Statistics and Probability Letters 81 (2011) 560–562

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Moment explosion in the LIBOR market model

Stefan Gerhold

Vienna University of Technology, Institute of Mathematical Methods in Economics, Wiedner Hauptstr. 8/105-1, A-1040 Vienna, Austria

ARTICLE INFO

Article history:
Received 18 August 2010
Received in revised form 15 December 2010
Accepted 14 January 2011
Available online 22 January 2011

MSC:
91G30

Keywords:
LIBOR market model
Moment explosion
Distribution tails

ABSTRACT

In the LIBOR market model, forward interest rates are log-normal under their respective forward measures. This note shows that their distributions under the other forward measures of the tenor structure have approximately log-normal tails.

1. Introduction

The LIBOR market model (Brace et al., 1997) is one of the most popular models for pricing and hedging interest rate derivatives. Its state variables are forward interest rates $F_n(t) := F(t; T_{n-1}, T_n)$, spanning time periods $[T_{n-1}, T_n]$, where

$$0 < T_0 < T_1 < \cdots < T_M$$

is a fixed tenor structure. Under the T_M-forward measure $\mathbb{Q}^M$, which has as numeraire the zero-coupon bond maturing at T_M, the dynamics of the forward rates are

$$\mathrm{d}F_n(t) = -\sigma_n(t)F_n(t)\sum_{j=n+1}^{M}\frac{\rho_{nj}\tau_j\sigma_j(t)F_j(t)}{1+\tau_jF_j(t)}\mathrm{d}t + \sigma_n(t)F_n(t)\mathrm{d}W_n(t), \quad 1 \le n < M,$$

$$\mathrm{d}F_M(t) = \sigma_M(t)F_M(t)\mathrm{d}W_M(t).$$

Here, the σ_n are some positive deterministic volatility functions, and W is a vector of standard Brownian motions with instantaneous correlations $\mathrm{d}W_i(t)\mathrm{d}W_j(t) = \rho_{ij}\mathrm{d}t$. Moreover, $\tau_n = \tau(T_{n-1}, T_n)$ denotes the year fraction between the tenor dates T_{n-1} and T_n.

Note that F_M is a geometric Brownian motion under $\mathbb{Q}^M$. More generally, each rate F_n is a geometric Brownian motion under its own forward measure, while it has a stochastic drift under the other forward measures. Therefore, pricing derivatives with the LIBOR market model often requires a Monte Carlo simulation. To avoid this, a popular approximation of the above dynamics is obtained by "freezing the drift" (Brace et al., 2001; Brigo and Mercurio, 2006):

$$\mathrm{d}F_n^{\mathrm{fd}}(t) = -\sigma_n(t)F_n^{\mathrm{fd}}(t)\sum_{j=n+1}^{M}\frac{\rho_{nj}\tau_j\sigma_j(t)F_j(0)}{1+\tau_jF_j(0)}\mathrm{d}t + \sigma_n(t)F_n^{\mathrm{fd}}(t)\mathrm{d}W_n(t), \quad 1 \le n < M,$$

$$\mathrm{d}F_M^{\mathrm{fd}}(t) = \sigma_M(t)F_M^{\mathrm{fd}}(t)\mathrm{d}W_M(t).$$

E-mail address: sgerhold@fam.tuwien.ac.at.

0167-7152/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.spl.2011.01.009

Since the log-drifts are now deterministic, the new rates F_n^{fd} are just geometric Brownian motions, which allow for explicit pricing formulas for many interest-linked products. As a piece of evidence for the quality of this approximation, we show in the present note that, for fixed $t > 0$, the distribution of $F_n^{\mathrm{fd}}(t)$ has roughly the same tail heaviness as the distribution of $F_n(t)$.

2. Main result

If X is any log-normal random variable, and so $\log X \sim \mathcal{N}(\mu, \sigma^2)$ for some real μ and positive σ, then

$$\sup\{v : \mathbf{E}[\mathrm{e}^{v \log^2 X}] < \infty\} = \frac{1}{2\sigma^2}. \tag{1}$$

This follows from

$$\mathbf{E}[\mathrm{e}^{v \log^2 X}] = \frac{1}{\sqrt{1-2\sigma^2 v}} \exp\left(\frac{\mu^2 v}{1-2\sigma^2 v}\right), \quad v < \frac{1}{2\sigma^2}.$$

Our main result shows that $F_n(t)$ has approximately log-normal tails, in the sense that the left-hand side of (1) is finite and positive if X is replaced by $F_n(t)$. Furthermore, this "critical moment" is the same for $F_n(t)$ and the frozen drift approximation $F_n^{\mathrm{fd}}(t)$. We write $\mathbf{E}^n$ for the expectation w.r.t. the T_n-forward measure $\mathbb{Q}^n$.

Theorem 1. *In the log-normal LIBOR market model, we have for all $t > 0$ and all $1 \le n \le M$*

$$\begin{aligned} \sup\{v : \mathbf{E}^M[\mathrm{e}^{v \log^2 F_n(t)}] < \infty\} &= \sup\{v : \mathbf{E}^M[\mathrm{e}^{v \log^2 F_n^{\mathrm{fd}}(t)}] < \infty\} \\ &= \frac{1}{2\int_0^t \sigma_n(s)^2 \mathrm{d}s}. \end{aligned}$$

Proof. Note that the latter equality is obvious from (1), since $F_n^{\mathrm{fd}}(t)$ is log-normal with log-variance parameter $\sigma^2 = \int_0^t \sigma_n(s)^2 \mathrm{d}s$. We now show the first equality. Recall that the measure change from the T_n-forward measure to the T_{n-1}-forward measure is effected by the likelihood process (Björk, 2004)

$$\left.\frac{\mathrm{d}\mathbb{Q}^n}{\mathrm{d}\mathbb{Q}^{n-1}}\right|_{\mathcal{F}_t} = \frac{1+\tau_n F_n(0)}{1+\tau_n F_n(t)}.$$

Therefore, putting $\phi(x) = \exp(\log^2 x)$, we obtain

$$\begin{aligned} \mathbf{E}^M[\phi(F_n(t))^v] &= \mathbf{E}^{M-1}\left[\phi(F_n(t))^v \times \frac{1+\tau_M F_M(0)}{1+\tau_M F_M(t)}\right] \\ &= \cdots = \\ &= \mathbf{E}^n\left[\phi(F_n(t))^v \prod_{i=n+1}^{M} \frac{1+\tau_i F_i(0)}{1+\tau_i F_i(t)}\right] \\ &\le \mathbf{E}^n[\phi(F_n(t))^v] \prod_{i=n+1}^{M} (1+\tau_i F_i(0)), \end{aligned}$$

and hence

$$\sup\{v : \mathbf{E}^n[\phi(F_n(t))^v] < \infty\} \le \sup\{v : \mathbf{E}^M[\phi(F_n(t))^v] < \infty\}.$$

On the other hand, for $1 < k \le M$ and $v \in \mathbb{R}$ we have

$$\begin{aligned} \mathbf{E}^{k-1}[\phi(F_n(t))^v] &= \mathbf{E}^k\left[\phi(F_n(t))^v \times \frac{1+\tau_k F_k(t)}{1+\tau_k F_k(0)}\right] \\ &= \frac{1}{1+\tau_k F_k(0)} (\mathbf{E}^k[\phi(F_n(t))^v] + \tau_k \mathbf{E}^k[F_k(t)\phi(F_n(t))^v]). \end{aligned}$$

Now let $\varepsilon > 0$ be arbitrary, and define q by $\frac{1}{q} + \frac{1}{1+\varepsilon} = 1$. Then Hölder's inequality yields

$$\mathbf{E}^k[F_k(t)\phi(F_n(t))^v] \le \mathbf{E}^k[F_k(t)^q]^{1/q} \times \mathbf{E}^k[\phi(F_n(t))^{v(1+\varepsilon)}]^{1/(1+\varepsilon)}.$$

By the finite moment assumption, we obtain the implication

$$\mathbf{E}^k[\phi(F_n(t))^{v(1+\varepsilon)}] < \infty \implies \mathbf{E}^{k-1}[\phi(F_n(t))^v] < \infty, \quad v \in \mathbb{R}.$$

(Note that the left-hand side implies $\mathbf{E}^k[\phi(F_n(t))^v] < \infty$.)

Inductively, this leads to the implication

$$\mathbf{E}^M[\phi(F_n(t))^{v(1+\varepsilon)^{M-n}}] < \infty \Longrightarrow \mathbf{E}^n[\phi(F_n(t))^v] < \infty, \quad v \in \mathbb{R}.$$

Therefore, we find

$$\begin{aligned}\sup\{v : \mathbf{E}^n[\phi(F_n(t))^v] < \infty\} &\geq \sup\{v : \mathbf{E}^M[\phi(F_n(t))^{v(1+\varepsilon)^{M-n}}] < \infty\} \\ &= \frac{1}{(1+\varepsilon)^{M-n}} \sup\{v : \mathbf{E}^M[\phi(F_n(t))^v] < \infty\}.\end{aligned}$$

Since ε was arbitrary,

$$\sup\{v : \mathbf{E}^n[\phi(F_n(t))^v] < \infty\} \geq \sup\{v : \mathbf{E}^M[\phi(F_n(t))^v] < \infty\}$$

follows, which finishes the proof. $\square$

The following corollary explicitly shows the implication of Theorem 1 on the distribution tails of $F_n(t)$: At least one of the two tails is log-normal, and neither tail is heavier than log-normal. We are talking here about the dominating factor of the tail asymptotics; a lower order factor (a power of x, for example) could make the tails (slightly) heavier or lighter than log-normal. These refined asymptotics are left for future research. Note also that our result does not so far exclude the possibility that *one* of the two tails is (significantly) lighter than log-normal.

Corollary 2. *Under the assumptions of Theorem 1, the supremum of all real v that satisfy the condition*

$$\mathbb{Q}^M[F_n(t) > x] + \mathbb{Q}^M\left[F_n(t) < \frac{1}{x}\right] = \mathrm{O}(\mathrm{e}^{-v \log^2 x}), \quad x \to \infty,$$

is given by

$$v^* = \frac{1}{2\int_0^t \sigma_n(s)^2 \mathrm{d}s}.$$

The same holds for $F_n^{\mathrm{fd}}(t)$ instead of $F_n(t)$.

Proof. From Theorem 1 we obtain

$$v^* = \sup\{v : \mathbb{Q}^M[\mathrm{e}^{\log^2 F_n(t)} > x] = \mathrm{O}(x^{-v})\}.$$

Now for $x > 1$ we have

$$\mathbb{Q}^M[\mathrm{e}^{\log^2 F_n(t)} > x] = \mathbb{Q}^M[F_n(t) > \mathrm{e}^{\sqrt{\log x}}] + \mathbb{Q}^M[F_n(t) < \mathrm{e}^{-\sqrt{\log x}}],$$

and the claim follows after replacing x by $\exp(\log^2 x)$. The same reasoning works for the frozen drift rate $F_n^{\mathrm{fd}}(t)$. $\square$

References

Björk, T., 2004. Arbitrage Theory in Continuous Time, second ed. Oxford university press.

Brace, A., Dun, T., Barton, G., 2001. Towards a central interest rate model. In: Option Pricing, Interest Rates and Risk Management. In: Handb. Math. Finance, Cambridge Univ. Press, Cambridge, pp. 278–313.

Brace, A., Gatarek, D., Musiela, M., 1997. The market model of interest rate dynamics. Math. Finance 7, 127–155.

Brigo, D., Mercurio, F., 2006. Interest rate models—theory and practice. In: Springer Finance, second ed. Springer-Verlag, Berlin, With smile, inflation and credit.