International Journal of Theoretical and Applied Finance
Vol. 16, No. 8 (2013) 1350044 (14 pages)

DOI: 10.1142/S0219024913500441

DIGITAL DOUBLE BARRIER OPTIONS: SEVERAL BARRIER PERIODS AND STRUCTURE FLOORS

SÜHAN ALTAY

Vienna University of Technology
Wiedner Hauptstrasse 8–10/105-1, A-1040 Vienna, Austria
suhan.altay@fam.tuwien.ac.at

STEFAN GERHOLD

Vienna University of Technology
Wiedner Hauptstrasse 8–10/105-1, A-1040 Vienna, Austria
sgerhold@fam.tuwien.ac.at

RAINER HAIDINGER

Raiffeisen Kapitalanlage-Gesellschaft m.b.H.
A-1010 Vienna, Austria
rainer.haidinger@gmail.com

KARIN HIRHAGER

Christian Doppler Laboratory for Portfolio Risk Management
Vienna University of Technology
Wiedner Hauptstrasse 8–10/105-1, A-1040 Vienna, Austria
karin.hirhager@fam.tuwien.ac.at

Received 21 September 2012
Revised 7 May 2013
Accepted 15 August 2013
Published 3 December 2013

We determine the price of digital double barrier options with an arbitrary number of barrier periods in the Black–Scholes model. This means that the barriers are active during some time intervals, but are switched off in between. As an application, we calculate the value of a structure floor for structured notes whose individual coupons are digital double barrier options. This value can also be approximated by the price of a corridor put.

Keywords: Double barrier option; digital option; binary option; structure floor; occupation time; corridor option.

1. Introduction

We consider digital double barrier options with an arbitrary number of barrier periods. This means that the holder receives the payoff only if the underlying stays between the two barriers in certain specified time intervals. While such contracts might make sense by themselves (as a weather or energy derivative with seasonal barriers, for example), our motivation is to use them for the pricing of certain structured notes with several coupons. Such trades often feature an aggregate floor at the final coupon date, which increases the total payoff to a guaranteed amount if the sum of the coupons is less than this amount. Pricing this terminal premium requires the law of the sum of the coupons, which can be recovered from its moments. If the individual coupons of the note are digital barrier options, then these moments can be computed from the prices of options of the kind described above, where the sets of barrier periods are subsets of the coupon periods of the note.

Recall that Monte Carlo pricing of barrier contracts is tricky, because the discretization produces a downward bias for the barrier hitting probability. For single barrier options, this difficulty can be overcome using the explicit law of the maximum of the Brownian bridge [1, 4]. For double barrier options, the exit probability of the Brownian bridge is not known; see [3] for an approximate approach using sample path large deviations. These numerical challenges led us to investigate exact valuation formulas. Among the many other works dealing with double barrier options, let us mention [19], (double barrier calls with curved boundaries), [14], (double barrier digitals), [10], (onion options, see also [20]), and [17], (new barriers activated at barrier hitting time).

The paper is structured as follows. In Sec. 2, we define the payoffs we are interested in and price them for a single barrier period. Section 3 extends the result to arbitrarily many periods of active barriers, which is illustrated numerically in Sec. 4. Our main application, namely the pricing of structure floors, is presented in Sec. 5. Since our exact pricing formula is fairly involved, we consider an asymptotic approximation for a large number of periods in Sec. 6.

2. Preliminaries and Pricing for One Period

We assume that the underlying $(S_t)_{t\geq 0}$ has the risk-neutral dynamics

$$dS_t/S_t = r dt + \sigma dW_t$$

with constant interest rate $r > 0$, volatility $\sigma > 0$ and a standard Brownian motion W. Consider a digital barrier option with two barriers B_{low} and B_{up} that are activated at time $T_0 > 0$ and stay active for a time period of length $P > 0$. At maturity $T_0 + P$, the payoff is one unit of currency if the underlying has stayed between the two barriers:

$$C_1 := \mathbf{1}_{\{B_{\mathrm{low}} < S_t < B_{\mathrm{up}},\ t \in [T_0, T_0+P]\}}. \tag{2.1}$$

Let us denote the price of this "one-period double barrier digital" by

$$\mathrm{BD}(S_t, t; \{T_0\}, P, B_{\mathrm{low}}, B_{\mathrm{up}}, r) := e^{-r(T_0+P-t)}\mathbf{E}[C_1|\mathcal{F}_t], \tag{2.2}$$

where $\mathbf{E}$ is the expectation w.r.t. the pricing measure $\mathbf{P}$. In the terminology of [15], this is a *rear-end* barrier option, because the two barriers are alive only towards the end of the contract, namely between T_0 and maturity $T_0 + P$. Hui in [15] has determined the price for a barrier *call* of this kind. The digital case is a simple modification, but we go through it to prepare the calculation of the price for several barrier periods (see Sec. 3). In probabilistic terms, we are integrating the probability to stay between the barriers (see [5, p. 616]), with S_{T_0} viewed as a parameter, against the law of S_{T_0}. For the multi-period case, the PDE approach seems more appropriate, though. The value function

$$f(S,t) := \mathrm{BD}(S,t;\{T_0\},P,B_{\mathrm{low}},B_{\mathrm{up}},r)$$

satisfies the Black–Scholes PDE

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf = 0$$

with the terminal condition $f(S, T_0+P) = 1$, for $S \in (B_{\mathrm{low}}, B_{\mathrm{up}})$, and the boundary conditions $f(B_{\mathrm{low}}, t) = f(B_{\mathrm{up}}, t) = 0$ for $t \in [T_0, T_0 + P]$. We use the standard transformation $f(S,t) = e^{\alpha x + \beta\tau} U(x,\tau)$, where

$$\begin{aligned} x &:= \log(S/B_{\mathrm{low}}), & \tau &:= \frac{1}{2}\sigma^2(T_0 + P - t), \\ \alpha &:= -\frac{1}{2}\left(\frac{2}{\sigma^2}r - 1\right), & \beta &:= -\frac{2r}{\sigma^2} - \alpha^2, \end{aligned} \tag{2.3}$$

to transform the Black–Scholes PDE into the heat equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial \tau}. \tag{2.4}$$

The time points $(0, T_0, T_0+P)$ are thus converted to $(\frac{1}{2}\sigma^2(T_0+P), p, 0)$, where $p := \frac{1}{2}\sigma^2 P$ is the barrier period length in the new time scale. The boundary conditions in the new coordinates are

$$U(0,\tau) = U(L,\tau) = 0, \quad \tau \in [0,p], \tag{2.5}$$

where $L := \log(B_{\mathrm{up}}/B_{\mathrm{low}})$. The terminal condition translates to the initial condition

$$U(x,0) = e^{-\alpha x}, \quad x \in (0,L). \tag{2.6}$$

Proposition 2.1. *For $0 < t < T_0$, the price of a barrier digital with barrier period $[T_0, T_0 + P]$ and payoff C_1 at $T_0 + P$ (see (2.1)) is*

$$\begin{aligned} &\mathrm{BD}(S,t;\{T_0\},P,B_{\mathrm{low}},B_{\mathrm{up}},r) \\ &\quad = \sqrt{2\pi}\left(\frac{S}{B_{\mathrm{low}}}\right)^{\alpha} \sum_{k=1}^{\infty} k \frac{1-(-1)^k e^{-\alpha L}}{\alpha^2 L^2 + k^2\pi^2} e^{-(\frac{k\pi}{L})^2 p + \beta\tau} \\ &\qquad \times \int_{-\frac{x}{\sqrt{2(\tau-p)}}}^{\frac{L-x}{\sqrt{2(\tau-p)}}} \sin\left(\frac{k\pi}{L}(x + y\sqrt{2(\tau-p)})\right) e^{-y^2/2} dy. \end{aligned} \tag{2.7}$$

Proof. We have to solve the problems (2.4)–(2.6). First consider the rectangle $(0,L)\times(0,p)$. There the solution, which is unique [11, p. 358], can be found by separation of variables [11, Sec. 4.1]:

$$U(x,\tau)=\sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L}x\right)e^{-(\frac{k\pi}{L})^2\tau}, \quad (x,\tau)\in(0,L)\times(0,p), \tag{2.8}$$

where

$$b_k := \frac{2}{L}\int_0^L e^{-\alpha x_1}\sin\left(\frac{k\pi}{L}x_1\right)dx_1 = 2k\pi\frac{1-(-1)^k e^{-\alpha L}}{\alpha^2L^2+k^2\pi^2}$$

are the Fourier coefficients of the boundary function $U(x,0)=e^{-\alpha x}$. To verify that this is indeed the solution, it suffices to appeal to the standard criterion for exchanging derivative and series [21, p. 152].

At $\tau=p$, the solution is given by (2.8) for $0<x<L$ and vanishes otherwise. Inserting $\tau=p$ into (2.8) yields

$$U(x,p)=\begin{cases}\displaystyle\sum_{k=1}^{\infty}2k\pi\frac{1-(-1)^ke^{-\alpha L}}{\alpha^2L^2+k^2\pi^2}\sin\left(\frac{k\pi}{L}x\right)e^{-(\frac{k\pi}{L})^2p}, & 0<x<L\\ 0, & x\le 0 \text{ or } x\ge L.\end{cases} \tag{2.9}$$

Now we solve for U in the region $\mathbb{R}\times(p,\frac{1}{2}\sigma^2(T_0+P))$. There are no boundary conditions here, since the barriers are not active in the interval $(0,T_0)$ (in the original time scale). A solution is found by convolving the initial condition (2.9) with the heat kernel [11, p. 47]:

$$\begin{aligned}U(x,\tau)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}U(x+y\sqrt{2(\tau-p)},p)e^{-y^2/2}dy\\&=\frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2(\tau-p)}}}^{\frac{L-x}{\sqrt{2(\tau-p)}}}U(x+y\sqrt{2(\tau-p)},p)e^{-y^2/2}dy.\end{aligned} \tag{2.10}$$

Inserting (2.9) and rearranging yields (2.7). It remains to argue that the solution (2.10) is the right one, i.e. that it indeed equals the transformation of the value function (2.2). By Tikhonov's classical uniqueness theorem [16, p. 216ff], the solution in the strip $\mathbb{R}\times(p,\frac{1}{2}\sigma^2(T_0+P))$ is unique if we restrict attention to functions admitting bounds of the form $c_1\exp(c_2|x|^2)$ with positive constants c_1 and c_2. Now note that (2.9), and hence also (2.10), is bounded by a constant, and that the solution U we seek is of at most exponential growth, since our value function $f(S,t)=e^{\alpha x+\beta\tau}U(x,\tau)$ is bounded. □

3. Double Barrier Digitals with Arbitrarily Many Periods

For n tenor dates

$$0<T_0<\cdots<T_{n-1}$$

and a fixed period length $P>0$, we consider a contract that pays one unit of currency at time $T_{n-1}+P$, if the underlying has remained between the two barriers

B_{low} and B_{up} during each of the time intervals $[T_i, T_i + P]$, $i = 0, \dots, n-1$. We assume that these do not overlap, i.e. $T_i + P \le T_{i+1}$ for $i = 0, \dots, n-2$. By the risk-neutral pricing formula, the price of this "multi-period double barrier digital" is given by

$$\mathrm{BD}(S_t, t; \{T_0, \dots, T_{n-1}\}, P, B_{\text{low}}, B_{\text{up}}, r) := e^{-r(T_{n-1}+P-t)}\mathbf{E}\left[\prod_{i=1}^{n} C_i \,\middle|\, \mathcal{F}_t\right], \quad (3.1)$$

where

$$C_i := \mathbf{1}_{\{B_{\text{low}} < S_t < B_{\text{up}},\ t \in [T_{i-1}, T_{i-1}+P]\}}.$$

To calculate the price, we use the coordinate change (2.3) again (with T_{n-1} in place of T_0). The n barrier periods $[T_i, T_i + P]$ are mapped to $[\tau_i, \tau_i + p]$, where

$$\tau_i := \frac{1}{2}\sigma^2(T_{n-1} - T_{i-1}), \quad i = n, \dots, 1,$$

are the images of the barrier period endpoints under the coordinate change (see Fig. 1). Define the following auxiliary functions:

$$\begin{aligned} &h_j(k_1, \dots, k_{j+1}; x_1, \dots, x_{j+1}; y_1, \dots, y_{j+1}; x, \tau) \\ &\quad := \frac{1}{\sqrt{2\pi}} e^{-y_{j+1}^2/2} \mathbf{1}_{\left[-\frac{x}{\sqrt{2(\tau-(\tau_{n-j}+p))}}, \frac{L-x}{\sqrt{2(\tau-(\tau_{n-j}+p))}}\right]}(y_{j+1}) \\ &\qquad \times g_j(k_1, \dots, k_{j+1}; x_1, \dots, x_{j+1}; y_1, \dots, y_j; \\ &\qquad x + y_{j+1}\sqrt{2(\tau - (\tau_{n-j}+p))}, \tau_{n-j} + p) \end{aligned}$$

and

$$\begin{aligned} &g_j(k_1, \dots, k_{j+1}; x_1, \dots, x_{j+1}; y_1, \dots, y_j; x, \tau) \\ &\quad := \frac{2}{L} \sin\frac{k_{j+1}\pi x_{j+1}}{L} \sin\frac{k_{j+1}\pi x}{L} e^{-(k_{j+1}\pi/L)^2(\tau - \tau_{n-j})} \\ &\qquad \times h_{j-1}(k_1, \dots, k_j; x_1, \dots, x_j; y_1, \dots, y_j; x_{j+1}, \tau_{n-j}), \end{aligned}$$

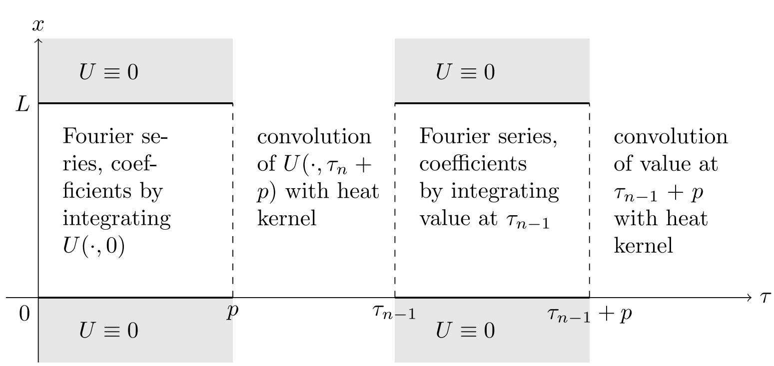

Fig. 1. For an arbitrary number of barrier periods consider the coordinate change (2.3) and solve the boundary value problem by Fourier series within the barrier period and by convolving with the heat kernel when the barriers are not active.

with the recursion starting at

$$g_0(k_1; x_1; ; x, \tau) := \frac{2}{L} e^{-\alpha x_1} \sin \frac{k_1 \pi x_1}{L} \sin \frac{k_1 \pi x}{L} e^{-(k_1\pi/L)^2 \tau}. \tag{3.2}$$

The following theorem contains our pricing formula. The first formula (3.3) is for time points inside a barrier period, whereas the second expression (3.4) holds for valuation times where the barriers are not active.

Theorem 3.1. *The value function* (3.1) *equals* $e^{\alpha x + \beta \tau} U(x, \tau)$, *where for* $0 \le j < n$, $\tau_{n-j} \le \tau \le \tau_{n-j} + p$, $0 < x < L$, *we have*

$$\begin{aligned} U(x, \tau) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{j} \underbrace{\int_0^L \cdots \int_0^L}_{j+1} \sum_{k_1=0}^{\infty} \cdots \sum_{k_{j+1}=0}^{\infty} \\ g_j(k_1, \ldots, k_{j+1}; x_1, \ldots, x_{j+1}; y_1, \ldots, y_j; x, \tau) \\ \times dx_1 \ldots dx_{j+1} dy_1 \ldots dy_j, \end{aligned} \tag{3.3}$$

whereas for $0 \le j < n$, $\tau_{n-j} + p < \tau < \tau_{n-(j+1)}$ (*with* $\tau_0 := \infty$), $x \in \mathbb{R}$, *we have*

$$\begin{aligned} U(x, \tau) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{j+1} \underbrace{\int_0^L \cdots \int_0^L}_{j+1} \sum_{k_1=0}^{\infty} \cdots \sum_{k_{j+1}=0}^{\infty} \\ h_j(k_1, \ldots, k_{j+1}; x_1, \ldots, x_{j+1}; y_1, \ldots, y_{j+1}; x, \tau) \\ \times dx_1 \ldots dx_{j+1} dy_1 \ldots dy_{j+1}. \end{aligned} \tag{3.4}$$

Proof. The idea is to iterate the argument of Proposition 2.1 (see Fig. 1). We use separation of variables in the barrier periods, and convolution with the heat kernel for the periods in between. The required initial condition at the left boundary comes from the previous step of the iteration (for $j = 0$ is also from the payoff). The discussion of existence and uniqueness is analogous to the proof of Proposition 2.1, and we omit the details.

For $j = 0$, formula (3.3) is identical to (2.8). To show (3.4) for $j = 0$, let $p < \tau < \tau_{n-1}$ (recall that $\tau_n = 0$) and $x \in \mathbb{R}$, and use (2.10) and (2.8) to obtain

$$\begin{aligned} U(x, \tau) &= \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2(\tau-p)}}}^{\frac{L-x}{\sqrt{2(\tau-p)}}} U(x + y_1\sqrt{2(\tau - p)}, p) e^{-y_1^2/2} dy_1 \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{1}_{\left[-\frac{x}{\sqrt{2(\tau-p)}}, \frac{L-x}{\sqrt{2(\tau-p)}}\right]}(y_1) \\ &\quad \times \int_0^L \sum_{k_1=0}^{\infty} g_0(k_1; x_1; ; x + y_1\sqrt{2(\tau - p)}, p) e^{-y_1^2/2} dx_1 dy_1 \\ &= \int_{-\infty}^{\infty} \int_0^L \sum_{k_1=0}^{\infty} h_0(k_1; x_1; y_1; x, \tau) dx_1 dy_1. \end{aligned}$$

This is (3.4) for $j = 0$.

Next consider a rectangle

$$(\tau, x) \in (\tau_{n-j}, \tau_{n-j} + p) \times (0, L), \quad 1 \le j < n. \tag{3.5}$$

At the left boundary, the solution is $x_{j+1} \mapsto U(x_{j+1}, \tau_{n-j})$. By the induction hypothesis, it equals (3.4) with j replaced by $j-1$:

$$\begin{aligned} U(x_{j+1}, \tau_{n-j}) = &\underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{j} \underbrace{\int_0^L \cdots \int_0^L}_{j} \sum_{k_1=0}^{\infty} \cdots \sum_{k_j=0}^{\infty} \\ &h_{j-1}(k_1, \ldots, k_j; x_1, \ldots, x_j; y_1, \ldots, y_j; x_{j+1}, \tau_{n-j}) \\ &\times dx_1 \ldots dx_j dy_1 \ldots dy_j. \end{aligned} \tag{3.6}$$

The solution in the rectangle (3.5) is thus obtained by separation of variables as

$$U(x, \tau) = \sum_{k_{j+1}=0}^{\infty} b_{k_{j+1}} \sin\left(\frac{k_{j+1}\pi}{L} x\right) e^{-\left(\frac{k_{j+1}\pi}{L}\right)^2 (\tau - \tau_{n-j})}, \tag{3.7}$$

where

$$b_{k_{j+1}} := \frac{2}{L} \int_0^L U(x_{j+1}, \tau_{n-j}) \sin\left(\frac{k_{j+1}\pi}{L} x_{j+1}\right) dx_{j+1} \tag{3.8}$$

denote now the Fourier coefficients of $x_{j+1} \mapsto U(x_{j+1}, \tau_{n-j})$. Inserting (3.6) into (3.8) and then (3.8) into (3.7) yields (3.3), by the definition of g_j.

Finally, consider a strip

$$(\tau, x) \in (\tau_{n-j} + p, \tau_{n-(j+1)}) \times \mathbb{R}, \quad 1 \le j < n. \tag{3.9}$$

At the left boundary, we use (3.3) as induction hypothesis. The solution thus vanishes for $x \notin (0, L)$, and for $\tau = \tau_{n-j} + p$ and $x \in (0, L)$ it is

$$\begin{aligned} U(x, \tau_{n-j} + p) = &\underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{j} \underbrace{\int_0^L \cdots \int_0^L}_{j+1} \sum_{k_1=0}^{\infty} \cdots \sum_{k_{j+1}=0}^{\infty} \\ &g_j(k_1, \ldots, k_{j+1}; x_1, \ldots, x_{j+1}; y_1, \ldots, y_j; x, \tau_{n-j} + p) \\ &\times dx_1 \ldots dx_{j+1} dy_1 \ldots dy_j. \end{aligned} \tag{3.10}$$

As above, the solution in the strip (3.9) is found by convolution with the heat kernel:

$$\begin{aligned} U(x, \tau) = &\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{1}_{\left[-\frac{x}{\sqrt{2(\tau - (\tau_{n-j} - p))}}, \frac{L-x}{\sqrt{2(\tau - (\tau_{n-j} - p))}}\right]}(y_{j+1}) \\ &\times U(x + y_{j+1}\sqrt{2(\tau - (\tau_{n-j} - p))}, \tau_{n-j} - p) e^{-y_{j+1}^2/2} dy_{j+1}. \end{aligned}$$

Now insert (3.10), with x replaced by $x + y_{j+1}\sqrt{2(\tau - (\tau_{n-j} + p))}$, and use the definition of h_j to conclude (3.4). $\square$

Note that Proposition 2.1 corresponds to (3.4) for $j = 0$. As seen there, the integral $\int_0^L dx_1$ can be done in closed form. We have not included this evaluation in Theorem 3.1 to increase its readability.

If a different option (a call, say) with the same barrier conditions is to be priced instead of a digital payoff, the quantity $e^{-\alpha x_1}$ in (3.2) should be replaced by the appropriate payoff $U(x_1, 0)$.

The deficiencies of modeling assets by geometric Brownian motion are well known, and indeed, double barrier option pricing has been studied for other models; see, e.g., [6] and the references therein. However, our approach at multi-period barriers (i.e. the algorithm emanating from Theorem 3.1) does not seem to lend itself to such generalization respectively variation: We consider the valuation PDE period by period, and exploit the fact that each subsequent solution can be obtained from the previous period by summation and integration. No such explicit form seems to be available for the PDEs of stochastic volatility models or the PIDEs (partial integro-differential equations) arising from Lévy models.

4. Numerical Implementation

For the implementation of the value function we have chosen the computer algebra system Mathematica. It offers symbolic capabilities for defining the auxiliary functions h_j and g_j, as well as fast multidimensional numerical integration. Thus, the pricing algorithm consists of the following steps: (i) define the auxiliary functions h_j, g_j as well as the first two functions g_0 and h_0, (ii) apply the coordinate change (2.3) to the input variables, (iii) determine in which period the valuation time lies, i.e. calculate j, (iv) define the integration variables and the integration limits, (v) plug in the values and integrate the functions.

In the last step we used `NIntegrate[]` to avoid symbolic integration and speed up the calculations. Furthermore, memoization should be used to save computation time when calculating the recursion. (This means storing function return values instead of repeating function calls for the same input; see, e.g., [8].) For the infinte sums we found that truncating after five summands suffices to get a reasonably accurate value.

In Fig. 2, the value of a double barrier digital with two barrier periods, varying time-to-maturity and underlying price, is shown.[1] The parameters are $r = 0.01$, $\sigma = 0.15$, $B_{\text{low}} = 80$, $B_{\text{up}} = 120$, $\{T_0, T_1\} = \{1, 6\}$, and $P = 2$. It is clearly visible that if the valuation takes place during a barrier period, e.g., $t \in [T_1, T_1 + P]$, then the value of the option is zero as soon as the underlying moves beyond the barriers.

Otherwise, between the periods, for example at $t = 4$, we have a positive value even if the price process is outside the barriers.

[1]With Mathematica 8.0 the calculation of one value takes about 15 s on a 2.83 GHz machine with four cores and 4 GB memory.

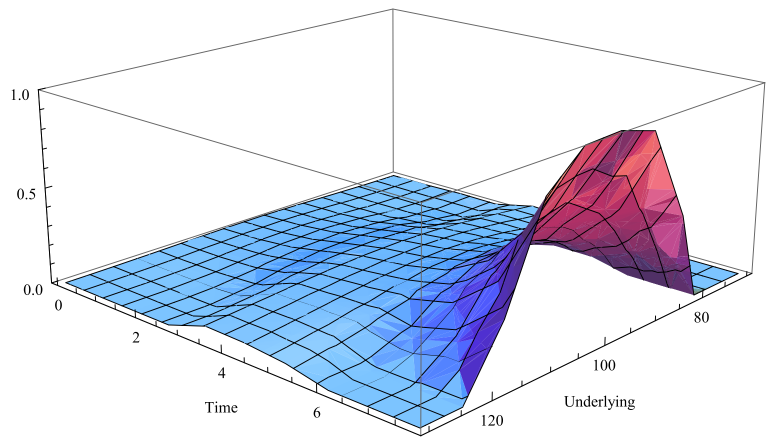

Fig. 2. Value function of a double barrier digital with two barrier periods, $[1, 3]$ and $[6, 8]$. Observe that the value outside of the barriers is zero during a barrier period and takes on positive values if the underlying stays within the barrier.

5. Structure Floors

In this section we assume that our tenor structure satisfies $T_{i-1} + P = T_i$ for $1 \le i < n$, and define $T_n := T_{n-1} + P$. We consider a structured note with n coupons, where the ith coupon consists of a payment of

$$C_i = \mathbf{1}_{\{B_{\mathrm{low}} < S_t < B_{\mathrm{up}},\ t \in [T_{i-1}, T_i]\}}, \quad 1 \le i \le n, \tag{5.1}$$

at time T_i. These coupons can be priced by Proposition 2.1 (replace T_0 by T_{i-1}). In addition, the holder receives the terminal premium

$$\left(F - \sum_{i=1}^{n} C_i\right)^{+} \tag{5.2}$$

at T_n, where $F > 0$. This means that the aggregate payoff $A := \sum_{i=1}^{n} C_i$ of the note is floored at F, which is a popular feature of structured notes. While the individual coupons are straightforward to valuate, it is less obvious how to get a handle on the law of A. We now show that this law is linked to barrier options with several barrier periods. Indeed, the following result is based on the fact that the moments

$$\mathbf{E}[A^{\nu}] = \sum_{i=0}^{n} i^{\nu} \mathbf{P}[A = i], \quad 1 \le \nu < n, \tag{5.3}$$

of A are linear combinations of multi-period double barrier option prices, with coefficients

$$c(\nu, J) := \sum_{\substack{0 \le i_1, \ldots, i_n \le \nu \\ \mathrm{supp}(\mathbf{i}) = J}} \binom{\nu}{i_1, \ldots, i_n}, \quad J \subseteq \{1, \ldots, n\}. \tag{5.4}$$

(The notation $\operatorname{supp}(\mathbf{i}) = J$ means that J is the set of indices such that the corresponding components of the vector $\mathbf{i} = (i_1, \ldots, i_n)$ are non-zero.) W.l.o.g. we assume that the valuation time is $t = 0$.

Theorem 5.1. *The price of the structure floor* (5.2) *at time* $t = 0$ *can be expressed as*

$$e^{-rT_n}\mathbf{E}[(F - A)^+] = e^{-rT_n} \sum_{i=0}^{n\wedge\lfloor F\rfloor} (F - i)\mathbf{P}[A = i], \tag{5.5}$$

where

$$\mathbf{P}[A = n] = \mathrm{BD}(S_0, 0; \{T_0\}, T_n - T_0, B_{\mathrm{low}}, B_{\mathrm{up}}, 0). \tag{5.6}$$

The other point masses $\mathbf{P}[A = i]$ *in* (5.5) *can be recovered from the moments of* A *by solving* (5.3) (*including* $\nu = 0$, *of course*). *The moments in turn can be computed from barrier digital prices by* $(1 \le \nu < n)$

$$\mathbf{E}[A^\nu] = \sum_{J\subseteq\{1,\ldots,n\}} c(\nu, J) \cdot \mathrm{BD}(S_0, 0; \{T_j : j \in J\}, P, B_{\mathrm{low}}, B_{\mathrm{up}}, 0), \tag{5.7}$$

where the coefficients $c(\nu, J)$ *are defined in* (5.4).

Proof. The expression (5.5) is clear. The event in (5.6) means that all of the n coupons (5.1) are paid. By our assumption that $T_i = T_{i-1} + P$, its risk-neutral probability is the (undiscounted) price of a double barrier digital with one-barrier period $[T_0, T_n]$, which yields (5.6). To prove (5.7), we calculate

$$\begin{aligned}
\mathbf{E}[A^\nu] &= \mathbf{E}\left[\left(\sum_{i=1}^n C_i\right)^\nu\right] \\
&= \sum_{i_1,\ldots,i_n} \binom{\nu}{i_1, \ldots, i_n} \mathbf{E}[C_1^{i_1} \ldots C_n^{i_n}] \\
&= \sum_{i_1,\ldots,i_n} \binom{\nu}{i_1, \ldots, i_n} \mathbf{E}\left[\prod_{\substack{j=1 \\ i_j>0}}^n C_j\right] \\
&= \sum_{J\subseteq\{1,\ldots,n\}} \left(\sum_{\substack{i_1,\ldots,i_n \\ \operatorname{supp}(\mathbf{i})=J}} \binom{\nu}{i_1, \ldots, i_n}\right) \mathbf{E}\left[\prod_{j\in J} C_j\right].
\end{aligned}$$

Now observe that $\prod_{j\in J} C_j$ is the payoff of a double barrier digital with barrier periods $[T_j, T_j + P]$ for $j \in J$. $\square$

When calculating the value BD in (5.7) for, say, $J = \{1, 2, 4, 5, 6\}$, the adjacent barrier periods should be concatenated: Do not compute the price for five barrier periods of length P, but rather for two periods with lengths $2P$ and $3P$. We did not include this obvious extension (barrier periods of variable length) in Theorem 3.1 in order not to complicate an already heavy notation.

We also remark that [2] (see also [9]) gives an (involved) expression for the joint distribution of a vector $(C_1, \ldots, C_n)$ of (dependent) Bernoulli random variables, in terms of centralized mixed moments. It does not seem to offer any simplification in our case, though.

6. Approximation by a Corridor Option

Theorems 3.1 and 5.1 express the price of the structure floor (5.2) by iterated sums and integrals. Due to the factors of order $e^{-k_j^2}$, the infinite series $\sum_{k_j}$ may be truncated after just a few terms. Still, numerical quadrature may be too involved for a large number of coupons, so we present an approximation. Let us fix a maturity $T = T_n$ and assume that the n coupon periods

$$\mathcal{T}_i^n := \left] \frac{i-1}{n}T, \frac{i}{n}T \right], \quad 1 \le i \le n,$$

have length T/n. For large n, the proportion of intervals during which the underlying stays inside the barrier interval

$$\mathcal{B} := [B_{\text{low}}, B_{\text{up}}]$$

is similar to the proportion of time that the underlying spends inside $\mathcal{B}$, i.e. the occupation time. A somewhat related problem has been studied in [13] (continuous versus discrete monitoring for occupation time derivatives). Our reasoning is made precise in the following result, which holds not only for the Black–Scholes model, but for virtually any continuous model. Note that the level sets of geometric Brownian motion have a.s. measure zero (cf. [18, Theorem 2.9.6]).

Theorem 6.1. *Let $(S_t)_{t \ge 0}$ be a continuous stochastic process such that for each real c the level set $\{t \ge 0 : S_t = c\}$ has a.s. Lebesgue measure zero. Then we have a.s.*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{S_t \in \mathcal{B}\ \forall t \in \mathcal{T}_i^n\}} = \frac{1}{T} \int_0^T \mathbf{1}_{\mathcal{B}}(S_t)dt.$$

Proof. For $1 \le i \le n$, define processes $(X_{ni}(t))_{0 \le t \le T}$ by

$$X_{ni}(t) := \begin{cases} 1 & \text{if } t \in \mathcal{T}_i^n \text{ and } S_u \in \mathcal{B}\ \forall\, u \in \mathcal{T}_i^n \\ 0 & \text{otherwise.} \end{cases}$$

Put $X_n := \sum_{i=1}^n X_{ni}$. We claim that, a.s., the function $X_n(\cdot)$ converges pointwise on the set $[0, T] \backslash \{t : S_t = B_{\text{low}} \text{ or } S_t = B_{\text{up}}\}$, with limit $\mathbf{1}_{\mathcal{B}}(S_\cdot)$. Indeed, if $t \in [0, T]$ is such that $S_t \notin \mathcal{B}$, then $X_n(t) = 0$ for all n. If, on the other hand, $S_t \in \text{int}(\mathcal{B})$, then t has a neighborhood V such that $S_u \in \mathcal{B}$ for all $u \in V$, by continuity. Hence, $X_n(t) = 1$ for large n. Since we have pointwise convergence on a set of (a.s.) full measure, we can apply the dominated convergence theorem to conclude

$$\lim_{n\to\infty}\int_0^T X_n(t)dt = \int_0^T \mathbf{1}_{\mathcal{B}}(S_t)dt.$$

But this is the desired result, since

$$\begin{aligned}\int_0^T X_n(t)dt &= \sum_{i=1}^n \int_0^T X_{ni}(t)dt \\ &= \sum_{i=1}^n \int_{\mathcal{T}_i^n} X_{ni}(t)dt \\ &= \sum_{i=1}^n |\mathcal{T}_i^n| \ \mathbf{1}_{\{S_t\in\mathcal{B}\ \forall t\in\mathcal{T}_i^n\}} \\ &= \frac{T}{n}\sum_{i=1}^n \mathbf{1}_{\{S_t\in\mathcal{B}\ \forall t\in\mathcal{T}_i^n\}}.\end{aligned}$$

□

Theorem 6.1 suggests the approximation

$$e^{-rT}\mathbf{E}[(F-A)^+] \approx e^{-rT}\frac{n}{T}\mathbf{E}\left[\left(\frac{FT}{n} - \int_0^T \mathbf{1}_{\mathcal{B}}(S_t)dt\right)^+\right] \tag{6.1}$$

for the price of the structure floor (5.2). It is obtained from replacing F by F/n in the relation

$$\mathbf{E}[(nF-A)^+] \sim n\mathbf{E}\left[\left(F - \frac{1}{T}\int_0^T \mathbf{1}_{\mathcal{B}}(S_t)dt\right)^+\right], \quad n\to\infty,$$

which follows from Theorem 6.1 (recall that $A=\sum_{i=1}^n C_i$ denotes the sum of the coupons). On the right-hand side of (6.1) we recognize the price of a put on the occupation time of S, also called a corridor option. Fusai in [12] studied such options in the Black–Scholes model. In particular, his Theorem 2.1 gives an expression for the characteristic function of $\int_0^T \mathbf{1}_{\mathcal{B}}(S_t)dt$. Since the formula is rather involved, we do not reproduce it here. Section 4 of [12] explains how to compute the corridor option price from the characteristic function by numerical Laplace inversion.

We checked (6.1) numerically for up to 10 coupons, with reasonable results, see Table 1. The maturity is $T = 4$, and the structure floor is at $F = 10$. The other model parameters are the same as in Fig. 2. The left-hand side of (6.1) was evaluated by a Monte Carlo simulation with 10,000 paths, using the discretization bias correction of [3] (with a threshold probability of 0.55 for discarding a path).

The approximation (6.1) holds for period lengths tending to zero. One could also let the number of coupons tend to infinity for a fixed period length P, so that maturity increases linearly with n. As seen from their definition in (5.1), the

Table 1. Numerical approximation of the structure floor by the corridor option (6.1) with maturity $T = 4$, structure floor level $F = 10$, and n coupons. The other parameters are $r = 0.01$, $\sigma = 0.15$, $B_{\text{low}} = 80$, and $B_{\text{up}} = 120$. This results show a reasonable approximation to the corridor option for larger n.

Coupons	Structure floor	Corridor option	Relative error	MC standard error
$n = 1$	7.63696	9.91563	0.22980	0.14436
$n = 2$	7.52979	9.24883	0.18586	0.16975
$n = 3$	7.42262	8.66698	0.14357	0.14522
$n = 4$	7.31545	8.06291	0.09270	0.15126
$n = 5$	7.20827	7.44886	0.03229	0.18958
$n = 6$	7.10110	7.31880	0.02974	0.20350
$n = 7$	6.99393	7.18558	0.02667	0.21811
$n = 8$	6.88677	7.03704	0.02135	0.24764
$n = 9$	6.77962	6.92288	0.02069	0.26579
$n = 10$	6.67232	6.80399	0.01935	0.28013

dependence of the random variables C_i and C_j decreases for large $|i - j|$, and so it is a natural question whether a central limit theorem holds, i.e. whether

$$\frac{A - \mathbf{E}[A]}{\sqrt{\mathbf{Var}[A]}}$$

converges in law to a standard normal random variable as $n \to \infty$. Note that $\mathbf{E}[A] = \sum_{i=1}^{n} \mathbf{E}[C_i]$ and $\mathbf{E}[A^2] = \mathbf{E}[A] + 2\sum_{i<j} \mathbf{E}[C_i C_j]$ can be easily computed from Proposition 2.1, respectively Theorem 3.1. The structure floor (5.2) could then be approximately valuated by a Bachelier-type put price formula. We were not able, though, to verify any of the mixing conditions [7] that could lead to a central limit result. Numerical experiments also cast some doubt on the existence of a Gaussian limit law. This is therefore left for future research.

Acknowledgments

We thank Richard C. Bradley, Antoine Jacquier, Uwe Schmock, and the anonymous referees for their valuable comments and discussions.

This work was financially supported by the Christian Doppler Research Association (CDG). The authors gratefully acknowledge the fruitful collaboration and support by Bank Austria and COR & FJA through CDG and the CD-Laboratory for Portfolio Risk Management (PRisMa Lab) http://www.prismalab.at/. S. Altay also gratefully acknowledges support by funds of the Oesterreichische Nationalbank (Anniversary Fund, project number: 14977). S. Gerhold gratefully acknowledges financial support from the Austrian Science Fund (FWF) under grant P 24880-N25.

The contents of this paper are the authors' sole responsibility. They do not necessarily represent the views of Raiffeisen Kapitalanlage-Gesellschaft m.b.H.

References

[1] L. Andersen and R. Brotherton-Ratcliffe, Exact exotics, *Risk* **9** (1996) 85–89.
[2] R. R. Bahadur, A representation of the joint distribution of responses to n dichotomous items, in *Studies in Item Analysis and Prediction*, *Stanford Mathematical Studies in the Social Sciences*, Vol. VI (Stanford University Press, Stanford, Calif., 1961), pp. 158–168.
[3] P. Baldi, L. Caramellino and M. G. Iovino, Pricing general barrier options: A numerical approach using sharp large deviations, *Math. Finance* **9** (1999) 293–322.
[4] D. R. Beaglehole, P. H. Dybvig and G. Zhou, Going to extremes: Correcting simulation bias in exotic option valuation, *Financial Anal. J.* **53** (1997) 62–68.
[5] A. N. Borodin and P. Salminen, *Handbook of Brownian Motion — Facts and Formulae, Probability and its Applications*, 2nd edn. (Birkhäuser-Verlag, Basel, 2002).
[6] M. Boyarchenko and S. Levendorskiĭ, Valuation of continuously monitored double barrier options and related securities, *Math. Finance* **22** (2012) 419–444.
[7] R. C. Bradley, Basic properties of strong mixing conditions. A survey and some open questions, *Probab. Surv.* **2** (2005) 107–144.
[8] T. H. Cormen, C. E. Leiserson, R. L. Rivest and C. Stein, *Introduction to Algorithms*, 3rd edn. (MIT Press, Cambridge, MA, 2009).
[9] D. R. Cox, The analysis of multivariate binary data, *J. Roy. Stat. Soc. C Appl.* **21** (1972) 113–120.
[10] S. Ebenfeld, M. R. Mayr and J. Topper, An analysis of onion options and double-no-touch digitals, *Wilmott Magazine* (2002) 68–77.
[11] L. C. Evans, Partial differential equations, *Graduate Studies in Mathematics*, Vol. 19 (American Mathematical Society, Providence, RI, 1998).
[12] G. Fusai, Corridor options and arc-sine law, *Ann. Appl. Probab.* **10** (2000) 634–663.
[13] G. Fusai and A. Tagliani, Pricing of occupation time derivatives: Continuous and discrete monitoring, *J. Computational Finance* **5** (2001) 1–37.
[14] C. H. Hui, One-touch double barrier binary option values, *Applied Financial Economics* **6** (1996) 343–346.
[15] C. H. Hui, Time-dependent barrier option values, *J. Futures Markets* **17** (1997) 667–688.
[16] F. John, Partial differential equations, *Applied Mathematical Sciences*, Vol. 1, 4th edn. (Springer-Verlag, New York, 1982).
[17] D. Jun and H. Ku, Pricing chained options with curved barriers, *Math. Finance* **23**(4) (2013) 763–776.
[18] I. Karatzas and S. E. Shreve, Brownian motion and stochastic calculus, *Graduate Texts in Mathematics*, Vol. 113, 2nd edn. (Springer-Verlag, New York, 1991).
[19] N. Kunimoto and M. Ikeda, Pricing options with curved boundaries, *Math. Finance* **2** (1992) 275–298.
[20] X. S. Lin, Double barrier hitting time distributions with applications to exotic options, *Insur. Math. Econom.* **23** (1998) 45–58.
[21] W. Rudin, *Principles of Mathematical Analysis*, 3rd edn. (McGraw-Hill Book Co., New York, 1976).
[22] H. Solomon, Studies in item analysis and prediction, *Stanford Mathematical Studies in the Social Sciences*, Vol. VI (Stanford University Press, Stanford, Calif., 1961).