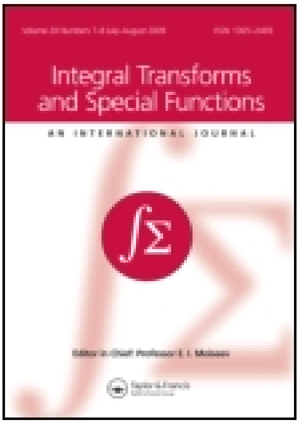

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Asymptotics for a variant of the Mittag-Leffler function

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Asymptotics for a variant of the Mittag–Leffler function

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We generalize the Mittag–Leffler function by attaching an exponent to its Taylor coefficients. The main result is an asymptotic formula valid in sectors of the complex plane, which extends the work by Le Roy [*Valeurs asymptotiques de certaines séries procédant suivant les puissances entières et positives d'une variable réelle*, Bull. des Sciences Math. 24, 1900] and Evgrafov [*Asimptoticheskie otsenki i tselye funktsii*, 3rd ed., Nauka, Moscow, 1979]. It is established by Plana's summation formula in conjunction with the saddle point method. As an application, we (re-)prove a non-holonomicity result about powers of the factorial sequence.

Keywords: entire function; Mittag–Leffler function; Plana's summation formula; non- holonomicity

1991 Mathematics Subject Classifications: 33E12; 41A60

1. Introduction

For $a, b, \alpha > 0$, the series

$$F_{a,b}^{(\alpha)}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(an+b)^{\alpha}} \tag{1}$$

defines an entire function of z. The Bessel function $\mathrm{I}_0(2\sqrt{z}) = \sum_{n=0}^{\infty} z^n/n!^2$ and the (generalized) Mittag–Leffler function $E_{a,b}(z) = \sum_{n=0}^{\infty} z^n/\Gamma(an+b)$ are special cases. If α is a natural number, then (1) is an instance of the multiple Mittag–Leffler function investigated by Yakubovich and Luchko [16] and Kiryakova [9], in connection with numerous applications to fractional calculus, but it seems that the asymptotic behaviour of the multiple Mittag–Leffler function has not been studied. The function (1) has the order $1/a\alpha$; see, e.g., Titchmarsh [15, Example 8.4], for $a = b = 1$. His argument trivially extends to $a, b > 0$.

We consider the asymptotic behaviour of $F_{a,b}^{(\alpha)}(z)$ as $z \to \infty$ in a sector of the complex plane, containing the positive real line. For the Mittag–Leffler function ($\alpha = 1$), this is usually analysed by an integral representation [2], which appears to have no immediate extension to $\alpha \neq 1$.

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Concerning $\alpha > 0$ and $a = b = 1$, note that Le Roy [10] has obtained the asymptotics of

$$F_{1,1}^{(\alpha)}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!^\alpha} \tag{2}$$

as z tends to infinity along the real line. His result (a special case of formula (4)) appears also in Hardy [7, p. 55]. To establish it, Le Roy approximates the sum by an integral and then uses the Laplace method. Note that the latter can also be applied directly to the sum: For real z, the summands in (2) are positive and concentrated near $n \approx z^{1/\alpha}$, and the Laplace method is easily carried out. The contribution of the present note is an extension of Le Roy's asymptotic formula to $a, b > 0$ and complex values of z, which is presented in Section 2. (For $a = b = 1$, this question is also discussed in Evgrafov's book [3]; see the end of Section 2 for detailed comments.)

As a small application, we prove in Section 3 that the (possibly formal) series (2) is not D-finite [14] for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. (This is a special case of a known result [1].)

We note in passing that the function (2) satisfies the integral relation

$$\int_0^\infty \mathrm{e}^{-t/z} F_{1,1}^{(\alpha+1)}(t)\,\mathrm{d}t = z F_{1,1}^{(\alpha)}(z), \qquad z \neq 0. \tag{3}$$

Indeed

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-t/z} \sum_{n=0}^{\infty} \frac{t^n}{n!^{\alpha+1}}\,\mathrm{d}t &= \sum_{n=0}^{\infty} \frac{1}{n!^{\alpha+1}} \int_0^\infty t^n\,\mathrm{e}^{-t/z}\,\mathrm{d}t \\ &= \sum_{n=0}^{\infty} \frac{z}{n!^{\alpha+1}} \int_0^\infty (zs)^n\,\mathrm{e}^{-s}\,\mathrm{d}s = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n!^\alpha}.\end{aligned}$$

2. Main result

THEOREM 1 *Let $\alpha, a, b > 0$ and $\varepsilon > 0$ be arbitrary. Then, for $z \to \infty$ in the sector*

$$|\arg(z)| \le \begin{cases} \frac{1}{2}a\alpha\pi - \varepsilon, & 0 < a\alpha < 2, \\ (2 - \frac{1}{2}a\alpha)\pi - \varepsilon, & 2 \le a\alpha < 4, \\ 0, & 4 \le a\alpha, \end{cases}$$

we have the asymptotics

$$F_{a,b}^{(\alpha)}(z) \sim \frac{1}{a\sqrt{\alpha}} (2\pi)^{(1-\alpha)/2} z^{(\alpha - 2b\alpha + 1)/2a\alpha}\,\mathrm{e}^{\alpha z^{1/a\alpha}}. \tag{4}$$

Applying the Laplace method directly does not work for non-real z; the absolute values of the summands in (1) are peaked near $n \approx a^{-1}|z|^{1/a\alpha}$, but it seems that one cannot balance the local expansion and the tails. This is caused by oscillations in the summands, which can be dealt with by shifting the problem to the asymptotic evaluation of an integral. The Laplace method then succeeds, after moving the integration contour through a saddle point located approximately at $a^{-1}z^{1/a\alpha}$.

LEMMA 2 *Let $\alpha, a, b > 0$ and $\varepsilon > 0$ be arbitrary. Then, as $z \to \infty$ in the sector $|\arg(z)| \le \max\{0, (2 - \frac{1}{2}a\alpha)\pi\}$, we have*

$$F_{a,b}^{(\alpha)}(z) = \int_0^\infty \frac{z^t}{\Gamma(at+b)^\alpha}\,\mathrm{d}t + \mathrm{O}(z). \tag{5}$$

Proof First, let us fix a z in this sector. Put $f(t)=z^t/\Gamma(at+b)^\alpha$. Plana's summation formula [8, Theorem 4.9c] yields

$$\sum_{n=0}^{\infty} f(n) = \int_0^\infty f(t)\,\mathrm{d}t + \frac{1}{2}f(0) + \mathrm{i}\int_0^\infty \frac{f(\mathrm{i}t)-f(-\mathrm{i}t)}{\mathrm{e}^{2\pi t}-1}\,\mathrm{d}t. \tag{6}$$

To check the validity of (6), we have to verify that

$$\lim_{y\to\infty} |f(x\pm \mathrm{i}y)|\,\mathrm{e}^{-2\pi y} = 0,$$

uniformly for x in finite intervals in $[0,\infty[$, and that

$$\int_0^\infty |f(x\pm\mathrm{i}y)|\,\mathrm{e}^{-2\pi y}\,\mathrm{d}y$$

exists for $x\geq 0$ and tends to zero for $x\to\infty$. To do so, first note that

$$|z^{x\pm\mathrm{i}y}| = |z|^x\,\mathrm{e}^{\mp y\arg(z)} \leq |z|^x\,\mathrm{e}^{y|\arg(z)|}.$$

Furthermore, by Stirling's formula, we have

$$|\Gamma(x\pm\mathrm{i}y)^{-\alpha}| \leq \mathrm{e}^{(\alpha+\delta)x}x^{-\alpha x}\,\mathrm{e}^{(\alpha\pi/2+\delta)y}$$

for large $|x+\mathrm{i}y|$, where $\delta>0$ is arbitrary. Hence

$$|\Gamma(ax+b\pm\mathrm{i}ay)^{-\alpha}| \leq \mathrm{e}^{(a\alpha+\delta)x}(ax+b)^{-\alpha(ax+b)}\,\mathrm{e}^{(a\alpha\pi/2+\delta)y},$$

and thus

$$|f(x\pm\mathrm{i}y)| \leq \frac{\mathrm{e}^{(a\alpha+\delta)x}|z|^x}{(ax+b)^{\alpha(ax+b)}}\,\mathrm{e}^{(|\arg(z)|+a\alpha\pi/2+\delta)y}, \tag{7}$$

which implies both required conditions. Finally, putting $x=0$, we see from (7) that the second integral in (6) is $\mathrm{O}(z)$ as $z\to\infty$. Since $f(0)=\mathrm{O}(1)$, we are done. ∎

Proof of Theorem 1. We apply the saddle point method to the integral in (5). To locate the saddle point, we equate the derivative of the logarithm of the integrand to zero, which leads to the equation

$$0 = \log z - a\alpha\frac{\Gamma'(at+b)}{\Gamma(at+b)} = \log z - a\alpha\log(at+b) + \frac{a\alpha}{2(at+b)} + \mathrm{O}(t^{-2}).$$

By bootstrapping, we find that there is an approximate saddle point at

$$t_0 := a^{-1}z^{1/a\alpha} + \frac{1-2b}{2a}.$$

We change the integration contour to a line $\mathcal{L}$ that begins at 0 and passes through t_0. Note that this change of contour is valid for large $|z|$, by Stirling's formula, as long as $|\arg(t_0)|$ is bounded away from $\pi/2$. But this follows from our assumption on $\arg(z)$.

The dominant contribution of the integral arises from the range

$$|t-t_0| \leq |t_0|^\beta$$

around the saddle point, where β is an arbitrary member of the interval $]\frac{1}{2}, \frac{2}{3}[$. We write

$$t = t_0(y+1), \quad -1 \le y < \infty,$$

and divide the integral as follows:

$$\begin{aligned}
\int_{\mathcal{L}} \frac{z^t}{\Gamma(at+b)^\alpha}\,\mathrm{d}t &= t_0 \int_{-1}^{\infty} \frac{z^{t_0(y+1)}}{\Gamma(at_0(y+1)+b)^\alpha}\,\mathrm{d}y \\
&= t_0 \left(\int_{-1}^{-|t_0|^{\beta-1}} + \int_{-|t_0|^{\beta-1}}^{|t_0|^{\beta-1}} + \int_{|t_0|^{\beta-1}}^{\infty} \right) \frac{z^{t_0(y+1)}}{\Gamma(at_0(y+1)+b)^\alpha}\,\mathrm{d}y \\
&=: I_1 + I_2 + I_3. \qquad (8)
\end{aligned}$$

First we investigate the central integral I_2. From Stirling's formula we obtain the following uniform local expansion of the logarithm of the integrand:

$$\begin{aligned}
&(t_0(y+1)) \log z - \alpha \log \Gamma((at_0(y+1)+b) \\
&\quad = t_0 \log z - a\alpha t_0 \log t_0 + a\alpha(1-\log a)t_0 + \alpha(\tfrac{1}{2}-b)\log t_0 \\
&\qquad + \alpha(\tfrac{1}{2}-b)\log a - \alpha \log\sqrt{2\pi} - \tfrac{1}{2}a\alpha t_0 y^2 + \mathrm{o}(1). \qquad (9)
\end{aligned}$$

Since

$$\begin{aligned}
\int_{-|t_0|^{\beta-1}}^{|t_0|^{\beta-1}} \exp\left(-\frac{1}{2}a\alpha t_0 y^2\right)\mathrm{d}y &= \frac{1}{\sqrt{|t_0|}} \int_{-|t_0|^{\beta-1/2}}^{|t_0|^{\beta-1/2}} \exp\left(-\frac{1}{2}a\alpha t_0|t_0|^{-1}u^2\right)\mathrm{d}u \\
&\sim \frac{1}{\sqrt{|t_0|}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}a\alpha t_0|t_0|^{-1}u^2\right)\mathrm{d}u \\
&= \sqrt{\frac{2\pi}{a\alpha t_0}},
\end{aligned}$$

the central part I_2 thus satisfies

$$\begin{aligned}
I_2 &\sim t_0 z^{t_0} t_0^{-a\alpha t_0}\, \mathrm{e}^{a\alpha(1-\log a)t_0} (at_0)^{\alpha(1/2-b)} (2\pi)^{-\alpha/2} \sqrt{\frac{2\pi}{a\alpha t_0}} \\
&\sim \frac{1}{a\sqrt{\alpha}} (2\pi)^{(1-\alpha)/2} z^{(\alpha-2b\alpha+1)/2a\alpha}\, \mathrm{e}^{\alpha z^{1/a\alpha}}. \qquad (10)
\end{aligned}$$

This is the right-hand side of (4).

It remains to show that the integrals I_1 and I_3 are negligible. By our assumption on $\arg(z)$, there is a constant $c_1 > 0$ (independent of z) such that

$$|\Im(t_0)| \le c_1 \Re(t_0),$$

hence

$$\Re(t_0) \ge \frac{|t_0|}{\sqrt{c_1^2+1}} =: c_2|t_0|. \qquad (11)$$

Now we divide the integral I_3 further into

$$\begin{aligned}
I_3 &= t_0 \left(\int_{|t_0|^{\beta-1}}^{u} + \int_{u}^{\infty} \right) \frac{z^{t_0(y+1)}}{\Gamma(at_0(y+1)+b)^\alpha}\,\mathrm{d}y \\
&=: I_{31} + I_{32},
\end{aligned}$$

where

$$u := \exp(\tfrac{1}{3}a\alpha c_2|t_0|^{2\beta-1}).$$

By Stirling's formula, there is a positive constant c_3 such that

$$\left|\frac{z^t}{\Gamma(at+b)^\alpha}\right| \le c_3 \exp \Re((at+b)\log z - \alpha(at+b)\log(at+b) + \alpha(at+b) \\ + \tfrac{1}{2}\alpha\log(at+b)), \quad t \in \mathcal{L}. \tag{12}$$

An elementary calculation shows that the right-hand side of (12) decreases w.r.t. $|y|$ for large $|z|$ and $|y| \ge |t_0|^{\beta-1}$. Therefore, we can estimate I_{31} by inserting $y = |t_0|^{\beta-1}$ into (12) and multiplying by the length of the integration path, which is $u - |t_0|^{\beta-1} < u$. Using (9), and writing $A = A(z)$ for the factor in front of the square root in the first line of (10), we obtain

$$\left|\frac{z^t}{\Gamma(at+b)^\alpha}\right|_{y=|t_0|^{\beta-1}} \le c_3|A\exp(-a\alpha t_0|t_0|^{2\beta-2}/2)| \\ = c_3|A|\exp(-a\alpha\Re(t_0)|t_0|^{2\beta-2}/2) \le c_3|A|\exp(-a\alpha c_2|t_0|^{2\beta-1}/2).$$

The latter inequality follows from (11). Hence

$$|I_{31}| \le c_3 u|A|\exp(-a\alpha c_2|t_0|^{2\beta-1}/2) = c_3|A|\exp(-a\alpha c_2|t_0|^{2\beta-1}/6).$$

Now we compare this estimate with (10). Since

$$\exp(-a\alpha c_2|t_0|^{2\beta-1}/6) \ll |t_0|^{-1/2},$$

the integral I_{31} is indeed negligible. As for I_{32}, it easily follows from Stirling's formula that

$$\left|\frac{z^t}{\Gamma(at+b)^\alpha}\right| \le \mathrm{e}^{-y}, \quad y \ge u,$$

for large $|z|$. We thus obtain

$$|I_{32}| \le t_0 \int_u^\infty \mathrm{e}^{-y}\,\mathrm{d}y \\ = t_0\,\mathrm{e}^{-u} \ll I_2.$$

Finally, the integral I_1 in (8) can be estimated analogously to I_{31}. ■

A full asymptotic expansion can be obtained easily by pushing the local expansion around the saddle point further.

Evgrafov [3, Section 4.2] offers a similar asymptotic treatment of $F_{1,1}^{(\alpha)}(z)$ in sectors of the complex plane. For $\alpha < 2$ and $|\arg(z)| < \frac{1}{2}\alpha\pi - \varepsilon$, his result agrees with ours. (Except that a factor $\alpha^{-1/2}$, or $\rho^{1/2}$ in Evgrafov's notation, is missing from the formula.) However, he gives few details on how to carry out the saddle point analysis, in particular, on how to do the tail estimates. For $\alpha \ge 2$, Evgrafov [3, p. 294] appears to go beyond our Theorem 1, in that he claims (4) (with $a = b = 1$) for *any* sector that stays away from the negative real axis. There seems to be a serious gap in the proof, though.

To be specific, we switch to Evgrafov's notation. On p. 292, he writes that $\sum_{n=0}^\infty t^n n!^{-1/\rho}$ satisfies the assumptions of Theorem 4.2.2 for all $\rho > 0$. (The text says Theorem 3.2.2 instead, but this is certainly a typo.) This means that $\mu(z) = \Gamma(z+1)^{-1/\rho}$ should satisfy $|\mu(x+\mathrm{i}y)| <$

$M_A \exp(-Ax)$, for arbitrary A and some other constant M_A, and for $x + \mathrm{i}y$ in a domain D containing a horizontal strip that contains the positive real line. But due to the exponential decrease of the Gamma function towards $\pm\mathrm{i}\infty$, this can hold only if the elements of D have bounded imaginary part. Then also the contours C_{-1}^{+} and C_{-1}^{-} on p. 292 must have a bounded imaginary part. On p. 293, the saddle point method is applied to the second integral on p. 292 (over the contour C_{-1}^{+}), where the location of the saddle point is $t^{\rho}\exp(2\pi\mathrm{i}\rho)$. The imaginary part of this point is *not* bounded for large t. Hence the validity of the necessary change of integration contour is not proved. Besides the generalization of Evgrafov's work to $a, b > 0$, these remarks seem to justify another study of the problem, provided by the present note.

We close the section with a possible question for future research. For complex α and fixed z, one might ask whether the function defined by (2) has an analytic continuation for $\Re(\alpha) \le 0$. (The relation (3) does not seem to be useful in this respect.)

3. An application: non-holonomicity

An analytic function (or formal power series) is called *D*-finite [14], or holonomic, if it satisfies a linear ODE with polynomial coefficients. An equivalent condition is that its power-series coefficients satisfy a linear recurrence with polynomial coefficients.

There has been some interest recently in showing that certain series (resp. sequences) are *not* holonomic [1,4–6,12,13]. Lipshitz [11, Example 3.4(i)] mentions (without proof) that the sequence $(n!^{\alpha})$, which satisfies

$$(n+1)!^{\alpha} = (n+1)^{\alpha} n!^{\alpha},$$

is holonomic if and only if α is an integer. See [1, Theorem 4.1] for a proof that there is indeed no recurrence with polynomial coefficients for $\alpha \in \mathbb{C} \setminus \mathbb{Z}$.

From Theorem 1, we can conclude the weaker result that $F_{1,1}^{(\alpha)}(z)$, and thus $(n!^{\alpha})$, is not holonomic for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Indeed, since multiplication by the holonomic sequence $n!^{-\lfloor\alpha\rfloor}$ preserves holonomicity, we may assume that $0 \le \alpha < 1$. Hence Theorem 1 yields the asymptotics of $F_{\alpha}(z)$. But a function that features the element $\exp(\alpha z^{1/\alpha})$ in its asymptotic expansion at infinity, in a sector of positive opening angle, can be holonomic only for rational α. This follows from a classical result on the asymptotic behaviour of solutions of linear ODEs. (For details on this method of showing non-holonomicity, see [4,5].) Including the parameters a and b, one can infer other non-holonomicity results from Theorem 1, but they are going to be weaker (in terms of parameter ranges) than the corresponding results deduced by the method of [1].

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