u^{\eta_2}.\end{aligned}$$

Therefore, if we want to simplify the integral in (14) by replacing the first term on the left-hand side of (15) with the second term, we have to argue that

$$T^{1-\eta_2}\int_{Tg(C)}^\infty e^{-c_1T^{1-\eta}(u/b)^\eta}e^{CT^{1-\eta_2}u^{\eta_2}}u^{\eta_2-1}\frac{\sin u}{u}du \tag{16}$$

and

$$\int_{Tg(C)}^\infty e^{-c_1T^{1-\eta}(u/b)^\eta}(e^{CT^{1-\eta_2}u^{\eta_2}}-1)\frac{\sin u}{u}du \tag{17}$$

are both $o(1)$. For (16) this holds by dominated convergence, whereas the argument for (17) is an easy modification of Lemma 5 below, which we omit for brevity. We thus obtain

$$\begin{aligned}\mathbb{P}[X_T \ge 0] &= \frac{1}{2} + \frac{1}{\pi}\int_{Tg(C)}^\infty e^{-c_1T^{1-\eta}(u/b)^\eta}\frac{\sin u}{u}du + o(1) \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty e^{-c_1T^{1-\eta}(u/b)^\eta}\frac{\sin u}{u}du + o(1). \qquad (18)\end{aligned}$$

If $\eta < 1$, we apply Lemma 5 to deduce $\mathbb{P}[X_T \ge 0] \to 1$. For $\eta = 1$, the integral in (18) evaluates to

$$\int_0^\infty e^{-c_1u/b}\frac{\sin u}{u}du = \arctan\frac{b}{c_1}.$$

Finally, if $\eta > 1$, then the integral in (18) tends to zero by dominated convergence, which finishes the proof of Theorem 1. □

In the preceding proof, we needed the limit of the following integral. Since the standard convergence theorems do not apply, we analyze it by hand.

Lemma 5. *For $c_1, b > 0$ and $0 < \eta < 1$, we have*

$$\lim_{T\to 0}\int_0^\infty e^{-c_1T^{1-\eta}(u/b)^\eta}\frac{\sin u}{u}du = \int_0^\infty \frac{\sin u}{u}du = \frac{\pi}{2}.$$

Proof. For simplicity of notation, we assume that $b = c_1 = 1$. We subtract the known integral $\int_0^\infty \frac{\sin u}{u} du$, divide the integration range into intervals of length π, and group together adjacent summands to control the sign of $\sin u$:

$$\int_0^\infty \frac{\sin u}{u}(1 - e^{-T^{1-\eta}u^\eta})du \tag{19}$$

$$= \sum_{k=0}^\infty \left(\int_{2k\pi}^{(2k+1)\pi} + \int_{(2k+1)\pi}^{(2k+2)\pi} \right) \frac{\sin u}{u}(1 - e^{-T^{1-\eta}u^\eta})du$$

$$= \sum_{k=0}^\infty \int_{2k\pi}^{(2k+1)\pi} \sin u \left(\frac{1 - e^{-T^{1-\eta}u^\eta}}{u} - \frac{1 - e^{-T^{1-\eta}(u+\pi)^\eta}}{u+\pi} \right) du. \tag{20}$$

We now estimate the term in parentheses for $2k\pi \le u \le (2k+1)\pi$ from above and below. It is

$$\le \frac{1 - e^{-T^{1-\eta}((2k+1)\pi)^\eta}}{2k\pi} - \frac{1 - e^{-T^{1-\eta}((2k+1)\pi)^\eta}}{(2k+2)\pi}$$

$$= \frac{1 - e^{-T^{1-\eta}((2k+1)\pi)^\eta}}{2k(k+1)\pi}, \tag{21}$$

and

$$\ge \frac{1 - e^{-T^{1-\eta}(2k\pi)^\eta}}{(2k+1)\pi} - \frac{1 - e^{-T^{1-\eta}((2k+2)\pi)^\eta}}{(2k+1)\pi}$$

$$= \frac{e^{-T^{1-\eta}((2k+2)\pi)^\eta} - e^{-T^{1-\eta}(2k\pi)^\eta}}{(2k+1)\pi}. \tag{22}$$

By the upper estimate (21), and the fact that $0 \le \sin u \le 1$ in (20), we see that the integral in (19) satisfies

$$\int_0^\infty \frac{\sin u}{u}(1 - e^{-T^{1-\eta}u^\eta})du \le \sum_{k=0}^\infty \frac{1 - e^{-T^{1-\eta}((2k+1)\pi)^\eta}}{2k(k+1)\pi},$$

which is $o(1)$ for $T \to 0$ by dominated convergence. On the other hand, the lower estimate (22) yields

$$\int_0^\infty \frac{\sin u}{u}(1 - e^{-T^{1-\eta}u^\eta})du \ge \sum_{k=1}^\infty \frac{e^{-T^{1-\eta}(2k\pi)^\eta}}{(2k-1)\pi} - \sum_{k=1}^\infty \frac{e^{-T^{1-\eta}(2k\pi)^\eta}}{(2k+1)\pi} - \frac{1}{\pi}$$

$$= \sum_{k=1}^\infty e^{-T^{1-\eta}(2k\pi)^\eta} \frac{2}{(4k^2-1)\pi} - \frac{1}{\pi}$$

$$= \sum_{k=1}^\infty \frac{2}{(4k^2-1)\pi} - \frac{1}{\pi} + o(1) = o(1), \tag{23}$$

where the penultimate equality follows from dominated convergence. Note that the series in (23) has the value $1/\pi$, as can be easily seen by applying partial fraction decomposition to its partial sums:

$$\sum_{k=1}^{n} \frac{2}{4k^2-1} = \sum_{k=1}^{n} \frac{1}{2k-1} + \sum_{k=1}^{n} \frac{1}{2k+1} = 1 - \frac{1}{2n+1}.$$

□

References

[1] R. A. Doney, *Fluctuation theory for Lévy processes*, vol. 1897 of Lecture Notes in Mathematics, Springer, Berlin, 2007. Lectures from the 35th Summer School on Probability Theory held in Saint-Flour, July 6–23, 2005, Edited and with a foreword by Jean Picard.

[2] J. E. Figueroa-López and C. Houdré, *Small-time expansions for the transition distributions of Lévy processes*, Stochastic Process. Appl., 119 (2009), pp. 3862–3889.

[3] S. Gerhold and I. C. Gülüm, *The small-maturity implied volatility slope for Lévy models.* Preprint, 2014.

[4] R. W. Lee, *Option pricing by transform methods: Extensions, unification, and error control*, Journal of Computational Finance, 7 (2004), pp. 51–86.

[5] M. Rosenbaum and P. Tankov, *Asymptotic results for time-changed Lévy processes sampled at hitting times*, Stochastic Process. Appl., 121 (2011), pp. 1607–1632.

[6] W. Schoutens, *Meixner processes: Theory and applications in finance*, EURANDOM Report 2002-004, EURANDOM, Eindhoven, 2002.

Vienna University of Technology, Wiedner Hauptstrasse 8–10/105-1, A-1040 Vienna, Austria

E-mail address: sgerhold@fam.tuwien.ac.at