Extrapolation Analytics for Dupire's Local Volatility

Peter Friz and Stefan Gerhold

Abstract We consider wing asymptotics of local volatility surfaces. While our recent paper in the journal Risk (De Marco et al. Risk 2:82–87, 2013, [3]) discusses our approximation formula from a practical and numerical perspective, the present paper focuses on rigorous proofs of the approximations. We apply the saddle point method (Heston model) and Hankel contour integration (variance gamma model).

Keywords Local volatility · Saddle point methods · Contour integration

1 Introduction

One of the main objectives in option pricing theory is to price exotic derivatives consistently with observed vanilla prices. According to the seminal work of Dupire [5], this can in principle be achieved, for a one-dimensional underlying, by a model with dynamics $dS_t/S_t = \sigma(S_t, t)dW_t$. As opposed to stochastic volatility models, here the volatility is a deterministic function of time and current underlying price. Any given smooth call price surface $C(K, T)$, for strikes $K > 0$ and maturities $T > 0$,

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can be recovered by a so-called local volatility model $dS_t/S_t = \sigma_{\mathrm{loc}}(S_t, t)dW_t$, where the volatility function is given by Dupire's formula [5]

$$\sigma^2_{\mathrm{loc}}(K, T) = \frac{2\partial_T C}{K^2 \partial_{KK} C}. \tag{1}$$

Exotic options can then be priced by Monte Carlo simulation. Local volatility models are of considerable practical importance, and serve as building blocks for more advanced models, e.g. local-stochastic-volatility (LSV) models.

In the present paper, we consider local volatility surfaces that arise from call prices that are generated by some model for the underlying. Our aim is to turn the knowledge of that model's mgf (moment generating function; of log-spot X_T) into asymptotic results of the corresponding local volatility surface. In [3], we described two applications of such approximations. One is to the design of local volatility parametrizations, whose asymptotic behavior may be matched to our results. Another application concerns model risk. Consider pricing under an "advanced" model (affine stochastic volatility, Lévy, etc.; anything with known mgf) versus a local volatility model. The relative differences between the prices has been named "toxicity index" in [13]. Roughly speaking, it measures the distance of the trade from vanilla options. The most consistent way to calculate this index is to use the local volatility model generated by the "advanced" model, because only then all vanillas will have zero toxicity. When computing the local volatility surface, our accurate approximations can then profitably replace other numerical methods in regimes where the latter become unstable (see [3] for details).

We suppose that the underlying price process $S_t = \exp(X_t)$ is a martingale under the pricing measure $\mathbb{P}$ and write $C(K, T)$ for its call price surface. For simplicity we assume zero interest rate throughout. If C is sufficiently smooth, then the associated local volatility function is given by Dupire's formula (1). Recall the main asymptotic formula from [3]:

$$\sigma^2_{\mathrm{loc}}(K, T) \sim \left. \frac{2\frac{\partial}{\partial T} m(s, T)}{s(s-1)} \right|_{s=\hat{s}(k,T)}, \tag{2}$$

where k denotes log-strike, and $\hat{s} = \hat{s}(k, T)$ is determined as solution of the saddle point equation

$$\frac{\partial}{\partial s} m(s, T) = k. \tag{3}$$

Here, $m(s, T) := \log M(s, T)$ is the logarithm of the moment generating function (mgf) M, which is defined by $M(s, T) := \mathbb{E}\exp(sX_T)$ and is analytic in the (maximal) strip $s_-(T) < \mathrm{Re}(s) < s_+(T)$. The numbers s_- and s_+ are called critical exponents. In this note, we will use (2) for $K \to \infty$, but other asymptotic regimes can also be covered [3, 8]; it is thus not only a local-volatility analogue of Lee's moment formula [11], but works also for maturity (or joint) asymptotics.

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As described in [3], formula (2) results from saddle point approximations of numerator and denominator of Dupire's formula, after inserting the Fourier representation of the call price:

$$\sigma^2_{\mathrm{loc}}(K,T) = \frac{2\partial_T C}{K^2 \partial_{KK} C} = \frac{2\int_{-i\infty}^{i\infty} \frac{\partial_T m(s,T)}{s(s-1)} e^{-ks} M(s,T)ds}{\int_{-i\infty}^{i\infty} e^{-ks} M(s,T)ds}. \tag{4}$$

(The real parts of the contours are in $(1, s_+)$.)

Whereas the focus of [3] is on numerical tests and applications, the present note gives proofs for the validity of (2), in the setting of the Heston and of the variance gamma model. As regards methodology, the proof for the Heston model uses a classical saddle point approach. Its most interesting ingredient, similarly to [7], is the use of ODE comparison results to furnish the necessary tail estimates, without taking recourse to the explicit form of the Heston mgf. The analysis is thus well suited to extension towards other affine stochastic volatility models. For the variance gamma model, the saddle point method is not appropriate. We apply another classical contour integration approach, based on Hankel contours, which seems to be new in mathematical finance.

2 The Heston Model

Even though practitioners seem to prefer local-stochastic-volatility models nowadays over the classical Heston model, it might still be useful for the two applications outlined in the introduction (model risk and parametrization design; recall that the large maturity Heston smile motivates the popular SVI parametrization of *implied* volatility [9]). The dynamics of the Heston model are

$$\begin{aligned} dS_t &= S_t\sqrt{Y_t}dW_t, \qquad S_0 = s_0 > 0, \\ dV_t &= (a + bV_t)dt + c\sqrt{V_t}dZ_t, \qquad V_0 = v_0 > 0, \end{aligned}$$

with $a \geq 0$, $b \leq 0$, $c > 0$, and $d\langle W, Z\rangle_t = \rho dt$ with $\rho \in (-1, 1)$.

Theorem 1 *In the Heston model with $\rho \leq 0$ (the relevant regime in practice, at least for equity models), the asymptotic equivalence (2) holds for $k \to \infty$. The explicit leading term is*

$$\sigma^2_{\mathrm{loc}}(K,T) \sim \frac{2}{s_+(s_+ - 1)R_1/R_2} \times k, \quad k \to \infty, \tag{5}$$

where $k = \log(K/S_0)$, $s_+ \equiv s_+(T)$ *and*

$$R_1 = Tc^2 s_+(s_+ - 1)\left[c^2(2s_+ - 1) - 2\rho c(s_+\rho c + b)\right] \tag{6}$$
$$- 2(s_+\rho c + b)\left[c^2(2s_+ - 1) - 2\rho c(s_+\rho c + b)\right]$$
$$+ 4\rho c\left[c^2 s_+(s_+ - 1) - (s_+\rho c + b)^2\right],$$
$$R_2 = 2c^2 s_+(s_+ - 1)\left[c^2 s_+(s_+ - 1) - (s_+\rho c + b)^2\right]. \tag{7}$$

Proof It was shown in [3] that the right hand side of (5) asymptotically equals the right hand side of (2). It thus remains to show that (2) holds for the Heston model as $k \to \infty$.

By the exponential decay of the Heston mgf towards $\pm i\infty$, the second equality in formula (4) is correct for the Heston model. For the saddle point analysis of (4), we employ the approximate saddle point

$$\hat{s}_{\text{approx}}(k) := s_+ - \beta k^{-1/2},$$

where $\beta = \frac{\sqrt{2v_0}}{c\sqrt{\sigma}}$, σ denotes the critical slope

$$\sigma(T) = -\frac{\partial T^*}{\partial s}(s_+(T)),$$

and

$$T^*(s) = \sup\{t \ge 0 : \mathbb{E}[e^{sX_t}] < \infty\}.$$

This is the same approximate saddle point as in [7]; see there for more details on its choice, and the definition of $\sigma(T)$ and $T^*(s)$. (In [7], our $\hat{s}_{\text{approx}}$ was called simply $\hat{s}$, since the *exact* saddle point of the denominator of (4), defined in (3), did not occur.) This approximate saddle may be used for both integrals in (4). As for the denominator, this was carried out in detail in [7], where an expansion of the Heston density $\partial_{KK}C$ was determined. The analysis of the numerator in (4) is similar, except that a new tail estimate is required. But first we discuss the local expansion around the saddle point. Let us fix a number $\alpha \in (\frac{2}{3}, \frac{3}{4})$ and define $h(k) = k^{-\alpha}$. Then, in the central range $|s - \hat{s}_{\text{approx}}(k)| \le h(k)$, we have

$$\frac{1}{s(s-1)} = \frac{1}{s_+(s_+ - 1)} + O(s_+ - s)$$
$$= \frac{1}{s_+(s_+ - 1)}\left(1 + O(k^{-1/2})\right)$$

and (cf. formula (19) in [3])

$$
\begin{aligned}
2\frac{\partial}{\partial T}m(s,T) &= \frac{2\beta^2}{\sigma(s_+-s)^2} + O\left(\frac{1}{s_+-s}\right) \\
&= \frac{2\beta^2}{\sigma}(\beta k^{-1/2} + O(k^{-\alpha}))^{-2} + O(k^{-1/2}) \\
&= \frac{2k}{\sigma}(1+O(k^{1/2-\alpha})).
\end{aligned}
$$

Therefore, the local expansions of the two integrands in (4) agree, up to a factor that is given by

$$
\frac{2\partial_T m(s,T)}{s(s-1)} = \frac{2k}{\sigma s_+(s_+-1)}(1+O(k^{1/2-\alpha})), \tag{8}
$$

where the error term holds uniformly w.r.t. the integration variable s. According to Theorem 1.2 of [7], we have

$$
\frac{1}{2i\pi}\int_{\hat{s}_{\text{approx}}-ih(k)}^{\hat{s}_{\text{approx}}+ih(k)} e^{-ks}M(s,T)ds \sim A_1 e^{(1-A_3)k+A_2\sqrt{k}}k^{-3/4+a/c^2} \tag{9}
$$

for certain constants A_1, $A_2 = 2\beta$, and $A_3 = s_+ + 1$. Analogously, we derive from (8) that

$$
\begin{aligned}
&\frac{1}{2i\pi}\int_{\hat{s}_{\text{approx}}-ih(k)}^{\hat{s}_{\text{approx}}+ih(k)} \frac{2\partial_T m(s,T)}{s(s-1)}e^{-ks}M(s,T)ds \\
&\qquad\qquad \sim \frac{2k}{\sigma s_+(s_+-1)} \times A_1 e^{(1-A_3)k+A_2\sqrt{k}}k^{-3/4+a/c^2}.
\end{aligned} \tag{10}
$$

Dividing (10) by (9) shows our claim (5), provided that the tails $|s-\hat{s}_{\text{approx}}(k)| > h(k)$ of the integrals can be discarded. For the denominator of (4), this was shown in Lemma A.3 of [7]. So we proceed with the numerator. We consider only the upper tail, as the lower one is handled by symmetry. By Lemma A.3 of [7], there is a constant $B > 0$ such that

$$
\left|\int_{\hat{s}_{\text{approx}}+ih(k)}^{\hat{s}_{\text{approx}}+iB} e^{-ks}M(s,T)ds\right| \le e^{(1-A_3)k}\exp(A_2\sqrt{k} - \tfrac{1}{2}\beta^{-1}k^{3/2-2\alpha} + O(\log k)). \tag{11}
$$

From formula (18) in [3] we obtain

$$
\left|\frac{\partial_T m(s,T)}{s(s-1)}\right| \le const \times k
$$

for all s on the contour in (11). This estimate can be absorbed into the factor $\exp(O(\log k))$ in (11), so that we conclude

$$\left|\int_{\hat{s}_{\text{approx}}+Ih(k)}^{\hat{s}_{\text{approx}}+iB} \frac{\partial_T m(s,T)}{s(s-1)} e^{-ks} M(s,T)ds\right| \\ \leq e^{(1-A_3)k} \exp(A_2\sqrt{k} - \tfrac{1}{2}\beta^{-1}k^{3/2-2\alpha} + O(\log k)). \tag{12}$$

This grows slower than the right hand side of (10) (compare the relevant factors $k^{-3/4+a/c^2}$ resp. $\exp(-\frac{1}{2}\beta^{-1}k^{3/2-2\alpha})$). As for $\mathrm{Im}(s) > B$, it was shown in [7] (Lemma A.2) that

$$\left|\int_{\hat{s}_{\text{approx}}+iB}^{\hat{s}_{\text{approx}}+i\infty} e^{-ks} M(s,T)ds\right| = O(\exp((1-A_3)k + \beta\sqrt{k})).$$

This was deduced from the exponential decay of $M(s,T)$ for large $\mathrm{Im}(s)$ (Lemma A.1 in [7]). The following lemma implies that the new factor $\partial_T m(s,T)/(s(s-1))$ grows only polynomially, so that the exponential decay of the integrand persists for the numerator of (4). This finishes the proof of Theorem 1. □

To state the lemma, recall that $m(s,t) = \phi(s,t) + v_0\psi(s,t)$, where ϕ and ψ satisfy the Riccati equations

$$\dot{\phi} = a\psi, \qquad \phi(0) = 0,$$
$$\dot{\psi} = \tfrac{1}{2}(s^2 - s) + \tfrac{1}{2}c^2\psi^2 + b\psi + s\rho c\psi, \qquad \psi(0) = 0.$$

We have to show that $\dot{m}$ grows only polynomially as $\mathrm{Im}(s) \to \infty$. Because of the Riccati equations, it suffices to show this for ψ. Let us write $\psi = f + ig$ and $s = \xi + iy$.

Lemma 2 *Let $T > 0$, and assume that the real part ξ of s stays bounded in some interval $1 \leq \xi \leq \xi_{\max}$. Then, there are positive constants $C_{i,T}$ ($i = 1, 2, 3$) such that for $y \geq y_0$, where y_0 depends only on $\xi_{\max}$ and the other (fixed) model parameters of the Heston model,*

$$-C_{3,T}y^2 \leq f(t) \leq -C_{1,T}y,$$
$$0 \leq g(t) \leq C_{2,T}\, y.$$

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In fact, we can take

$$
\begin{aligned}
C_{1,T} &= 1/\left(3c\right), \\
C_{2,T} &= \frac{1}{2}\left(2\xi_{\max} - 1\right)T, \\
C_{3,T} &= T\left(1 + \frac{c^2}{2}C_{2,T}^2\right).
\end{aligned}
$$

Proof It follows from the proof of Lemma A.1 in [7] that (e.g. with $C_{1,T} := T\theta = \frac{1}{c}\sqrt{1/6} \le \frac{1}{3c}$)

$$
f(t) \le -T\theta y = -\frac{1}{c}\sqrt{1/6}y \le -\frac{1}{3c}y =: -C_{1,T}y.
$$

We next provide a similar upper estimate for g. To this end we first show that $g = g(t)$ remains ≥ 0 for all times $t > 0$. The differential equation for g,

$$
\dot{g} = \frac{1}{2}\left(2\xi y - y\right) + c^2 fg - \gamma g, \qquad g(0) = 0,
$$

implies the first order Euler estimate

$$
\begin{aligned}
g\left(t\right) &= g(0) + \left\{\frac{1}{2}\left(2\xi y - y\right) + c^2 f(0)g(0) - \gamma g(0)\right\}t + o(t) \\
&= \underbrace{\frac{1}{2}\left(2\xi y - y\right)}_{>0}t + o(t),
\end{aligned}
$$

and hence g is positive (even strictly so) on some interval $(0, \varepsilon_1)$. Assume this interval is maximal in the sense that $g(\varepsilon_1) = 0$ and g is (strictly) negative on some further interval $(\varepsilon_1, \varepsilon_2)$. Clearly then $\dot{g}(\varepsilon_1) \le 0$, which contradicts the information from the differential equation: indeed, using $g(\varepsilon_1) = 0$, we obtain the contradiction

$$
\dot{g}(\varepsilon_1) = \underbrace{\frac{1}{2}\left(2\xi y - y\right)}_{>0}.
$$

The observation that $g \geq 0$ is useful to us, since it leads, together with $f \leq -C_{1,T}y$ and $\gamma \geq 0$, to the differential inequality

$$\begin{aligned}\dot{g} &= \frac{1}{2}(2\xi y - y) + c^2 fg - \gamma g \\ &\leq \frac{1}{2}(2\xi y - y) - \left(c^2 C_{1,T} + \gamma\right) g \\ &\leq \frac{1}{2}(2\xi y - y),\end{aligned}$$

and hence to the upper estimate

$$\forall 0 \leq t \leq T : g(t) \leq \frac{1}{2}(2\xi_{\max} - 1)\, T \times y =: C_{2,T}y.$$

We can feed this upper estimate on g back in the differential equation for f to obtain a lower estimate

$$\begin{aligned}\dot{f} &= \frac{1}{2}\left(\xi^2 - y^2 - \xi\right) + \frac{c^2}{2}\left(f^2 - g^2\right) - \gamma f \\ &\geq \frac{1}{2}\left(\xi^2 - y^2 - \xi\right) + \frac{c^2}{2} f^2 - \frac{c^2}{2} C_{2,T}^2 y^2 - \gamma f \\ &= -\frac{1}{2}\left(1 + c^2 C_{2,T}^2\right) y^2 + \frac{1}{2}\left(\xi^2 - \xi\right) - \gamma f + \frac{c^2}{2} f^2 \\ &\geq -\frac{1}{2}\left(1 + c^2 C_{2,T}^2\right) y^2 + \frac{1}{2}\left(\xi^2 - \xi\right) - \gamma f \\ &\geq -\left(1 + \frac{c^2}{2} C_{2,T}^2\right) y^2 - \gamma f,\end{aligned}$$

where in the last step we assume that y is large enough so that the extra amount subtracted (at least: $\frac{1}{2}y^2$) is larger than $\frac{1}{2}\left(\xi^2 - \xi\right)$, which remains bounded. We also know that $f(t) \leq -C_{1,T}y \leq 0$ for all $0 \leq t \leq T$. It follows that $-\gamma f \geq 0$ and omission leads to our final lower bound on $\dot{f}$, namely

$$\dot{f} \geq -\left(1 + \frac{c^2}{2} C_{2,T}^2\right) y^2.$$

This entails immediately

$$f(t) \geq -T\left(1 + \frac{c^2}{2} C_{2,T}^2\right) y^2 =: -C_{3,T}y^2. \qquad \square$$

3 The Variance Gamma Model

The mgf of the variance gamma model is

$$M(s,T) = e^{Tbs}(1-\theta\nu s - \tfrac{1}{2}\sigma^2\nu s^2)^{-T/\nu},$$

where $\sigma, \nu > 0$ and $\theta \in \mathbb{R}$. The "drift" $b = \nu^{-1}\log(1-\theta\nu-\frac{1}{2}\sigma^2\nu)$ is chosen such that $S = e^X$ becomes a martingale (w.l.o.g., $S_0 = 1$). For fixed T with $0 < T/\nu < \frac{1}{2}$, the density $\partial_{KK}C(K,T)$ of S_T has a singularity at the origin. Indeed, it behaves as $\approx |k|^{2T/\nu-1}$, which easily follows from the integral representation of the density [1] (as always, $k = \log K$). At the money, the denominator of the Dupire formula (1) thus explodes for small T. If $T/\nu > \frac{1}{2}$, then the density is continuous. This lack of smoothness is just an additional issue on top of a common feature of jump models: The associated local volatility surface explodes as $T \to 0$, and so the local volatility SDE

$$dS/S = \sigma_{\mathrm{loc}}(S,t)dW \tag{13}$$

does not make sense on $[0,\infty) \ni t$.

However, following [8], we can start a Monte Carlo simulation of (13) at a time $T_0 > 0$ (here, $T_0 > \nu/2$) instead of time zero. With the appropriate *stochastic* initial value, sampled from the density $\partial_{KK}C(K,T_0)$, we recover call prices from time T_0 on. (T_0 is called ε in [8].) This gives a meaning to the local volatility surface of a jump model, without appealing to the practically challenging approach of local Lévy models [2]. Our aim is not to make this fully rigorous for the variance gamma model (or other jump models), which would require to show that (13) admits a unique strong solution on $[T_0,\infty)$. Our focus, instead, is on a rigorous proof that (2) is valid in this setting. To ensure the validity of the Fourier representations of density and call price, we even assume $T/\nu > 1$ (instead of $T/\nu > \frac{1}{2}$).

Theorem 3 *In the variance gamma model, formula* (2) *holds for* $k = \log K \to \infty$. *The explicit leading term is*

$$\sigma^2_{\mathrm{loc}}(K,T) \sim \frac{2\log(k/T)}{\nu s_+(s_+-1)}, \quad k \to \infty. \tag{14}$$

Note that the numerator of (14) is $\sim 2\log k$. We kept the T-dependence, because the same analysis works for fixed k and $T \to 0$, and in fact for any asymptotic regime with $k/T \to \infty$. This is a common feature of Lévy models, since the right-hand side of (2) depends on k and T only through k/T.

Proof We write the moment generating function as

$$M(s,T) = e^{bTs}\Big(\tfrac{1}{2}\sigma^2\nu(s_+-s)(s-s_-)\Big)^{-T/\nu}$$

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where the critical moments are

$$s_{\pm} = \frac{-\nu\theta \pm \sqrt{2\nu\sigma^2 + \nu^2\theta^2}}{\nu\sigma^2}.$$

We analyze the denominator of (4), i.e., the density. The arguments for the numerator are analogous (see below). The shift $k \to k + bT$ makes it clear that we may w.l.o.g. assume that $b = 0$. The main part of the saddle point equation (3) is $T/(\nu(s_+ - s)) = k$, and so

$$\hat{s} = s_+ - \frac{T}{\nu k} + O(k^{-2}).$$

The saddle point approximation of the density then is

$$\frac{1}{2i\pi}\int_{-i\infty}^{i\infty} e^{-ks} M(s,T)ds \approx \frac{\exp(m(\hat{s},t) - k\hat{s})}{\sqrt{2\pi m''(\hat{s},T)}}. \tag{15}$$

The interesting point now is that (15) is *wrong* for the variance gamma model, inasmuch as asymptotic equality does not hold. The algebraic singularity of the mgf is not pronounced enough to make the saddle point method work; see also the remark after the proof. For a correct analysis, we use an integration contour as in Fig. 1. The U-shaped notch, denoted by $\mathcal{C}(k)$, extends a bit to the right of the singularity s_+, and captures enough asymptotic information from it. By transformation into a so called Hankel path, Hankel's representation of the Gamma function can be invoked after termwise integration of a local expansion. This "Hankel contour approach" is well known in analytic combinatorics, in particular, from the so-called singularity analysis of generating functions [6].

Let us first argue that the integrals over the dashed lines in Fig. 1 can be discarded. By symmetry, it suffices to consider the upper one. The real part of s is then $\mathrm{Re}(s) = s_+ + (\log k)/k$. First suppose that s is away from the singularity, say $\mathrm{Im}(s) > 1$. The

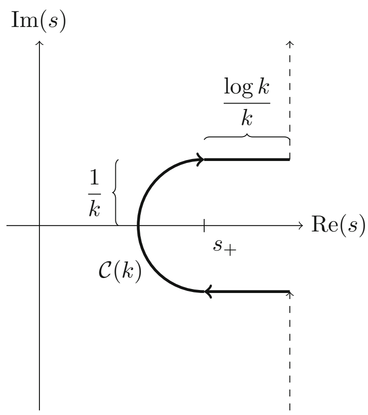

Fig. 1 The contour $\mathcal{C}(k)$, a small notch embracing the critical moment s_+

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integral of $((s_+ - s)(s - s_-))^{-T/\nu}$ over this part of the contour is $O(1)$, and so we get the bound $O(e^{-k\mathrm{Re}(s)}) = O(e^{-ks_+}/k)$. Now consider s with $1/k \le \mathrm{Im}(s) < 1$. We estimate the resulting integral by the length of the contour, which is $O(1)$, times the absolute value of the integrand at the lower endpoint $s = s_+ + (\log k)/k + i/k$. The latter is easily seen to be $O(e^{-ks_+}k^{T/\nu-1}(\log k)^{-T/\nu})$.

We will now show that the integral over $\mathcal{C}(k)$ is of order $e^{-ks_+}k^{T/\nu-1}$, so that the tail estimates we have just derived are good enough. The factor $(s - s_-)$ is locally almost constant; we have, uniformly for $s \in \mathcal{C}(k)$,

$$M(s,T) \sim c_1(s_+ - s)^{-T/\nu}, \quad k \to \infty,$$

where $c_1 = c_1(T) = (\sigma^2\nu(s_+ - s_-)/2)^{-T/\nu}$. Therefore,

$$\int_{\mathcal{C}(k)} e^{-ks} M(s,T)ds \sim \int_{\mathcal{C}(k)} e^{-ks} \frac{c_1}{(s_+ - s)^{T/\nu}} ds.$$

The change of variables $s = s_+ - w/k$ transforms this into

$$\begin{aligned}\frac{e^{-ks_+}}{k}\int_{\mathcal{H}(k)} e^w c_1 \left(\frac{k}{w}\right)^{T/\nu} dw &= c_1 \frac{e^{-ks_+}}{k^{1-T/\nu}} \int_{\mathcal{H}(k)} e^w w^{-T/\nu} dw \\ &\sim c_1 \frac{e^{-ks_+}}{k^{1-T/\nu}} \int_{\mathcal{H}(\infty)} e^w w^{-T/\nu} dw.\end{aligned}$$

The integration paths are displayed in Fig. 2. The right one, $\mathcal{H}(\infty)$, is called a Hankel contour; $\mathcal{H}(k)$ is a Hankel contour truncated at $\mathrm{Re}(s) = -\log k$. Now recall Hankel's representation for the Gamma function [12]:

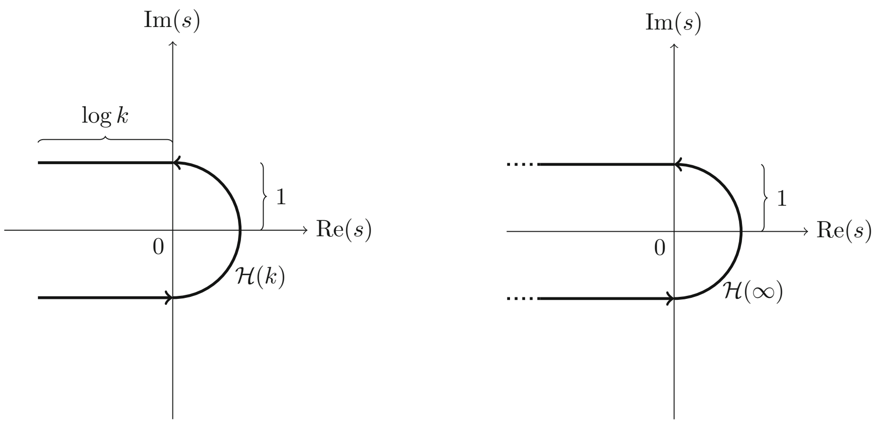

Fig. 2 The integration contours $\mathcal{H}(k)$ and $\mathcal{H}(\infty)$. The *dots* should indicate that the contour $\mathcal{H}(\infty)$ extends to $-\infty$

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$$\frac{1}{2i\pi}\int_{\mathcal{H}(\infty)} e^w w^{-z} dw = \frac{1}{\Gamma(z)}.$$

We thus arrive at

$$\frac{1}{2i\pi}\int_{-i\infty}^{i\infty} e^{-ks} M(s,T)ds \sim \frac{c_1}{\Gamma(T/\nu)} e^{-ks_+} k^{T/\nu-1}. \tag{16}$$

The numerator of (4) can be treated analogously, with a very similar tail estimate. The contribution of the new factor to the local expansion is

$$\begin{aligned} 2\frac{\partial_T m(s,T)}{s(s-1)} &\sim \frac{2/\nu}{s_+(s_+-1)} \log \frac{1}{s_+-s} \\ &= \frac{2/\nu}{s_+(s_+-1)} \log \frac{k}{w} \\ &\sim \frac{2\log k}{\nu s_+(s_+-1)}, \end{aligned}$$

and so

$$2\int_{-i\infty}^{i\infty} \frac{\partial_T m(s,T)}{s(s-1)} e^{-ks} M(s,T)ds \sim \frac{2\log k}{\nu s_+(s_+-1)} \times \frac{c_1}{\Gamma(T/\nu)} e^{-ks_+} k^{T/\nu-1}. \tag{17}$$

Dividing (17) by (16) yields the desired result. □

As mentioned in the preceding proof, the saddle point formula (15) is not an asymptotic equivalence for the variance gamma model. But, as we have shown, our formula (2) is still correct. What happens is that (15), and its counterpart for the numerator of (4), are *almost* correct: They are only off by a constant factor. (This phenomenon has already been observed for similar integrals in [4].) This constant factor is the same for both integrals, and thus cancels in the quotient (4). Therefore, our asymptotic formula (2) extends well beyond models where the saddle point method is applicable. In fact, we conjecture that the formula holds whenever the mgf explodes close to the singularity s_+.

4 Other Jump Models

Without giving proofs, we briefly discuss local volatility asymptotics for two other jump models. The mgf of Kou's double exponential Lévy jump diffusion is given by

$$M(s,T) = \exp\left(T\left(bs + \frac{\sigma^2 s^2}{2} + \lambda\left(\frac{\lambda_+ p}{\lambda_+ - s} + \frac{\lambda_-(1-p)}{\lambda_- + s} - 1\right)\right)\right).$$

The critical moment is $s_+ = \lambda_+$, and the saddle point is located at

$$\hat{s} \approx s_+ - \sqrt{\frac{\lambda\lambda_+ pT}{k}}.$$

The singularity type, the same as in the Heston model, is amenable to the saddle point method. Formula (2) can thus certainly be verified, and yields

$$\sigma^2_{\text{loc}}(K,T) \sim \frac{2\sqrt{\lambda p}}{\sqrt{\lambda_+ T}(\lambda_+ - 1)} k^{1/2}, \quad k \to \infty.$$

For $T \to 0$, the blowup of local volatility is of order $T^{-1/2}$. (Just as the Hankel contour analysis in the proof of Theorem 3 can be carried out for any asymptotic regime with $k/T \to \infty$, the same is true when applying the saddle point method to the local volatility surface of a Lévy model.)

Finally, we consider the normal inverse Gaussian (NIG) model. The mgf

$$M(s,T) = \exp\left(Tbs + \delta T\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + s)^2}\right)\right)$$

has no blow-up at the critical moment

$$s_+ = \alpha - \beta,$$

but a square-root type singularity, with local expansion

$$M(s,T) \approx e^{Tbs_+ + \delta T\sqrt{\alpha^2-\beta^2}}\left(1 - \delta T\sqrt{2\alpha}\sqrt{s_+ - s}\right). \tag{18}$$

It is still true that $\sigma^2_{\text{loc}}(K,T)$ asymptotically depends, via (4), on the local behavior of $M(s,T)$ near s_+. However, the approximation (2) hinges on the *first* term of the local expansion of $M(s,T)$. It therefore fails to capture the asymptotics of $\sigma^2_{\text{loc}}(K,T)$ here, which depend on the first *singular* term (the term $\sqrt{s_+ - s}$ in (18)). The NIG model is thus one of the few examples where (2) is wrong. (It gives the qualitatively correct result of convergence to a constant, but a wrong one.) The Hankel contour analysis in the proof of Theorem 3 can be adapted to handle this situation. The result is that local volatility tends to a constant for $k \to \infty$. This fact may be understood by comparing the NIG marginals with those of Heston's in the time $T \to \infty$ regime (this link is made precise in [10]). In particular, the result is then consistent with the Heston asymptotics (5) of local vol, given that the $O(k)$ term carries a factor $\approx 1/T$ which tends to zero as $T \to \infty$.

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