Quantitative Finance
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/rquf20

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Available online: 28 Jul 2011

Quantitative Finance, Vol. 11, No. 8, August 2011, 1151–1164

On refined volatility smile expansion in the Heston model

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(*Received 11 March 2010; in final form 16 November 2010*)

It is known that Heston's stochastic volatility model exhibits moment explosion, and that the critical moment s_+ can be obtained by solving (numerically) a simple equation. This yields a leading-order expansion for the implied volatility at large strikes: $\sigma_{\mathrm{BS}}(k,T)^2T \sim \Psi(s_+ - 1)\times k$ (Roger Lee's moment formula). Motivated by recent 'tail-wing' refinements of this moment formula, we first derive a novel tail expansion for the Heston density, sharpening previous work of Drăgulescu and Yakovenko [*Quant. Finance*, 2002, **2**(6), 443–453], and then show the validity of a refined expansion of the type $\sigma_{\mathrm{BS}}(k,T)^2T = (\beta_1 k^{1/2} + \beta_2 + \cdots)^2$, where all constants are explicitly known as functions of s_+, the Heston model parameters, the spot vol and maturity T. In the case of the 'zero-correlation' Heston model, such an expansion was derived by Gulisashvili and Stein [*Appl. Math. Optim.*, 2010, **61**(3), 287–315]. Our methods and results may prove useful beyond the Heston model: the entire quantitative analysis is based on affine principles and at no point do we need knowledge of the (explicit, but cumbersome) closed-form expression of the Fourier transform of $\log S_T$ (equivalently the Mellin transform of S_T). What matters is that these transforms satisfy ordinary differential equations of the Riccati type. Secondly, our analysis reveals a new parameter (the '*critical slope*'), defined in a model-free manner, which drives the second- and higher-order terms in tail and implied volatility expansions.

Keywords: Volatility smile fitting; Stochastic volatility; Differential equations; Derivatives pricing

1. Introduction

The Heston (1993) model is one of the most popular stochastic volatility models used in the financial industry. Furthering its understanding, and in particular the understanding of its implied volatility surface, is of particular interest in light of the recent financial crisis: the volatility smile (underlying: SPX) did steepen after September 2008, then flattened again; it also steepened substantially after the flash crash in May 2010 and has since flattened again.‖ It is also worth recalling that the very existence of the volatility smile as we know it was triggered by the events of 1987.

This general motivation is complemented by an everyday question in the financial industry: how to (smoothly) extrapolate the smile seen in the market (typically a stepping stone towards the robust construction of a local volatility surface). Theorem 1.3 below contributes precisely in this direction and we derive new expansions for the implied volatility in the Heston model. Recall that its dynamics under the forward measure are given by

$$\mathrm{d}S_t = S_t\sqrt{V_t}\,\mathrm{d}W_t, \quad S_0 = 1,$$

$$\mathrm{d}V_t = (a + bV_t)\mathrm{d}t + c\sqrt{V_t}\,\mathrm{d}Z_t, \quad V_0 = v_0 > 0, \qquad (1.1)$$

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‖From a private communication with a derivative trader at a major investment bank.

Quantitative Finance
ISSN 1469–7688 print/ISSN 1469–7696 online © 2011 Taylor & Francis
http://www.informaworld.com
DOI: 10.1080/14697688.2010.541486

where $a \geq 0$, $b \leq 0$, $c > 0$, and $\mathrm{d}\langle W, Z\rangle_t = \rho\,\mathrm{d}t$ with $\rho \in (-1, 1)$. Observe that our choice $S_0 = 1$, as well as zero drift, entails no loss of generality. As is well-known (Andersen and Piterbarg 2007, Lions and Musiela 2007, Benaim and Friz 2008, Friz and Keller-Ressel 2010, Keller-Ressel 2011), the Heston model, as many other stochastic volatility models, exhibits *moment explosion* in the sense that

$$T^*(s) = \sup\{t \geq 0 : E[S_t^s] < \infty\}$$

is finite for sufficiently large s. (Here and throughout the paper, $E[\cdot]$ denotes the risk-neutral expectation.) In other words, for fixed maturity T, there will be a (finite) *critical moment*

$$s_+ := \sup\{s \geq 1 : E[S_T^s] < \infty\}.$$

(In the Heston model, and many other affine stochastic volatility models, T^* is explicitly known. The critical moment, for fixed T, is then found numerically from $T^*(s_+) = T$). A model-free result due to Lee, known as the moment formula (Lee 2004a, Benaim *et al.* 2008) (see also Benaim and Friz (2008, 2009), Friz (2010) and Gulisashvili (2010)), then yields

$$\limsup_{k\to\infty} \sigma_{\mathrm{BS}}(k, T)^2 T = \Psi(s_+ - 1) \times k, \tag{1.2}$$

where $k = \log(K/S_0)$ denotes the log-strike, σ_{BS} the Black–Scholes implied volatility, and

$$\Psi(x) = 2 - 4\left(\sqrt{x^2 + x} - x\right) \in [0, 2].$$

We remark that, subject to some 'regularity' of the moment blowup (fulfilled in all practical cases (Benaim and Friz 2008)), lim sup can be replaced by a genuine limit. Thus, the *total implied variance* $\sigma_{\mathrm{BS}}(k, T)^2 T$ is asymptotically linear in k with slope $\Psi(s_+ - 1)$. Similar results apply in the small strike limit $k \to -\infty$.

Parametric forms of the implied volatility smile used in the industry respect this behavior; a widely used parametrization is the following.

Example 1.1 (Gatheral's SVI parametrization (Gatheral 2004))**:** For fixed T, a parametric form of $\sigma_{\mathrm{BS}}(k, T)^2 T$ is given by

$$\begin{aligned} k \mapsto \mathfrak{a} + \mathfrak{b}[(-\mathfrak{m} + k)\mathfrak{r} + \sqrt{(-\mathfrak{m} + k)^2 + \mathfrak{s}}] \\ \equiv \mathrm{SVI}(k; \mathfrak{a}, \mathfrak{b}, \mathfrak{r}, \mathfrak{m}, \mathfrak{s}). \end{aligned}$$

An expansion for $k \to \infty$ yields

$$\begin{aligned} \mathrm{SVI}(k) &= k\mathfrak{b}(1 + \mathfrak{r}) + (\mathfrak{a} - \mathfrak{b}\mathfrak{m}(1 + \mathfrak{r})) + O(k^{-1}), \\ \sqrt{\mathrm{SVI}(k)} &= k^{1/2}\sqrt{\mathfrak{b}(1 + \mathfrak{r})} + k^{-1/2}\frac{(\mathfrak{a} - \mathfrak{b}\mathfrak{m}(1 + \mathfrak{r}))}{2\sqrt{\mathfrak{b}(1 + \mathfrak{r})}} \\ &\quad + O(k^{-3/2}), \end{aligned} \tag{1.3}$$

and we see that $\mathrm{SVI}(k)$ is asymptotically linear. We remark that this parametrization is not *ad hoc* but has been obtained by a $T \to \infty$ analysis of the Heston smile (Gatheral 2004, Forde *et al.* 2010, Gatheral and Jacquier 2011).

Our main results are the following two theorems. Remark **10** in section 3.3 and formula (4.10) in section 4 complement them by left-tail asymptotics.

Theorem 1.2: *For every fixed $T > 0$, the distribution density D_T of the stock price S_T in a correlated Heston model with $\rho \leq 0$ satisfies the following asymptotic formula:*

$$D_T(x) = A_1 x^{-A_3} \mathrm{e}^{A_2\sqrt{\log x}} (\log x)^{-3/4 + a/c^2}(1 + O((\log x)^{-1/2})), \tag{1.4}$$

as $x \to \infty$. The constants A_3 and A_2 are expressed explicitly in terms of critical moment s_+ and critical slope

$$\sigma := -\left.\frac{\partial T^*(s)}{\partial s}\right|_{s=s_+} \tag{1.5}$$

as

$$A_3 = s_+ + 1, \quad A_2 = 2\frac{\sqrt{2v_0}}{c\sqrt{\sigma}}. \tag{1.6}$$

An expression for A_1 is presented in remark **7** *below.*

Theorem 1.3: *Under the assumptions of theorem* 1.2, *the Black–Scholes implied volatility admits the expansion*

$$\sigma_{\mathrm{BS}}(k, T)^2 T = \left(\beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{1}{k^{1/2}}\right)\right)^2, \tag{1.7}$$

as $k \to \infty$, where

$$\begin{aligned} \beta_1 &= \sqrt{2}\left(\sqrt{A_3 - 1} - \sqrt{A_3 - 2}\right), \\ \beta_2 &= \frac{A_2}{\sqrt{2}}\left(\frac{1}{\sqrt{A_3 - 2}} - \frac{1}{\sqrt{A_3 - 1}}\right), \\ \beta_3 &= \frac{1}{\sqrt{2}}\left(\frac{1}{4} - \frac{a}{c^2}\right)\left(\frac{1}{\sqrt{A_3 - 1}} - \frac{1}{\sqrt{A_3 - 2}}\right). \end{aligned}$$

Remark 1: The restriction to $\rho \leq 0$ is (mathematically) not essential, but allows us to streamline the presentation. As is commonly noted, this covers essentially all practical applications of the Heston model. We also note that, since $(a + bV_t) = -b(a/(-b) - V_t)$, it can be helpful to look at $-b$ (respectively $\bar{v} = a/(-b)$) as the speed of mean-reversion (respectively mean-reversion level) of the Heston variance process.

Let us now draw attention to the main predecessors of this paper: Drăgulescu and Yakovenko (2002) apply a saddle point argument to deduce the leading order behavior of the density in the stationary variance regime: Essentially, $D_T(x) \approx x^{-A_3}$. Gulisashvili and Stein (2010) study the 'uncorrelated' Heston model ($\rho = 0$) and find the same functional form as in (1.4) and (1.7), with (more involved) explicit expressions for A_i and β_i. (Their method relies on representing call prices as averages of Black–Scholes prices and does not apply when $\rho \neq 0$.) While it is easy to see that, in the case $\rho = 0$, our expressions for A_3 agree, we check in appendix B (for the reader's peace of mind) that our $A_2 = 2(\sqrt{2v_0}/c\sqrt{\sigma})|_{\rho=0}$ coincides with their expression for A_2. In appendix C we present a numerical example that shows the accuracy of our asymptotic formula for the density, and of the resulting implied volatility expansion.

An interesting feature of our approach, somewhat in contrast to most analytic treatments of the Heston model,† is that our entire quantitative analysis is based on affine principles—at no point do we need knowledge of the (explicit, but cumbersome) closed-form expression of the Fourier transform of $\log S_T$ or, equivalently, the Mellin transform of S_T. (With one inconsequential exception, namely a simplification of the formula for the constant factor A_1.) Instead, we are able to extract all the necessary information on the transform by analysing the corresponding Riccati equations near criticality, using higher-order Euler estimates.‡ In conjunction with a classical saddle point computation we then 'implement' the Tauberian principle that the precise behavior of the transformed function near the singularity (the leading order of which is exactly described by the critical slope!) contains all the asymptotic information concerning the original function. At this heuristic level, we would expect that the critical slope σ, as defined in (1.5), is the key quantity that drives the second-order terms in tail and implied volatility expansions of general stochastic volatility models (even in the presence of jumps). Back to a rigorous level, it appears that the key ingredients of our analysis are applicable to general affine stochastic volatility models (Keller-Ressel 2011), and we hope to take this up in future work.

The explicit constants A_i and β_i for $i=1,2,3$ in the above theorem are clearly tied to the Heston model itself. In fact, it is the explicit nature of how these constants depend on the Heston parameters (a,b,c,ρ), as well as spot vol v_0 and maturity T, that furthers our understanding. Let us be explicit. It follows from equation (2.4) below that $s_+=s_+(b,c,\rho,T)$ does not depend on a, v_0 (equivalently, does not depend on $\bar{v}, v_0$). Furthermore, $s_+(T)\to s_+(\infty)\in(1,\infty)$ as $T\to\infty$. Moreover, the critical slope is explicitly computable: σ/T will be seen to be an explicit fraction involving only b, c, ρ and s_+, but not a, v_0 (equivalently, $\bar{v}, v_0$). We see, furthermore, that $1/\sigma=(T/\sigma)/T=O(1/T)$ as $T\to\infty$. As a consequence of all this, we see that changes in spot vol $\sqrt{v_0}$ are second-order effects: β_1 does not depend on $\sqrt{v_0}$, whereas β_2 has a linear dependence. Practically put, we see that increasing the spot vol allows us to up-shift the smile (intuitively obvious!), but does not affect its slopes at the extremes. We also note that changes in $\bar{v}$ are not seen until looking at β_3. No such information could be extracted from (1.2) and previous works.

Another application concerns the design of parametrizations of the implied volatility: the SVI expansion (1.3) is *not* compatible with the correct expansion (1.7), the latter having a constant term, β_2, which is not present in (1.3). (We are grateful to J. Gatheral for pointing this out.) The solution to this apparent contradiction (recall that SVI was obtained by a $T\to\infty$ analysis of the Heston smile) is simply that $\beta_2\propto A_2=O(\sigma^{-1/2})=O(T^{-1/2})\to 0$. In fact, this suggests that SVI-type parametrizations could well benefit from additional terms corresponding to such a β_2 term, essentially accounting for the fact that $T\neq\infty$.

2. Moment explosion in the Heston model

2.1. *Heston model as an affine model and moment explosion*

Consider the correlated Heston model given by (1.1), and set $X_t=\log S_t$. From basic principles of affine diffusions (see, e.g., Keller-Ressel (2011)) we know that

$$\log E[\mathrm{e}^{sX_t}] = \phi(s,t) + v_0\psi(s,t), \tag{2.1}$$

where the functions ϕ and ψ satisfy the following Riccati equations:

$$\dot{\phi} = F(s,\psi), \quad \phi(0)=0, \tag{2.2}$$

$$\dot{\psi} = R(s,\psi), \quad \psi(0)=0, \tag{2.3}$$

with $F(s,v)=av$ and $R(s,v)=\frac{1}{2}(s^2-s)+\frac{1}{2}c^2v^2+bv+s\rho cv$. In equation (2.3), $\dot{\phi}$ and $\dot{\psi}$ are the partial derivatives with respect to t of the functions ϕ and ψ, respectively. Our goal in section 2 is to identify the smallest positive singularity, $s=s_+$, of (2.1), and to analyse the asymptotic behavior of (2.1) in its vicinity. The estimates found will be put to use in section 3, where we perform the asymptotic inversion of the Mellin transform $E[\mathrm{e}^{(u-1)X_t}]$ of the Heston model.

Remark 2: The symbol s denotes a real parameter. The Riccati ODEs in (2.2) and (2.3) are also valid when s is replaced by a complex parameter $u=s+iy$.

Given $s\geq 1$, define the explosion time for the moment of order s by

$$T^*(s) = \sup\{t\geq 0 : E[\mathrm{e}^{sX_t}]<\infty\}.$$

An elementary computation gives

$$2c^2 \min_{\eta\in[0,\infty]} R(s,\eta) = -[(s\rho c+b)^2 - c^2(s^2-s)] =: -\Delta(s).$$

Let us also set $\chi(s)=s\rho c+b$. A typical situation in applications (a correlation parameter satisfying $\rho\leq 0$, and a non-zero mean reversion $b<0$) implies that χ is negative for $s\geq 0$. We thus assume in the sequel that

$$\chi(s)<0, \quad \text{for all } s\geq 0.$$

This assumption allows us to use the following formula from Keller-Ressel (2011, theorem 4.2):

$$T^*(s) = \begin{cases} +\infty, & \text{if } \Delta(s)\geq 0, \\ \int_0^\infty 1/R(s,\eta)\,\mathrm{d}\eta, & \text{if } \Delta(s)<0. \end{cases} \tag{2.4}$$

†Exceptions include Fahrner (2007) and Keller-Ressel (2011).
‡Such higher-order Euler estimates are studied in—and form the foundation of—rough path theory; see Friz and Victoir (2008) and Chapter 3, 10 in Friz and Victoir (2010).

Remark 3: The integral in (2.4) can be represented as follows. For $\Delta(s)<0$, we have

$$T^*(s) = \frac{2}{\sqrt{-\Delta(s)}}\left(\arctan\frac{\sqrt{-\Delta(s)}}{\chi(s)} + \pi\right). \tag{2.5}$$

The derivative

$$\partial_s T^* = \int_0^\infty -\frac{\partial_s R}{R^2}(s,\eta)\mathrm{d}\eta$$

can be computed explicitly. Indeed, from (2.5) we obtain

$$\partial_s T^*(s) = -T^*(s)\frac{2\rho c(s\rho c + b) - c^2(2s-1)}{2\Delta(s)} - \frac{[c^2(2s-1) - 2\rho c(s\rho c+b)](s\rho c+b) + 2\rho c\Delta(s)}{\Delta(s)[(s\rho c+b)^2 - \Delta(s)]}. \tag{2.6}$$

2.2. Moment explosion

For $t>0$, let $s_+(t)\geq 1$ be the (generalized) inverse of the (decreasing) function $T^*(\cdot)$, that is

$$s_+(t) = \sup\{s \geq 1 : E[\mathrm{e}^{sX_t}] < \infty\}.$$

Definition 2.1: Given $T>0$, we call

$$s_+ := s_+(T) = \sup\{s\geq 1 : E[S_T^s] < \infty\}$$

the 'critical moment'. The quantities

$$\sigma := -\partial_s T^*|_{s_+} \geq 0, \quad \kappa := \partial_s^2 T^*|_{s_+}$$

are called the 'critical slope' and the 'critical curvature', respectively. Note that s_+, σ, and κ depend on T.

Since $T^*(s_+) = T$, formula (2.6) implies that

$$\sigma = -\frac{\partial T^*}{\partial s}(s_+) = \frac{R_1}{R_2}, \tag{2.7}$$

where

$$\begin{aligned} R_1 &= Tc^2 s_+(s_+-1)[c^2(2s_+-1) - 2\rho c(s_+\rho c+b)] \\ &\quad - 2(s_+\rho c+b)[c^2(2s_+-1) - 2\rho c(s_+\rho c+b)] \\ &\quad + 4\rho c[c^2 s_+(s_+-1) - (s_+\rho c+b)^2] \end{aligned}$$

and

$$R_2 = 2c^2 s_+(s_+-1)[c^2 s_+(s_+-1) - (s_+\rho c+b)^2].$$

Remark 4: The critical moment s_+ can (and, in general, must) be obtained by a simple numerical root-finding procedure.

Let $s\geq 1$. We know that $T^*(s)$ is the explosion time of ψ. On the other hand, using the Riccati ODE for ψ, we see that

$$(1/\psi)^{\cdot} = -\frac{\dot\psi}{\psi^2} = -\frac{R(s,\psi)}{\psi^2}.$$

Since $R(s,u)/u^2 \to c^2/2$ as $u\to\infty$, we obtain

$$\psi(s,t) \sim \frac{1}{(c^2/2)(T^*(s)-t)}, \quad \text{as } t\uparrow T^*(s), \tag{2.8}$$

uniformly on bounded subintervals of $[1,\infty)$. Next, fix $T>0$. Then we have $T = T^*(s_+)$ with $s_+ = s_+(T)$. Since the function T^* is continuously differentiable (and even C^2) in s, we have

$$\begin{aligned} T^*(s) - T &= T^*(s) - T^*(s_+) \\ &= (s_+ - s)(\sigma + O(s_+ - s)) \\ &\sim \sigma(s_+ - s), \quad \text{as } s\uparrow s_+, \end{aligned} \tag{2.9}$$

where $\sigma = -\partial_s T^*|_{s_+}$ is the critical slope. Hence,

$$\psi(s,T) \sim \frac{2}{(s_+-s)c^2\sigma}, \quad \text{as } s\uparrow s_+ = s_+(T). \tag{2.10}$$

It follows from (2.8) and (2.10) that $\phi(s,t) = \int_0^t a\psi(s,\vartheta)\mathrm{d}\vartheta$ has a logarithmic blowup:

$$\phi(s,t) \sim -\frac{2a}{c^2}\log(T^*(s)-t), \quad \text{as } t\uparrow T^*(s),$$

or

$$\phi(s,T) \sim -\frac{2a}{c^2}\log((s^*-s)\sigma), \quad \text{as } s\uparrow s_+ = s_+(T).$$

The following lemma refines these asymptotic results.

Lemma 2.2: *For every $T>0$ and for $s\uparrow s_+ = s_+(T)$, the following formulas hold*:

$$\psi(s,T) = \frac{2}{(s_+-s)c^2\sigma} - \frac{b+s_+\rho c}{c^2} - \frac{\kappa}{c^2\sigma^2} + O(s_+-s), \tag{2.11}$$

$$\begin{aligned} \phi(s,T) &= \frac{2a}{c^2}\log\frac{1}{s_+-s} + \frac{2a}{c^2}\log\frac{T}{\sigma} \\ &\quad + a\int_0^T\left(\psi(s_+,\vartheta) - \frac{2}{c^2(T-\vartheta)}\right)\mathrm{d}\vartheta + O(s_+-s). \end{aligned} \tag{2.12}$$

Proof: The idea is to use (second order) Euler estimates for the Riccati ODEs near criticality. This yields the limiting behavior of $\psi(s,t)$ and $\phi(s,t)$ as $t\uparrow T^*(s)$, and we complete the proof using (2.9). More precisely, let us introduce time-to-criticality $\tau = T^*(s) - t$, and set $\hat\psi(s,\tau) = \psi(s, T^*(s)-\tau)$. Observe that $1/\hat\psi(s,0) = 0$ and

$$\begin{aligned} (1/\hat\psi)^{\cdot} &= -\frac{(\hat\psi)^{\cdot}}{\hat\psi^2} = \frac{1}{\hat\psi^2}R(s,\hat\psi) \\ &= \frac{c^2}{2} + \frac{b+s\rho c}{\hat\psi} + \frac{s^2-s}{2\hat\psi^2} = W(s, 1/\hat\psi), \end{aligned}$$

where $W(s,u) = (c^2/2) + (b+s\rho c)u + [(s^2-s)/2]u^2$. A higher-order Euler scheme for this ODE yields

$$\begin{aligned} (1/\hat\psi)(s,\tau) &= (1/\hat\psi)(s,0) + W(s,0)\tau \\ &\quad + W(s,0)W'(s,0)\tau^2/2 + o(\tau^2), \end{aligned}$$

as $\tau \to 0$ and s stays in a bounded interval. Since $W(s,0) = c^2/2$ and $W'(s,0) = b + s\rho c$, we obtain

$$1/\hat{\psi}(s,\tau) = \frac{c^2}{2}\tau\left(1 + \frac{b+s\rho c}{2}\tau + O(\tau^2)\right)$$
$$= \frac{c^2}{2}\tau\left(1 - \frac{b+s\rho c}{2}\tau + O(\tau^2)\right)^{-1}.$$

It follows that

$$\hat{\psi}(s,\tau) = \frac{1}{(c^2/2)\tau}\left(1 - \frac{b+s\rho c}{2}\tau + O(\tau^2)\right)$$
$$= \frac{2}{c^2\tau} - \frac{b+s\rho c}{c^2} + O(\tau), \qquad (2.13)$$

as $\tau = T^*(s) - t \downarrow 0$. Note that

$$\frac{1}{\tau} = \left(\sigma(s_+ - s) + \frac{1}{2}\kappa(s_+ - s)^2 + O\big((s_+ - s)^3\big)\right)^{-1}$$
$$= \frac{1}{\sigma(s_+ - s)} - \frac{\kappa}{2\sigma^2} + O(s_+ - s).$$

Hence we obtain

$$\psi(s,T) = \frac{2}{c^2\sigma(s_+ - s)} - \frac{b + s_+\rho c}{c^2} - \frac{\kappa}{c^2\sigma^2} + O(s_+ - s),$$

as $s \uparrow s_+ = s_+(T)$. For the expansion of $\phi(s,t) = \int_0^t a\psi(s,\vartheta)\mathrm{d}\vartheta$, we find

$$\phi(s,t) = a\int_0^t \left(\psi(s,\vartheta) - \frac{2}{c^2(T^*(s)-\vartheta)}\right)\mathrm{d}\vartheta$$
$$+ \frac{2a}{c^2}\int_0^t \frac{1}{T^*(s)-\vartheta}\mathrm{d}\vartheta$$
$$= \frac{2a}{c^2}\log\frac{1}{T^*(s)-t} + \frac{2a}{c^2}\log T^*(s)$$
$$+ a\int_0^t \left(\psi(s,\vartheta) - \frac{2}{c^2(T^*(s)-\vartheta)}\right)\mathrm{d}\vartheta$$
$$= \frac{2a}{c^2}\log\frac{1}{T^*(s)-t} + \frac{2a}{c^2}\log T^*(s)$$
$$+ a\int_0^{T^*(s)} \left(\psi(s,\vartheta) - \frac{2}{c^2(T^*(s)-\vartheta)}\right)\mathrm{d}\vartheta$$
$$+ O(T^*(s) - t). \qquad (2.14)$$

To see the last equality, note that the integrand of

$$\int_t^{T^*(s)} \left(\psi(s,\vartheta) - \frac{2}{c^2(T^*(s)-\vartheta)}\right)\mathrm{d}\vartheta = O(T^*(s) - t)$$

has an expansion resulting from (2.13), which may be integrated termwise (de Bruijn 1981). It now suffices to use (2.9) and (2.14) to see that, as $s \uparrow s_+ = s_+(T)$, formula (2.12) holds. $\square$

Remark 5: It readily follows from the proof that lemma 2.2 also holds as s tends to s_+ in the complex plane, provided that $\Re(s) < s_+$.

3. Mellin inversion via the saddle point method

Our proof of theorem 1.2 proceeds by an asymptotic analysis of $E[\mathrm{e}^{(u-1)X_T}]$, where u is complex. This is the Mellin transform of the density of S_T. As noted in section 2.1, we can represent it in terms of the functions ϕ and ψ appearing in the Riccati ODEs:

$$\log E[\mathrm{e}^{(u-1)X_T}] = \phi(u-1,T) + v_0\psi(u-1,T).$$

The density can be recovered using the Mellin inversion formula, that is

$$D_T(x) = \frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty} \mathrm{e}^{-uL+\phi(u-1,T)+v_0\psi(u-1,T)}\mathrm{d}u, \qquad (3.1)$$

where $L = \log x$, provided that s is in the fundamental strip, $s \in (s_-(T), s_+(T))$.

Remark 6: The integral in (3.1) exists, since its integrand decays exponentially at $\pm i\infty$ (see lemma A.1 in appendix A). Moreover, if $u-1$ is imaginary, then the characteristic function of the random variable $X_T = \log(S_T)$ decays exponentially. It follows that X_T (and therefore S_T) admits a smooth density. Since S_T is (a component of) a locally elliptic diffusion with smooth coefficients, this can also be seen by employing classical stochastic or PDE methods (see de Marco (2009) for recent advances in this direction).

We will deduce the asymptotics of (3.1) by the saddle point (or steepest descent) method (de Bruijn 1981, Flajolet and Sedgewick 2009). The main idea is to deform the contour of integration into a path of steepest descent from a saddle point of the integrand. In cases where the method can be applied successfully, the saddle becomes steeper and more pronounced as the parameter (x in our case) increases. We then replace the integrand with a local expansion around the saddle point. The resulting integral, taken over a small part of the contour containing the saddle point, is easy to evaluate asymptotically. Finally, it suffices to show that the tails of the original integral are negligible, in order to establish the asymptotics of the original integral. Our treatment bears similarities to the Taylor expansions studied by Wright (1932) and to the saddle point analysis of certain Lindelöf integrals (Flajolet *et al.* 2010). The type of pertinent singularity (exponential of a pole) is the same in all cases.

3.1. Finding the saddle point

A (real) saddle point of the integrand in formula (3.1) can be found by equating its derivative to zero. Since it usually suffices to calculate an approximate saddle point, we note that lemma 2.2 and remark **5** imply the following expansion, as $u \to u^* := s_+ + 1 = A_3$ with $\Re(u) < u^*$:

$$\phi(u-1,T) + v_0\psi(u-1,T)$$
$$= \frac{\beta^2}{u^*-u} + \frac{2a}{c^2}\log\frac{1}{u^*-u} + \Gamma + O(u^*-u), \qquad (3.2)$$

where we put $\beta^2 = 2v_0/c^2\sigma$ and

$$\Gamma = -v_0\left(\frac{b+s_+\rho c}{c^2} + \frac{\kappa}{c^2\sigma^2}\right) + \frac{2a}{c^2}\log\frac{T}{\sigma} + a\int_0^T\left(\psi(s_+,\vartheta) - \frac{2}{c^2(T-\vartheta)}\right)\mathrm{d}\vartheta. \tag{3.3}$$

Retaining only the dominant term of (3.2), we obtain the approximate saddle point equation:

$$\left[x^{-u}\exp\left(\frac{\beta^2}{u^*-u}\right)\right]' = 0,$$

or, equivalently,

$$-L + \frac{\beta^2}{(u^*-u)^2} = 0.$$

The solution to the previous equation,

$$\hat{u} = \hat{u}(x) := u^* - \beta L^{-1/2},$$

is the approximate saddle point of the integrand.

3.2. *Local expansion around the saddle point*

Our next goal is to expand the function $\phi(u-1,T) + v_0\psi(u-1,T)$ at the point $u=\hat{u}$. Put $u=\hat{u}+iy$, and recall that we use the following notation: $\sigma = -\partial_s T^*|_{s_+}$ and $L=\log x$. Since the (approximate) saddle point $\hat{u}$ approaches u^* as $L\to\infty$, we may find the expansion of the integrand using (3.2). To make the expansion valid uniformly w.r.t. the new integration parameter y, we confine y to the following small interval:

$$|y| < L^{-\alpha}, \quad \frac{2}{3} < \alpha < \frac{3}{4}. \tag{3.4}$$

The choice of the upper bound on α in (3.4) will be clear from the tail estimates obtained in appendix A. Since $u^*-u = \beta L^{-1/2} - iy$, we have

$$\begin{aligned}\frac{1}{u^*-u} &= \beta^{-1}L^{1/2}(1 - i\beta^{-1}L^{1/2}y)^{-1}\\ &= \beta^{-1}L^{1/2}(1 + i\beta^{-1}L^{1/2}y - \beta^{-2}Ly^2 + O(L^{3/2-3\alpha}))\\ &= \beta^{-1}L^{1/2} + i\beta^{-2}Ly - \beta^{-3}L^{3/2}y^2 + O(L^{2-3\alpha}).\end{aligned} \tag{3.5}$$

It follows that

$$\begin{aligned}\log\frac{1}{u^*-u} &= \log[\beta^{-1}L^{1/2}(1 + O(L^{1/2-\alpha}))]\\ &= \frac{1}{2}\log L - \log\beta + O(L^{1/2-\alpha}).\end{aligned}$$

Next, plugging the previous expansions, with $u=\hat{u}+iy$, into (3.2), we obtain the following asymptotic formula:

$$\begin{aligned}&\phi(\hat{u}-1+iy,T) + v_0\psi(\hat{u}-1+it,T)\\ &\quad = \beta L^{1/2} + \frac{a}{c^2}\log L + iLy - \beta^{-1}L^{3/2}y^2\\ &\qquad - \frac{2a}{c^2}\log\beta + \Gamma + O(L^{2-3\alpha}).\end{aligned} \tag{3.6}$$

3.3. *Saddle point approximation of the density*

For the sake of simplicity, we will first obtain formula (1.4) with a weaker error estimate $O((\log x)^{-1/4+\varepsilon})$, where $\varepsilon>0$ is arbitrary. We then explain how to obtain the stronger estimate $O((\log x)^{-1/2})$.

We shift the contour in the Mellin inversion formula (3.1) through the saddle point $\hat{u}$, so that

$$D_T(x) = \frac{1}{2\pi i}\int_{\hat{u}-i\infty}^{\hat{u}+i\infty} \mathrm{e}^{-uL+\phi(u-1,T)+v_0\psi(u-1,T)}\mathrm{d}u \tag{3.7}$$

$$= x^{-\hat{u}}\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-iyL+\phi(\hat{u}+iy-1,T)+v_0\psi(\hat{u}+iy-1,T)}\mathrm{d}y. \tag{3.8}$$

The term

$$x^{-\hat{u}} \approx x^{-u^*} = x^{-A_3}$$

will yield the leading-order decay in (1.4); its exponent corresponds to the *location* of the dominating singularity of the Mellin transform. The lower-order factors are dictated by the *type* of the singularity at $u=u^*$, to be unveiled in the following.

The 'tail' of the last integral in (3.8), corresponding to $|y|>L^{-\alpha}$, can be estimated using lemma A.3 (see appendix A). Therefore,

$$\begin{aligned}D_T(x) &= x^{-\hat{u}}\frac{1}{2\pi}\int_{-L^{-\alpha}}^{L^{-\alpha}} \mathrm{e}^{-iyL+\phi(\hat{u}+iy-1,T)+v_0\psi(\hat{u}+iy-1,T)}\mathrm{d}y\\ &\quad + x^{-A_3}\exp(2\beta L^{1/2} - \beta^{-1}L^{3/2-2\alpha} + O(\log L)).\end{aligned}$$

Next, using (3.6) and the equality $x^{-\hat{u}}\exp(\beta L^{1/2}) = x^{-u^*}\exp(2\beta L^{1/2})$, we obtain

$$\begin{aligned}D_T(x) &= \frac{\exp(\Gamma)}{2\pi}x^{-u^*}\mathrm{e}^{2\beta L^{1/2}}\beta^{-2a/c^2}L^{a/c^2}\\ &\quad\times\int_{-L^{-\alpha}}^{L^{-\alpha}}\exp(-\beta^{-1}L^{3/2}y^2)\mathrm{d}y\\ &\quad\times(1+O(L^{2-3\alpha})) + x^{-A_3}\\ &\quad\times\exp(2\beta L^{1/2} - \beta^{-1}L^{3/2-2\alpha} + O(\log L)).\end{aligned} \tag{3.9}$$

Evaluating the Gaussian integral, we obtain

$$\begin{aligned}&\int_{-L^{-\alpha}}^{L^{-\alpha}}\exp(-\beta^{-1}L^{3/2}y^2)\mathrm{d}y\\ &\quad = \beta^{1/2}L^{-3/4}\int_{-\beta^{-1/2}L^{3/4-\alpha}}^{\beta^{-1/2}L^{3/4-\alpha}}\exp(-w^2)\mathrm{d}w\\ &\quad \sim \beta^{1/2}L^{-3/4}\int_{-\infty}^{\infty}\exp(-w^2)\mathrm{d}w = \sqrt{\pi}\beta^{1/2}L^{-3/4}.\end{aligned} \tag{3.10}$$

Here we use the fact that the tails of the Gaussian integral are exponentially small in L. Taking into account (3.9) and (3.10), we can compare the main part of the asymptotic expansion and the two error terms:

$\mathrm{const}\times x^{-A_3}L^{a/c^2-3/4}\exp(2\beta L^{1/2})$
(main part),
$x^{-A_3}L^{a/c^2-3/4}\exp(2\beta L^{1/2})O(L^{2-3\alpha})$
(error from local expansion),
$x^{-A_3}\exp(2\beta L^{1/2} - \beta^{-1}L^{3/2-2\alpha} + O(\log L))$
(error from tail estimate).

Since $2-3\alpha<0$, the expression on the second line is asymptotically smaller than the main part. In addition, since $3/2-2\alpha>0$, the quantity $\exp(-\beta^{-1}L^{3/2-2\alpha})$ decays faster than any power of L. This shows that the expression on the third line is negligible in comparison with the error term in the local expansion. Hence, it suffices to keep only the error term resulting from the local expansion. As a result, the error term in the asymptotic formula for D_T is $O(L^{2-3\alpha})=O(L^{-1/4+\varepsilon})$. (Take α close to 3/4.) More precisely, using (3.9) and (3.10), we obtain the following formula:

$$D_T(x)=\left[\frac{\exp(\Gamma)}{2\pi}\sqrt{\pi}\beta^{1/2-2a/c^2}\right]x^{-(s_++1)}\mathrm{e}^{2\beta L^{1/2}}L^{-3/4+a/c^2}\times(1+O(L^{-1/4+\varepsilon})). \tag{3.11}$$

It follows from (3.11) that formula (1.4), with a weaker error estimate, holds for the correlated Heston model of interest.

Remark 7: The integral on the right-hand side of (3.3) can easily be calculated from the closed-form expression (Heston 1993, Gatheral 2006) of ψ. From (3.11), we thus obtain the explicit expression

$$A_1=\frac{1}{2\sqrt{\pi}}(2v_0)^{1/4-a/c^2}c^{2a/c^2-1/2}\sigma^{-a/c^2-1/4}\times\exp\left(-v_0\left(\frac{b+s_+\rho c}{c^2}+\frac{\kappa}{c^2\sigma^2}\right)-\frac{aT}{c^2}(b+c\rho s_+)\right)\times\left(\frac{2\sqrt{b^2+2bc\rho s_++c^2s_+(1-(1-\rho^2)s_+)}}{\left\{\begin{matrix}c^2s_+(s_+-1)\sinh\\ \frac{1}{2}\sqrt{b^2+2bc\rho s_++c^2s_+(1-(1-\rho^2)s_+)}\end{matrix}\right\}}\right)^{2a/c^2},$$

for the constant factor in (1.4).

Our next goal is to show how to obtain the relative error $O((\log x)^{-1/2})$ in formula (1.4). Taking two more terms in the expansion (3.5) of $1/(u^*-u)$, we obtain

$$\begin{aligned}\frac{1}{u^*-u}&=\beta^{-1}L^{1/2}(1-i\beta^{-1}L^{1/2}y)^{-1}\\&=\beta^{-1}L^{1/2}(1+i\beta^{-1}L^{1/2}y-\beta^{-2}Ly^2\\&\quad-i\beta^{-3}L^{3/2}y^3+\beta^{-4}L^2y^4+O(L^{5/2-5\alpha}))\\&=\beta^{-1}L^{1/2}+i\beta^{-2}Ly-\beta^{-3}L^{3/2}y^2\\&\quad-i\beta^{-4}L^2y^3+\beta^{-5}L^{5/2}y^4+O(L^{3-5\alpha}).\end{aligned}$$

Expanding the logarithm, we obtain

$$\begin{aligned}\log\frac{1}{u^*-u}&=\log(\beta^{-1}L^{1/2}\\&\quad\times(1+i\beta^{-1}L^{1/2}y-\beta^{-2}Ly^2+O(L^{3/2-3\alpha})))\\&=\frac{1}{2}\log L-\log\beta+i\beta^{-1}L^{1/2}y\\&\quad-\frac{1}{2}\beta^{-2}Ly^2+O(L^{3/2-3\alpha}).\end{aligned}$$

We insert these two expansions into (3.2) to obtain a refined expansion of the integrand:

$$\begin{aligned}&x^{-\hat{u}-iy}\exp(\phi(\hat{u}-1+iy,T)+v_0\psi(\hat{u}-1+it,T))\\&\quad=x^{-u^*}\exp\left(2\beta L^{1/2}+\frac{a}{c^2}\log L-\beta^{-1}L^{3/2}y^2-\frac{2a}{c^2}\log\beta+\Gamma\right)\\&\quad\times(1+c_1L^2y^3+c_2L^{5/2}y^4+c_3L^{1/2}y+c_4Ly^2\\&\quad+c_5L^{-1/2}+O(L^{-3/4+\varepsilon})),\end{aligned}\tag{3.12}$$

for constants $c_1,\dots,c_5$. Note that the terms with c_1 and c_2 come from $(u^*-u)^{-1}$, those involving c_3 and c_4 from $\log(u^*-u)^{-1}$, and that with c_5 from u^*-u. (To be precise, we have used the fact that the $O(\ )$ term in (3.2) is of the form $c(u^*-u)+O((u^*-u)^2)$, as is readily seen by a third-order Taylor expansion along the lines of section 2.2.)

We will now reason as in the proof of the weaker error estimate. The main term and the error term from the tail estimate remain the same. The error term from the local expansion can be obtained as follows. Integrate the functions on both sides of formula (3.12) and take into account that

$$\int_{L^{-\alpha}}^{L^{-\alpha}}y^3\exp(-\beta^{-1}L^{3/2}y^2)\mathrm{d}y=\int_{L^{-\alpha}}^{L^{-\alpha}}y\exp(-\beta^{-1}L^{3/2}y^2)\mathrm{d}y=0.$$

The two integrals resulting from the y^2 and y^4 terms in (3.12) are easily calculated; they yield a relative contribution of $L^{-1/2}$, which merges with the term $c_5L^{-1/2}$. Hence we see that the absolute error term from the local expansion is

$$x^{-A_3}L^{a/c^2-3/4}\exp(2\beta L^{1/2})\times O(L^{-1/2}).$$

This completes the proof of theorem 1.2.

Remark 8: Note that the preceding argument can be extended by taking more terms in the local expansion of the integrand. A full asymptotic expansion in descending powers of $L=\log x$ can thus be obtained, which replaces the error term $(1+O((\log x)^{-1/2}))$ in (1.4) by

$$1+C_1(\log x)^{-1/2}+C_2(\log x)^{-3/4}+\cdots+O((\log x)^{-m/4}),$$

with constants C_k and arbitrarily large m. This is a typical feature of the saddle point method (Flajolet and Sedgewick 2009, section VIII.3).

Remark 9: By a standard result on integrating functions of regular variation (Bingham *et al.* 1987, proposition 1.5.10), formula (1.4) yields the estimate

$$\mathbb{P}[S_T>x]=\frac{A_1}{A_3-1}x^{-A_3+1}\mathrm{e}^{A_2\sqrt{\log x}}(\log x)^{-3/4+a/c^2}\times(1+O((\log x)^{-1/2})),$$

as $x\to\infty$, for the tail of the distribution of S_T. Note that the main factor x^{-A_3+1} was obtained by Drăgulescu and Yakovenko (2002, section 6).

Remark 10: We briefly discuss the behavior of the Heston density $D_T(x)$ near zero. Define the lower critical moment by

$$s_- := \inf\{s \le 0 : E[S_T^s] < \infty\},$$

and the corresponding slope and curvature by

$$\sigma_- := \partial_s T^*|_{s_-} \ge 0, \quad \kappa_- := \partial_s^2 T^*|_{s_-}.$$

As $x \downarrow 0$, the integrand in (3.1) has a saddle point that approaches the singularity $s_- + 1$ at a speed of $(-\log x)^{-1/2}$. All steps of the subsequent analysis precisely parallel the case $x \to \infty$ treated above. The net result is

$$D_T(x) = B_1 x^{B_3} e^{B_2\sqrt{-\log x}}(-\log x)^{a/c^2 - 3/4} \times (1 + O((-\log x)^{-1/2})), \tag{3.13}$$

as $x \downarrow 0$, where

$$B_3 = -(s_- + 1), \quad B_2 = 2\frac{\sqrt{2v_0}}{c\sqrt{\sigma_-}},$$

$$B_1 = \frac{1}{2\sqrt{\pi}}(2v_0)^{1/4 - a/c^2} c^{2a/c^2 - 1/2} \sigma_-^{-a/c^2 - 1/4} \times \exp\left(-v_0\left(\frac{b + s_-\rho c}{c^2} + \frac{\kappa_-}{c^2\sigma_-^2}\right) - \frac{aT}{c^2}(b + c\rho s_-)\right) \times \left(\frac{2\sqrt{b^2 + 2bc\rho s_- + c^2 s_-(1 - (1-\rho^2)s_-)}}{\left\{\begin{array}{l} c^2 s_-(s_- - 1)\sinh \\ \frac{1}{2}\sqrt{b^2 + 2bc\rho s_- + c^2 s_-(1 - (1-\rho^2)s_-)} \end{array}\right\}}\right)^{2a/c^2}.$$

Remark 11: The density $D_T^{\log}$ of the log-spot price $\log S_T$ is given by

$$D_T^{\log}(x) = e^x D_T(e^x).$$

Its asymptotics readily follow from (1.4) and (3.13):

$$D_T^{\log}(x) = A_1 e^{-(A_3 - 1)x} e^{A_2\sqrt{x}} x^{a/c^2 - 3/4}(1 + O(x^{-1/2})), \quad x \to \infty,$$

and

$$D_T^{\log}(x) = B_1 e^{-(B_3 + 1)|x|} e^{B_2\sqrt{|x|}} |x|^{a/c^2 - 3/4}(1 + O(|x|^{-1/2})), \quad x \to -\infty.$$

Figure 1 shows the numerical fit of these approximations, using the set

$$a = \bar{v}\lambda, \quad b = -\lambda, \quad c = 0.2928, \quad v_0 = 0.0654,$$
$$\rho = -0.7571, \quad \bar{v} = 0.0707, \quad \lambda = 0.6067 \tag{3.14}$$

of typical market parameters (Schoutens *et al.* 2004).

4. Call pricing functions and smile asymptotics

Recall that our main result (theorem 1.2) is the following asymptotic formula for the stock price distribution density in a correlated Heston model with $S_0 = 1$:

$$D_T(x) = A_1 x^{-A_3} e^{A_2\sqrt{\log x}}(\log x)^{-(3/4)+(a/c^2)} \times (1 + O((\log x)^{-1/2})), \tag{4.1}$$

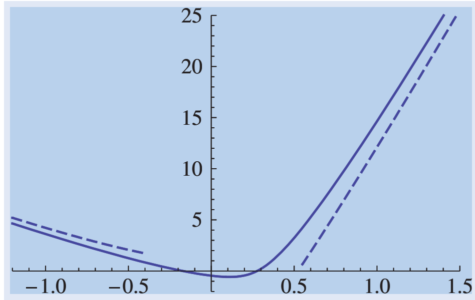

Figure 1. $-\log D_T^{\log}(x)$ with its asymptotic approximations, where $D_T^{\log}$ is the density of $\log S_T$.

as $x \to \infty$. In the present section we will characterize the asymptotic behavior of the call pricing function $K \mapsto C(K)$ in such a model, and then prove theorem 1.3. The following formula is a generalization of a similar result obtained for uncorrelated Heston models by Gulisashvili (2010):

$$C(K) = \frac{A_1}{(-A_3 + 1)(-A_3 + 2)} K^{-A_3 + 2} e^{A_2\sqrt{\log K}}(\log K)^{-(3/4)+(a/c^2)} \times \left(1 + O\left((\log K)^{-1/4}\right)\right), \tag{4.2}$$

as $K \to \infty$. Formula (4.2) follows from (4.1), Gulisashvili (2010, theorem 7.1) and Gulisashvili (2010, remark 6.1). Note that $A_3 > 2$.

We will now use the tail-wing formulas obtained by Gulisashvili (2010) to study the asymptotic behavior of the Black–Scholes-implied volatility $K \mapsto \sigma_{\text{BS}}(K, T)$ in a correlated Heston model in the case where the maturity T is fixed and the strike K approaches infinity or zero. The following statement was given by (Gulisashvili 2010, section 7). Suppose that the stock price density D_T in a general stock price model satisfies the condition

$$c_1 x^{-\xi} h(x) \le D_T(x) \le c_2 x^{-\xi} h(x), \tag{4.3}$$

for all large x, where $\xi > 2$, h is a slowly varying function, and c_1 and c_2 are positive constants. Then for every positive function φ on $(0, \infty)$ with $\lim_{x\to\infty} \varphi(x) = \infty$, we have the following:

$$\begin{aligned}
\sigma_{\text{BS}}(K, T)\frac{\sqrt{T}}{\sqrt{2}}
&= \sqrt{\log K + \log\frac{1}{K^2 D_T(K)} - \frac{1}{2}\log\log\frac{1}{K^2 D_T(K)}} \\
&\quad - \sqrt{\log\frac{1}{K^2 D_T(K)} - \frac{1}{2}\log\log\frac{1}{K^2 D_T(K)}} \\
&\quad + O((\log K)^{-1/2}\varphi(K)) \\
&= \sqrt{\log K + \log\frac{1}{K^{-\xi+2}h(K)} - \frac{1}{2}\log\log\frac{1}{K^{-\xi+2}h(K)}} \\
&\quad - \sqrt{\log\frac{1}{K^{-\xi+2}h(K)} - \frac{1}{2}\log\log\frac{1}{K^{-\xi+2}h(K)}} \\
&\quad + O((\log K))^{-1/2}\varphi(K)),
\end{aligned} \tag{4.4}$$

as $K \to \infty$.

A similar assertion holds for small values of the strike price (Gulisashvili 2010, section 7) and can be formulated as follows. Suppose that the stock price density D_T is such that

$$c_1 x^\gamma h(x^{-1}) \le D_T(x) \le c_2 x^\gamma h(x^{-1}), \qquad (4.5)$$

for all sufficiently small $x > 0$, where $\gamma > -1$, h is a slowly varying function, and c_1 and c_2 are positive constants. Let τ be a positive function on $(0, \infty)$ with $\lim_{K\to 0} \tau(K) = \infty$. Then

$$\begin{aligned}\sigma_{\mathrm{BS}}(K, T)\frac{\sqrt{T}}{\sqrt{2}} &= \sqrt{\log\frac{1}{K^2 D_T(K)} - \frac{1}{2}\log\log\frac{1}{K D_T(K)}} \\ &\quad - \sqrt{\log\frac{1}{K D_T(K)} - \frac{1}{2}\log\log\frac{1}{K D_T(K)}} \\ &\quad + O\left(\left(\log\frac{1}{K}\right)^{-1/2}\tau(K)\right) \\ &= \sqrt{\log\frac{1}{K^{\gamma+2}h(K^{-1})} - \frac{1}{2}\log\log\frac{1}{K^{\gamma+1}h(K^{-1})}} \\ &\quad - \sqrt{\log\frac{1}{K^{\gamma+1}h(K^{-1})} - \frac{1}{2}\log\log\frac{1}{K^{\gamma+1}h(K^{-1})}} \\ &\quad + O\left(\left(\log\frac{1}{K}\right)^{-1/2}\tau(K)\right), \qquad (4.6)\end{aligned}$$

as $K \to 0$.

Remark 12: The asymptotic formulas in (4.4) and (4.6) are equivalent to similar formulas with $\varphi(K) = 1$ and $\tau(K) = 1$, respectively. Indeed, if for some function f and all functions g, which tend to infinity, we have $f(K) = O(g(K))$ as $K \to \infty$, then $f(K) = O(1)$ as $K \to \infty$. This can be shown as follows. If the function f is not bounded near infinity, then there exists a sequence $K_n \uparrow \infty$ such that $f(K_n) \ge 2^n$ for all $n \ge 1$. Put $g(K_n) = n$, and define the function g by linear interpolation. Then $g(K) \to \infty$ as $K \to \infty$, but $f(K) \ne O(g(K))$ as $K \to \infty$. The proof for $K \to 0$ is similar. The authors thank Roger Lee for bringing this simple fact to their attention.

Now let us apply (4.4) and (4.6) to the Heston model. It is easy to see from (4.1) that (4.3) holds with $\xi = A_3$ and the slowly varying function

$$h(x) = \mathrm{e}^{A_2\sqrt{\log x}}(\log x)^{a/c^2 - 3/4}.$$

It follows from (4.4) and remark 12 that

$$\begin{aligned}\sigma_{\mathrm{BS}}(K, T)\frac{\sqrt{T}}{\sqrt{2}} &= \sqrt{(A_3 - 1)\log K - A_2\sqrt{\log K} - \left(\frac{a}{c^2} - \frac{3}{4}\right)\log\log K - \frac{1}{2}\log\log\frac{1}{K^{-A_3+2}h(K)}} \\ &\quad - \sqrt{(A_3 - 2)\log K - A_2\sqrt{\log K} - \left(\frac{a}{c^2} - \frac{3}{4}\right)\log\log K - \frac{1}{2}\log\log\frac{1}{K^{-A_3+2}h(K)}} \\ &\quad + O((\log K)^{-1/2}), \qquad (4.7)\end{aligned}$$

as $K \to \infty$. Next, using the mean value theorem, we see that it is possible to replace the term $\frac{1}{2}\log\log[1/K^{-A_3+2}h(K)]$ under the square roots in formula (4.7) by the term $\frac{1}{2}\log\log K$. Therefore,

$$\begin{aligned}\sigma_{\mathrm{BS}}(K, T)\frac{\sqrt{T}}{\sqrt{2}} &= \sqrt{(A_3 - 1)\log K - A_2\sqrt{\log K} - \left(\frac{a}{c^2} - \frac{1}{4}\right)\log\log K} \\ &\quad - \sqrt{(A_3 - 2)\log K - A_2\sqrt{\log K} - \left(\frac{a}{c^2} - \frac{1}{4}\right)\log\log K} \\ &\quad + O((\log K)^{-1/2}), \qquad (4.8)\end{aligned}$$

as $K \to \infty$. Since $\sqrt{1-h} = 1 - \frac{1}{2}h + O(h^2)$ as $h \to 0$, formula (4.8) implies that

$$\begin{aligned}\sigma_{\mathrm{BS}}(K, T)\frac{\sqrt{T}}{\sqrt{2}} &= \left(\sqrt{A_3 - 1} - \sqrt{A_3 - 2}\right)\sqrt{\log K} \\ &\quad + \frac{A_2}{2}\left(\frac{1}{\sqrt{A_3 - 2}} - \frac{1}{\sqrt{A_3 - 1}}\right) \\ &\quad + \frac{1}{2}\left(\frac{a}{c^2} - \frac{1}{4}\right)\left(\frac{1}{\sqrt{A_3 - 2}} - \frac{1}{\sqrt{A_3 - 1}}\right) \\ &\quad \times \frac{\log\log K}{\sqrt{\log K}} + O((\log K)^{-1/2}, \qquad (4.9)\end{aligned}$$

as $K \to \infty$. Next, using (4.9), we obtain the expansion (1.7) for the implied volatility $k \mapsto \sigma_{\mathrm{BS}}(k, T)$, considered as a function of the forward-log-in-moneyness $k = \log K$. Theorem 1.3 is thus proved. In the case where $\rho = 0$, formula (1.7) was obtained by Gulisashvili and Stein (2010) (see Gulisashvili and Stein (2010) and Gulisashvili (2010) for more details). Note that already the leading-order term,

$$\sigma_{\mathrm{BS}}(k, T)\sqrt{T} \sim \beta_1 k^{1/2}, \quad k \to \infty,$$

gives very good numerical approximation results. This term was obtained by Benaim and Friz (2008). As a 'lim sup' statement, based on Lee's moment formula, it appeared in the work of Andersen and Piterbarg (2007).

Let us denote by W_{BS} the Black–Scholes-implied total variance defined by

$$W_{\mathrm{BS}}(k, T) = \sigma_{\mathrm{BS}}(k, T)^2 T.$$

Then formula (1.7) implies the following expansion for $W_{\rm BS}$:

$$W_{\rm BS}(k,T) = \beta_1^2 k + 2\beta_1\beta_2 k^{1/2} + 2\beta_1\beta_3 \log k + O(\varphi(k)), \quad \text{as } k \to \infty,$$

where β_1, β_2, β_3, and φ are the same as in (1.7).

Similar reasoning can be used in the case where $k \to -\infty$. Put $\gamma = B_3$ and

$$h(x) = \mathrm{e}^{B_2\sqrt{\log x}}(\log x)^{a/c^2-3/4},$$

where B_2 and B_3 are defined in remark 10. In addition, fix a positive function φ on $(0,\infty)$ with $\lim_{x\to\infty}\varphi(x)=\infty$. Then (3.13) shows that all the conditions under which formula (4.6) holds are satisfied. Next, using (4.6) and simplifying, we obtain the following asymptotic formula for the implied volatility in the Heston model:

$$\sigma_{\rm BS}(k,T)\sqrt{T} = \rho_1(-k)^{1/2} + \rho_2 + \rho_3\frac{\log(-k)}{(-k)^{1/2}} + O\left(\frac{\varphi(-k)}{(-k)^{1/2}}\right), \tag{4.10}$$

as $k \to -\infty$. The constants in (4.10) are given by

$$\rho_1 = \sqrt{2}\left(\sqrt{B_3+2} - \sqrt{B_3+1}\right),$$

$$\rho_2 = \frac{B_2}{\sqrt{2}}\left(\frac{1}{\sqrt{B_3+1}} - \frac{1}{\sqrt{B_3+2}}\right),$$

$$\rho_3 = \frac{1}{\sqrt{2}}\left(\frac{1}{4} - \frac{a}{c^2}\right)\left(\frac{1}{\sqrt{B_3+2}} - \frac{1}{\sqrt{B_3+1}}\right).$$

For the total implied variance, we have

$$W_{\rm BS}(k,T) = \rho_1^2(-k) + 2\rho_1\rho_2(-k)^{1/2} + 2\rho_1\rho_3\log(-k) + O(\varphi(-k)),$$

as $k \to -\infty$.

Acknowledgements

P.F. and S.S. (affiliated to TU Berlin while this work was started) acknowledge support by MATHEON. S.G. was partially supported by the Austrian Federal Financing Agency and the Christian-Doppler-Gesellschaft.

References

Andersen, L.B.G. and Piterbarg, V.V., Moment explosions in stochastic volatility models. *Finance Stochast.*, 2007, **11**(1), 29–50.

Benaim, S. and Friz, P., Smile asymptotics II: Models with known moment generating functions. *J. Appl. Probab.*, 2008, **45**(1), 16–32.

Benaim, S. and Friz, P., Regular variation and smile asymptotics. *Math. Finance*, 2009, **19**(1), 1–12.

Benaim, S., Friz, P. and Lee, R.W., The Black Scholes implied volatility at extreme strikes. *Frontiers in Quantitative Finance: Volatility and Credit Risk Modeling*, 2008 (Wiley: New York). chapt. 2.

Bingham, N.H., Goldie, C.M. and Teugels, J.L., *Regular Variation, Encyclopedia of Mathematics and its Applications*, Vol. 27, 1987 (Cambridge University Press: Cambridge).

de Bruijn, N.G., *Asymptotic Methods in Analysis*, 3rd ed., 1981 (Dover: New York).

de Marco, S., Smoothness of densities and tail estimates for SDEs with locally smooth coefficients and applications to square-root type diffusions. Scuola Normale, Pisa, Preprint di Matematica No. 3, 2009.

del Baño Rollin, S., Ferreiro-Castilla, A. and Utzet, F., A new look at the Heston characteristic function, Preprint, 2009, (arxiv.org/abs/0902.2154).

Drăgulescu, A.D. and Yakovenko, V.M., Probability distribution of returns in the Heston model with stochastic volatility. *Quant. Finance*, 2002, **2**(6), 443–453.

Fahrner, I., Modern logarithms for the Heston model. *Int. J. Theor. Appl. Finance*, 2007, **10**(1), 23–30.

Flajolet, P., Gerhold, S. and Salvy, B., Lindelöf representations and (non-)holonomic sequences. *Electron. J. Combinator.*, 2010, Research Paper 3, 28pp.

Flajolet, P. and Sedgewick, R., *Analytic Combinatorics*, 2009 (Cambridge University Press: Cambridge).

Forde, M., Jacquier, A. and Mijatovic, A., Asymptotic formulae for implied volatility in the Heston model. *P. Roy. Soc. A.*, 2010, **466**, 3593–3620.

Friz, P., Implied volatility: Large strike asymptotics. In *Encyclopedia of Quantitative Finance*, edited by R. Cont, pp. 909–913, 2010 (John Wiley & Sons Ltd.: Chichester, UK).

Friz, P. and Keller-Ressel, M., Moment explosions. In *Encyclopedia of Quantitative Finance*, edited by R. Cont, pp. 1247–1253, 2010 (John Wiley & Sons Ltd: Chichester, UK).

Friz, P. and Victoir, N., Euler estimates for rough differential equations. *J. Different. Eqns*, 2008, **244**(2), 388–412.

Friz, P. and Victoir, N., *Multidimensional Stochastic Processes as Rough Paths. Theory and Applications*, Cambridge Studies of Advanced Mathematics, Vol. 120, 2010 (Cambridge University Press: Cambridge).

Gatheral, J., *A parsimonious arbitrage-free implied volatility parameterization with application to the valuation of volatility derivatives*, May 2004 (Presentation at Global Derivatives & Risk Management: Madrid). Available online at: http://www.faculty.baruch.cuny.edu/jgatheral/madrid2004.pdf

Gatheral, J., *The Volatility Surface, A Practitioner's Guide*, 2006 (Wiley: New York).

Gatheral, J. and Jacquier, A., Convergence of Heston to SVI. *Quant. Finance*, 2011, **11**(8), 1129–1132, this issue.

Gulisashvili, A., Asymptotic formulas with error estimates for call pricing functions and the implied volatility at extreme strikes. *SIAM J. Financial Math.*, 2010, **1**, 609–641.

Gulisashvili, A. and Stein, E.M., Asymptotic behavior of the stock price distribution density and implied volatility in stochastic volatility models. *Appl. Math. Optim.*, 2010, **61**(3), 287–315.

Heston, S., A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Financial Stud.*, 1993, **6**, 327–343.

Keller-Ressel, M., Moment explosions and long-term behavior of affine stochastic volatility models. *Math. Finance*, 2011, **21**(1), 73–98.

Lee, R.W., The moment formula for implied volatility at extreme strikes. *Math. Finance*, 2004a, **14**(3), 469–480.

Lee, R.W., Option pricing by transform methods: Extensions, unification, and error control. *J. Comput. Finance*, 2004b, **7**(3), 51–86.

Lions, P.-L. and Musiela, M., Correlations and bounds for stochastic volatility models. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 2007, **24**(1), 1–16.

Lucic, V., On singularities in the Heston model. Working Paper, 2007.

Schoutens, W., Simons, E. and Tistaert, J., A perfect calibration! Now what? *Wilmott Mag.*, 2004, **2**, 66–78.

Wright, E.M., The coefficients of a certain power series. *J. London Math. Soc.*, 1932, **7**, 256–262.

Appendix A: Tail estimates

It is known (Lucic 2007, del Baño Rollin *et al.* 2009) that all the singularities of the Mellin transform $E[\mathrm{e}^{(u-1)X_t}]$ of the stock price density D_T in the Heston model are located on the real line. Therefore, the function $u \mapsto \mathrm{e}^{\phi(u-1,T)+v_0\psi(u-1,T)}$ is analytic everywhere in the complex plane except the points of singularity on the real line. The next statement justifies the application of the Mellin inversion formula in (3.8), and will be useful in the tail estimate for the saddle point method. By symmetry, it clearly suffices to consider the upper tail ($\Im(u)>0$).

Lemma A.1: *Let $T>0$ and $1\le s_1\le \Re(s)\le s_2$. Then the following estimate holds as $\Im(s)\to\infty$:*

$$|\mathrm{e}^{\phi(s,T)+v_0\psi(s,T)}| = O(\mathrm{e}^{-C\Im(s)}),$$

where the constant $C>0$ depends on T, s_1, s_2, and v_0.

Proof: Let $s=\xi+iy$ and assume that $y>0$. We will first estimate the function ψ. Recall that

$$\dot{\psi} = \frac{1}{2}(s^2-s)+\frac{c^2}{2}\psi^2+b\psi+s\psi\rho c, \quad \text{with } \psi(\xi,0)=0.$$

Set $\psi=f+ig$ and $\gamma=-(b+\xi\rho c)$. Then $\gamma\ge 0$, and we have

$$\dot{f} = \frac{1}{2}(\xi^2-y^2-\xi)+\frac{c^2}{2}(f^2-g^2)-\gamma f, \quad f(s,0)=0,$$
$$\dot{g} = \frac{1}{2}(2\xi y-y)+c^2 fg-\gamma g, \quad g(s,0)=0.$$

Our goal is to show that there exists a positive continuously differentiable function $t\mapsto C(t)$ on $[0,T]$ such that

$$f(s,t)\le -C(t)\,y, \tag{A1}$$

where $s=\xi+iy$, $1\le s_1\le\xi\le s_2$, and y is sufficiently large. We first observe that f satisfies the differential inequality

$$\dot{f} \le \frac{1}{2}(\xi^2-y^2-\xi)+\frac{c^2}{2}f^2-\gamma f \tag{A2}$$

$$\le -\frac{1}{3}y^2+\frac{c^2}{2}f^2-\gamma f, \tag{A3}$$

for $y>y_0$, where y_0 depends only on s_1 and s_2. Set

$$V(y,r) = -\frac{1}{3}y^2+\frac{c^2}{2}r^2-\gamma r.$$

Then (A3) can be rewritten as

$$\dot{f}(s,t)\le V(y,f(s,t)), \tag{A4}$$

where $s=\xi+iy$.

We will next find a function $C(t)$, $t\in[0,T]$ with $C(0)=0$, strictly positive for $t>0$, and such that the function $F(y,t):=-C(t)y$ satisfies the differential inequality

$$V(y,F)\le \dot{F}. \tag{A5}$$

Let us first assume that such a function C exists. Then it is clear that given $s=\xi+iy$, the initial data $F(y,0)=f(s,0)=0$ match. Now we can use the ODE comparison results and derive from (A4) and (A5) that (A1) holds, which implies the following estimate:

$$|\mathrm{e}^{v_0\psi(s,T)}| = \mathrm{e}^{v_0 f(s,T)} \le \mathrm{e}^{-v_0 C(T)\Im(s)}, \tag{A6}$$

for all $s=\xi+iy$ with y sufficiently large and $s_1\le\xi\le s_2$.

We now look for the function C satisfying the equation

$$\dot{C}(t) = -\gamma C(t)+\theta,$$

where θ is a positive constant, and $C(0)=0$. The solution of this equation is given by

$$C(t) = \begin{cases} \theta\gamma^{-1}(1-\mathrm{e}^{-\gamma t}), & \text{if } \gamma>0, \\ \theta t, & \text{if } \gamma=0. \end{cases}$$

It follows that, for $t\in(0,T]$,

$$0<C(t)\le T\theta.$$

Next, choosing $\theta>0$ for which $-\frac{1}{3}+(c^2/2)T^2\theta^2=-\frac{1}{4}$, we obtain

$$\begin{aligned} V(y,F(y,t)) &\le -\frac{1}{3}y^2+\frac{c^2}{2}T^2\theta^2y^2+\gamma C(t)\,y \\ &= -\frac{1}{4}y^2+(\theta-\dot{C}(t))\,y \\ &\le -\dot{C}(t)\,y = \dot{F}(y,t). \end{aligned} \tag{A7}$$

In (A7), y is large enough and depends only on θ, and hence on the model parameter c and on T. This shows that the function F satisfies the differential inequality in (A5), and it follows that estimates (A1) and (A6) hold.

Finally, we note that

$$\Re(\phi(s,T)) = a\int_0^T f(s,t) \le ay\left(-\int_0^T C(t)\mathrm{d}t\right) = -ay\tilde{C}(T).$$

Therefore, for $\Im(s)$ large enough,

$$|\mathrm{e}^{\phi(s,T)+v_0\psi(s,T)}| \le \exp\{-(a\tilde{C}(T)+v_0C(T))\Im(s)\}.$$

The proof of lemma A.1 is thus completed. □

Lemma A.2: *If $B>0$ is any constant, then the portion of the integral* (3.7) *where $\Im(u)>B$ is $O(x^{-A_3}\exp(\beta L^{1/2}))$.* (*Recall that $L=\log x$.*)

Proof: If $\tilde{B}>B$ is a sufficiently large positive constant, then it readily follows from lemma A.1 that

$$\begin{aligned} &\left|\int_{\hat{u}+i\tilde{B}}^{\hat{u}+i\infty} \mathrm{e}^{-uL+\phi(u-1)+v_0\psi(u-1)}\mathrm{d}u\right| \\ &\quad\le Cx^{-A_3}\exp(\beta L^{1/2})\int_{\tilde{B}}^{\infty}\mathrm{e}^{-Cy}\mathrm{d}y \\ &\quad= O(x^{-A_3}\exp(\beta L^{1/2})). \end{aligned}$$

(The integral is clearly $O(1)$.) Moreover, since the Mellin transform of D_T does not have singularities outside the real line (Lucic 2007), we have

$$\left|\int_{\hat{u}+iB}^{\hat{u}+i\tilde{B}} \mathrm{e}^{-uL+\phi+v_0\psi}\mathrm{d}u\right| = O(\mathrm{e}^{-\hat{u}L}) = O(x^{-A_3}\exp(\beta L^{1/2})).$$

This completes the proof of lemma A.2. □

Lemma A.2 shows that the part of the tail integral where $\Im(u) > B$ is asymptotically much smaller than the central part. We will next estimate the whole tail integral.

Lemma A.3: *The following estimate holds for the tail integral*:

$$\left| \int_{\hat{u}+iL^{-\alpha}}^{\hat{u}+i\infty} \mathrm{e}^{-uL+\phi+v_0\psi} \mathrm{d}u \right| = x^{-A_3} \exp\left(2\beta L^{1/2} - \frac{1}{2}\beta^{-1} L^{3/2-2\alpha} + O(\log L) \right).$$

Proof: We will prove that there exists a constant $B > 0$ such that the absolute value of the part of the tail integral where $L^{-\alpha} < \Im(u) < B$ equals

$$x^{-A_3} \exp\left(2\beta L^{1/2} - \frac{1}{2}\beta^{-1} L^{3/2-2\alpha} + O(\log L) \right). \tag{A8}$$

It suffices to establish this statement, since lemma A.2 shows that the absolute value of the integral from $\hat{u} + iB$ to $\hat{u} + i\infty$ is asymptotically smaller than the expression in (A8). (Indeed: dividing (A8) by $x^{-A_3}\exp(\beta L^{1/2})$ yields $\exp(\beta L^{1/2} + O(L^{3/2-2\alpha}))$, which tends to infinity. Note that $3/2 - 2\alpha < 1/2$ by (3.4).)

It follows from lemma 2.2 and remark 5 that, for some constant $\gamma > 0$,

$$\mathrm{e}^{\phi(u-1,T)+v_0\psi(u-1,T)} = O\left(\exp\left(\frac{\beta^2}{A_3 - u} - \gamma \log(A_3 - u) \right) \right),$$

as u tends to $u^* = s_+ + 1 = A_3$ inside the analyticity strip. More verbosely, there exists a constant $C > 0$ such that, for a sufficiently small number $B > 0$ and for all u in the analyticity strip with $|\Im(u)| < B$ and $\Re(u) > u^* - B$, we have

$$|\mathrm{e}^{\phi(u-1)+v_0\psi(u-1)}| \leq C|A_3 - u|^{-\gamma} \exp\left(\Re\left(\frac{\beta^2}{A_3 - u} \right) \right).$$

Hence,

$$\begin{aligned}
&\left| \int_{\hat{u}+iL^{-\alpha}}^{\hat{u}+iB} \mathrm{e}^{-uL+\phi+v_0\psi} \mathrm{d}u \right| \\
&\leq Cx^{-A_3} \exp(\beta L^{1/2}) \int_{L^{-\alpha}}^{B} |A_3 - (\hat{u} + iy)|^{-\gamma} \\
&\quad \times \exp\left(\Re\left(\frac{\beta^2}{A_3 - (\hat{u} + iy)} \right) \right) \mathrm{d}y \\
&\leq Cx^{-A_3} \exp(\beta L^{1/2}) L^{\gamma/2} \exp\left(\frac{\beta^2 (A_3 - \hat{u})}{(A_3 - \hat{u})^2 + L^{-2\alpha}} \right) \\
&= Cx^{-A_3} \exp(2\beta L^{1/2} - \beta^{-1} L^{3/2-2\alpha} + O(\log L)).
\end{aligned}$$

We have used the fact that the factor $|A_3 - (\hat{u} + iy)|^{-\gamma}$ increases only like a power of L, since

$$\beta L^{-1/2} = A_3 - \hat{u} \leq |A_3 - (\hat{u} + iy)|.$$

Furthermore, the quantity

$$\Re\left(\frac{\beta^2}{A_3 - (\hat{u} + iy)} \right) = \frac{\beta^2 (A_3 - \hat{u})}{(A_3 - \hat{u})^2 + y^2} \tag{A9}$$

decreases w.r.t. $|y|$. Therefore, the integral $\int_{L^{-\alpha}}^{B}$ of (A9) can by estimated by the value of its integrand at $L^{-\alpha}$ times the length of the integration path. The latter is absorbed into C, and the former is given by

$$\begin{aligned}
\frac{\beta^2 (A_3 - \hat{u})}{(A_3 - \hat{u})^2 + L^{-2\alpha}} &= \beta L^{1/2} - \frac{\beta L^{1/2}}{\beta^2 L^{2\alpha-1} + 1} \\
&= \beta L^{1/2} - \beta^{-1} L^{3/2-2\alpha} + O(L^{5/2-4\alpha}).
\end{aligned}$$

(This can also be obtained by plugging $y = L^{-\alpha}$ into the singular expansion (3.5) computed above.) Finally, we write the factor $L^{\gamma/2}$ as exp $(O(\log L))$. □

Appendix B: Comparison of constants

Since s_+ is the order of the critical moment, it is not hard to see that if $\rho = 0$, then the constant A_3 defined by $A_3 = s_+ + 1$ is the same as the constant A_3 of Gulisashvili and Stein (2010).

We will now show that, for $\rho = 0$, the constant A_2 defined in (1.6) is the same as the corresponding constant of Gulisashvili and Stein (2010). It follows from (1.6) and (2.7) that the constant A_2 used in the present paper for $\rho = 0$ satisfies

$$A_2^2 = \frac{8v_0}{c^2\sigma}, \tag{B1}$$

with

$$\sigma = \frac{(2s_+ - 1)[Tc^2 s_+(s_+ - 1) - 2b]}{2s_+(s_+ - 1)[c^2 s_+(s_+ - 1) - b^2]}.$$

We will now turn our attention to the constant A_2 of Gulisashvili and Stein (2010). Lemmas 6.6 and 7.3 established by Gulisashvili and Stein (2010) provide an explicit expression for this constant. First note that the quantity $r = r_{(1/2)T|b|}$ in Gulisashvili and Stein (2010) and the quantity s_+ in the present paper are related by

$$r = \frac{T}{2}\left[c^2 s_+(s_+ - 1) - b^2 \right]^{1/2}. \tag{B2}$$

This follows from the formula for A_3 in (1.6) and from Gulisashvili and Stein (2010, lemmas 6.6 and 7.3).

It was shown by Gulisashvili and Stein (2010, lemmas 6.5, 6.6, and 7.3) that the following formula holds:

$$A_2 = \frac{B\sqrt{2}}{T^{1/4}(8C + T)^{1/4}},$$

with

$$\begin{aligned}
B &= \frac{\sqrt{2T}}{c} \left(\frac{Tv_0 \sin r}{2c^2(T^2/8r)\left|\left(1 + \frac{1}{2}T|b|\right)\cos r - r\sin r\right|} \right)^{1/2} \\
&\quad \times \left(b^2 + \frac{4}{T^2} r^2 \right)^{1/2} \\
&= \frac{2\sqrt{2}\sqrt{v_0}\sqrt{r\sin r}}{c^2\left|\left(1 + \frac{1}{2}T|b|\right)\cos r - r\sin r\right|^{1/2}} \left(b^2 + \frac{4}{T^2} r^2 \right)^{1/2}
\end{aligned}$$

and

$$C = \frac{T}{2c^2}\left(b^2 + \frac{4r^2}{T^2}\right).$$

Hence,

$$A_2 = \frac{4\sqrt{v_0}\sqrt{r\sin r}}{c^2\sqrt{T}\sqrt{2s_+ - 1}|(1+\frac{1}{2}T|b|)\cos r - r\sin r|^{1/2}} \times \left(b^2 + \frac{4}{T^2}r^2\right)^{1/2}.$$

Here we use the formulas for A_3 in (1.6) and in lemma 7.3 of Gulisashvili and Stein (2010). Since $r\cos r + \frac{1}{2}T|b|\sin r = 0$ and formula (B2) holds, we obtain the following relation between the constant A_2 of Gulisashvili and Stein (2010) and s_+:

$$\begin{aligned} A_2 &= \frac{4\sqrt{v_0}r}{c^2\sqrt{T}\sqrt{2s_+ - 1}\left[\frac{1}{2}T|b|(1+\frac{1}{2}T|b|) + r^2\right]^{1/2}} \times \left(b^2 + \frac{4}{T^2}r^2\right)^{1/2} \\ &= \frac{4\sqrt{v_0}\sqrt{s_+(s_+ - 1)}[c^2 s_+(s_+ - 1) - b^2]^{1/2}}{c\sqrt{2s_+ - 1}[Tc^2 s_+(s_+ - 1) - 2b]^{1/2}}. \end{aligned}$$

Therefore,

$$A_2^2 = \frac{16v_0 s_+(s_+ - 1)[c^2 s_+(s_+ - 1) - b^2]}{c^2(2s_+ - 1)[Tc^2 s_+(s_+ - 1) - 2b]}. \quad \text{(B3)}$$

Next, comparing (B1) and (B3), we see that the constant A_2 used in the present paper coincides with the corresponding constant of Gulisashvili and Stein (2010).

Appendix C: Numerical results

To conclude we illustrate the accuracy of (1.4) by a numerical example, and show plots of the corresponding smile approximations. We will use the parameter values (3.14). Note that (1.4) implies that

$$-\frac{\log D_T(x)}{\log x} \to A_3 \approx 33.2124, \quad \text{(C1)}$$

$$\frac{\log(x^{A_3}D_T(x))}{\sqrt{\log x}} \to A_2 \approx 12.3533, \quad \text{(C2)}$$

$$\frac{x^{A_3}D_T(x)}{\mathrm{e}^{A_2\sqrt{\log x}}(\log x)^{a/c^2 - 3/4}} \to A_1 \approx 2311.69, \quad \text{(C3)}$$

as $x \to \infty$. Figures C1–C3 plot the left- and right-hand sides of (C1–C3), with $\log x$ on the horizontal axis. The density D_T was evaluated by numerical integration of (3.8), using the explicit expressions (Heston 1993, del Baño Rollin *et al.* 2009) for ϕ and ψ.

Finally, to demonstrate the accuracy of the smile asymptotics, we plot the smile together with the asymptotic approximations. This is done by simply matching Heston prices with Black–Scholes prices by means of a root-finding procedure. To evaluate the Heston prices (with initial stock price $S_0 = 1$) we use Lee's (2004b) formula

$$C(T,k) = \frac{\mathrm{e}^{-\alpha k}}{\pi}\int_0^\infty \Re\left(\frac{\mathrm{e}^{-iuk}\phi(u - i(\alpha+1), T)}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u}\right)\mathrm{d}u,$$

where k is again the log-strike and α is a 'damping constant' which we are free to choose, noting only that for $\alpha > 0$ this formula gives us call prices, whereas for $\alpha < -1$ we obtain the prices of the respective puts. To optimize our results, we will use (following Lee) call options for

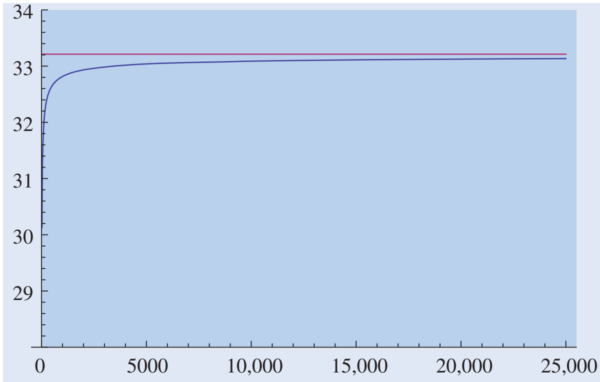

Figure C1. Numerical check for the constant A_3.

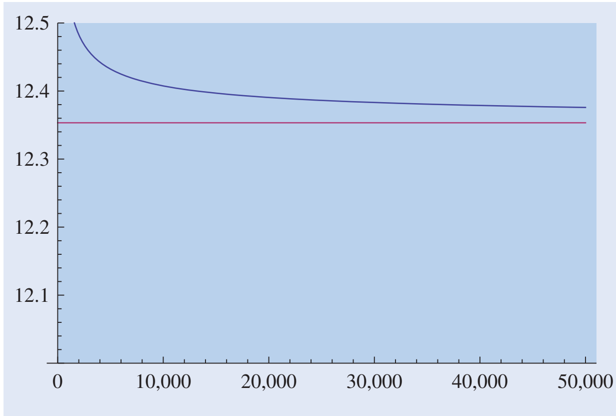

Figure C2. Numerical check for the constant A_2.

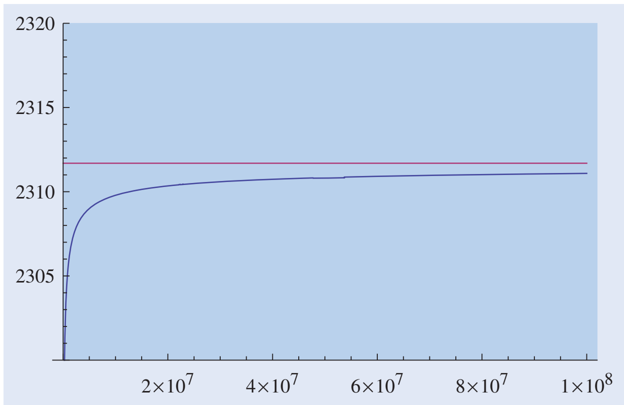

Figure C3. Numerical check for the constant A_1.

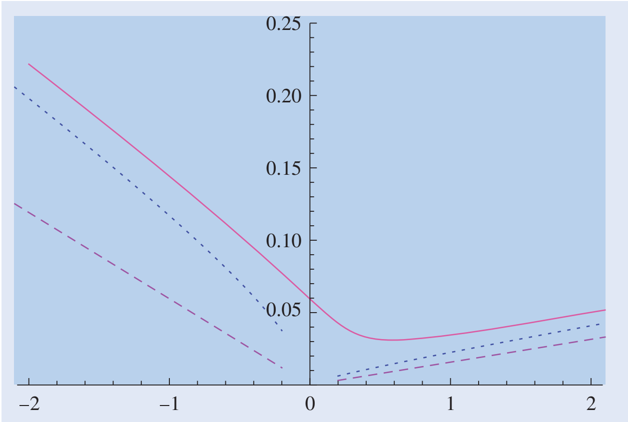

Figure C4. Implied variance $\sigma(k,1)^2$ in terms of log-strikes compared with the first-order (dashed) and third-order (dotted) approximations.

large strikes, and put options for small strikes, both with maturity $T = 1$. As a good choice for the damping constant α we suggest $\alpha = 29.1$ for the calls and $\alpha = -4.4$ for the puts.

The respective Black–Scholes prices are calculated by the Black–Scholes formula, evaluating the cumulative density function of the normal distribution by straightforward numerical integration.† To obtain good results for deep in-the-money/out-of-the-money options, we use as starting point for the root-finding procedure the value given by our third-order approximation. In the numerical example, this leads to a stable evaluation of the smile in a quite large interval, e.g. log-strikes ranging from -14 to 24. The results, compared with the first- and third-order asymptotics, are shown in figure C4. There, the log-strike is confined to the (more realistic) interval $[-2, 2]$.

†We thank Roger Lee for helpful comments on this numerical evaluation.