Finance Stoch (2010) 14: 81–128
DOI 10.1007/s00780-009-0104-1

A generalization of Panjer's recursion and numerically stable risk aggregation

Stefan Gerhold · Uwe Schmock · Richard Warnung

Received: 18 January 2008 / Accepted: 20 August 2008 / Published online: 20 August 2009

Abstract Portfolio credit risk models as well as models for operational risk can often be treated analogously to the collective risk model coming from insurance. Applying the classical Panjer recursion in the collective risk model can lead to numerical instabilities, for instance if the claim number distribution is extended negative binomial or extended logarithmic. We present a generalization of Panjer's recursion that leads to numerically stable algorithms. The algorithm can be applied to the collective risk model, where the claim number follows, for example, a Poisson distribution mixed over a generalized tempered stable distribution with exponent in (0, 1). De Pril's recursion can be generalized in the same vein. We also present an analogue of our method for the collective model with a severity distribution having mixed support.

Keywords Portfolio credit risk · CreditRisk$^+$ · Operational risk · Collective risk model · Extended negative binomial distribution · Extended logarithmic distribution · Compound distribution · Extended Panjer recursion · Numerical stability · De Pril's recursion · Poisson mixture distribution · Generalized tempered

This work was financially supported by the Christian Doppler Research Association (CDG). The authors gratefully acknowledge the fruitful collaboration and support by the Bank Austria and the Austrian Federal Financing Agency (ÖBFA) through CDG and the CD-Laboratory for Portfolio Risk Management (PRisMa Lab). The third author also gratefully acknowledges the financial support by WWTF MA13.

S. Gerhold · U. Schmock (✉)
Institute for Mathematical Methods in Economics, Vienna University of Technology,
Wiedner Hauptstraße 8–10/105-1, 1040 Vienna, Austria
e-mail: schmock@fam.tuwien.ac.at

S. Gerhold
e-mail: sgerhold@fam.tuwien.ac.at

R. Warnung
Risikomanagement, Raiffeisen Capital Management, Schwarzenbergplatz 3,
1010 Vienna, Austria
e-mail: richard.warnung@rcm.at

stable distribution · (Generalized) inverse Gaussian distribution · Reciprocal generalized inverse Gaussian distribution · Inverse gamma distribution · Severities with mixed support

Mathematics Subject Classification (2000) 91B30 · 65Q05 · 62P05

JEL Classification C63 · C16

1 Introduction

The aggregation of risks on a portfolio basis is a classical topic in insurance, which gained increasing importance in the context of credit risk (e.g. CreditRisk$^+$ [2, 9]) and the advanced measurement approach for operational risk ([18, Chap. 10] and [30]). By applying methods from the collective risk model to quantitative risk management, we work on the interface between actuarial sciences with their long history and financial mathematics with its steadily growing challenges. Modeling the above-mentioned losses on a portfolio basis usually leads to the problem to calculate the distribution of a compound sum

$$S = \sum_{n=1}^{N} X_n, \tag{1.1}$$

where the sequence $\{X_n\}_{n\in\mathbb{N}}$ of individual credit, operational or insurance losses, respectively, is i.i.d. and independent from the $\mathbb{N}_0$-valued number N of losses. The distribution of the random sum S is then called a compound distribution with primary distribution $\mathcal{L}(N)$ and secondary distribution $\mathcal{L}(X_1)$. According to the Basel II regulations [1], the loss distribution in credit portfolios as well as of the operational loss occurring to business lines of a bank has to be calculated up to high quantiles such as the 99% level and above [18, Sect. 10.1.3]. Recursive schemes such as Panjer's recursion offer a useful method to calculate the loss distribution, avoiding the stochastic error which is associated with a Monte Carlo approach (even when combined with variance reduction techniques). For calculating extreme quantiles as required by the Basel II regulations, it seems crucial to us to avoid the stochastic error.

Concerning the credit risk model CreditRisk$^+$, the need for numerically stable risk aggregation algorithms is repeatedly reported in the literature [7, 10]. It is one of the prime examples where a numerically unstable algorithm was recognized and remedies were proposed. We point out in Sect. 5.5 how a numerically stable algorithm based on an iterated application of Panjer's recursion can be constructed. Using our results, in particular Lemma 5.10 and Algorithm 5.3, we indicate how the CreditRisk$^+$ model can be extended, retaining its numerical stability.

In this article we present several algorithms to calculate the distribution of a random sum as given in (1.1), focusing on numerical stability. With the exception of Sect. 9, we assume that X_1 is $\mathbb{N}_0$-valued. If the distribution of N, denoted by $\{q_n\}_{n\in\mathbb{N}_0}$, belongs to a Panjer$(a, b, k)$ class, then the classical procedure to calculate the distribution of the aggregate loss S is to apply Panjer's recursion (cf. Theorem 4.1 below).

Recall that a probability distribution $\{q_n\}_{n\in\mathbb{N}_0}$ is said to belong to the Panjer(a,b,k) class with $a,b\in\mathbb{R}$ and $k\in\mathbb{N}_0$ if $q_0=q_1=\cdots=q_{k-1}=0$ and

$$q_n=\left(a+\frac{b}{n}\right)q_{n-1}\quad \text{for all } n\in\mathbb{N} \text{ with } n\geq k+1. \tag{1.2}$$

All distributions belonging to a Panjer(a,b,k) class were identified by Sundt and Jewell [29] for the case $k=0$, Willmot [33] for the case $k=1$, and finally Hess et al. [11] for general $k\in\mathbb{N}_0$. More general relations than (1.2) and corresponding recursion schemes have been considered in articles by Sundt [28], Hesselager [12], and Wang and Sobrero [31].

Panjer and Wang [21] show that for non-degenerate severity distributions, the numerical stability of Panjer's recursion for a claim number distribution belonging to the Panjer(a,b,k) class only depends on the values of a and b. They also establish the stability of infinite order linear recurrences with non-negative coefficients and non-negative starting values [21, Theorem 7]. The Poisson, the logarithmic, and the negative binomial distribution lead to recurrences of this kind, hence the computation of the aggregate loss is numerically stable. However, in the case of the extended negative binomial and the extended logarithmic distribution, quantities of opposite sign are added during Panjer's recursion, which can lead to numerical inaccuracies.

After recalling the definitions of these two distributions in Sect. 2, we present an illustrative example of a failed computation in Sect. 3. Section 4 contains our main result, which is a generalization of the classical Panjer recursion. It leads to stable algorithms for both distributions, which are presented in Sects. 5.1 and 5.2. Albeit slower than Panjer's recursion by a constant factor, they can reduce the numerical error substantially because cancellations cannot occur, cf. Table 1. The gist of our method is the reduction to a random sum with a different claim number distribution, whose computation by Panjer's recursion, for example, is numerically stable. Besides being of interest on its own as a claim number distribution, the extended negative binomial distribution occurs in the collective risk model with Poisson-mixed claim number distributions, which is the topic of Sects. 5.3 and 5.4. In particular, we introduce the generalized tempered α-stable distribution as Poisson mixture distribution and derive readily implementable, numerically stable recursive schemes, which to our knowledge are new. The Lévy distribution, the inverse gamma distribution with half-integer parameter and the (generalized) inverse Gaussian distribution are special examples for the case $\alpha=1/2$.

Sections 6 and 7 present some other claim number distributions to which our method can be applied. Similarly to our generalization of Panjer's recursion, De Pril's recursion for the moments of a compound distribution is extended in Sect. 8. Finally, we show in Sect. 9 how to adapt our method to severity distributions with mixed support.

2 Extended distributions from the Panjer class

As noted in the introduction, numerical stability of Panjer's recursion (4.2) cannot be guaranteed for certain claim number distributions as positive and negative terms are summed up. We briefly recall two families that will serve as typical examples of claim number distributions, where cancellation can arise and where our stable algorithm is applicable.

2.1 Extended negative binomial distribution

The extended negative binomial distribution[1] ExtNegBin(α, k, p) with parameters $k \in \mathbb{N}$, $\alpha \in (-k, -k+1)$ and $p \in [0, 1)$ is defined by $q_0 = \cdots = q_{k-1} = 0$ and, with $q = 1 - p$, by

$$q_n = \frac{\binom{\alpha+n-1}{n} q^n}{p^{-\alpha} - \sum_{j=0}^{k-1} \binom{\alpha+j-1}{j} q^j} \quad \text{for } n \in \mathbb{N} \text{ with } n \geq k. \tag{2.1}$$

The probability generating function (pgf) is given by

$$\varphi(s) = \sum_{n=k}^{\infty} q_n s^n = \frac{(1-qs)^{-\alpha} - \sum_{j=0}^{k-1} \binom{\alpha+j-1}{j} (qs)^j}{p^{-\alpha} - \sum_{j=0}^{k-1} \binom{\alpha+j-1}{j} q^j} \quad \text{for } |s| \leq \frac{1}{q}. \tag{2.2}$$

For the important case $k = 1$, hence $\alpha \in (-1, 0)$, this simplifies to

$$\varphi(s) = \frac{1 - (1-qs)^{-\alpha}}{1 - p^{-\alpha}} \quad \text{for } |s| \leq \frac{1}{q}. \tag{2.3}$$

It is well known [11] and straightforward to verify that ExtNegBin(α, k, p) is in the Panjer$(q, (\alpha-1)q, k)$ class.

The ordinary negative binomial distribution NegBin(α, p) with parameters $\alpha > 0$ and $p \in (0, 1)$ is also given by (2.1), its pgf by (2.2) for $|s| < 1/q$ and its Panjer class as above (with $k = 0$ in all three cases).

2.2 Extended logarithmic distribution

The second claim number distribution to which we want to apply our algorithm is the extended logarithmic distribution ExtLog(k, q) with parameters $k \in \mathbb{N} \setminus \{1\}$ and $q \in (0, 1]$. Its probability mass function $\{q_n\}_{n \in \mathbb{N}_0}$ is given by $q_0 = \cdots = q_{k-1} = 0$ and

$$q_n = \frac{\binom{n}{k}^{-1} q^n}{\sum_{\ell=k}^{\infty} \binom{\ell}{k}^{-1} q^\ell} \quad \text{for } n \geq k. \tag{2.4}$$

For a closed-form expression of the probability generating function we need

Lemma 2.1 *For $k \in \mathbb{N} \setminus \{1\}$ define*

$$\chi_k(x) = \frac{(-1)^k}{k} \sum_{n=k}^{\infty} \frac{x^n}{\binom{n}{k}}, \quad |x| \leq 1.$$

[1] See, for example, *Extended negative binomial distribution* at http://en.wikipedia.org/wiki/, version of July 2, 2008, for more detailed information.

Then

$$\chi_k(x) = (1-x)^{k-1}\log(1-x) + \sum_{i=1}^{k-1} a_{i,k} x^i, \quad |x| \le 1, \tag{2.5}$$

with the convention $0 \log 0 = 0$ *for the natural logarithm and*

$$a_{i,k} = \sum_{j=0}^{i-1} \binom{k-1}{j} \frac{(-1)^j}{i-j}, \quad i \in \{1, \dots, k-1\}. \tag{2.6}$$

Proof The representation for χ_k can be verified as follows: Insert the Taylor series $\log(1-x) = -\sum_{n\in\mathbb{N}} x^n/n$ to see that the coefficients of $x^1, \dots, x^{k-1}$ vanish on the right-hand side of (2.5). For the remaining coefficients, note that the kth derivative of the power series defining χ_k is proportional to the geometric series, more precisely

$$\chi_k^{(k)}(x) = (-1)^k \frac{(k-1)!}{1-x}, \quad |x| < 1,$$

and that the kth derivative of $(1-x)^{k-1}\log(1-x)$ leads to the same result. □

Note that (2.6) simplifies to $a_{1,k} = 1$ for $k \ge 2$, $a_{2,k} = \frac{3}{2} - k$ for $k \ge 3$, and $a_{3,k} = \frac{1}{3} + \frac{1}{2}(k-1)(k-3)$ for $k \ge 4$. Using Lemma 2.1, we see that the probability generating function of ExtLog(k, q) is given by

$$\varphi(s) = \frac{\sum_{n=k}^{\infty} \binom{n}{k}^{-1} (qs)^n}{\sum_{\ell=k}^{\infty} \binom{\ell}{k}^{-1} q^\ell} = \frac{\chi_k(qs)}{\chi_k(q)} \quad \text{for } |s| \le \frac{1}{q}. \tag{2.7}$$

We sometimes use ExtLog$(1, q)$ with $q \in (0, 1)$ to denote the logarithmic distribution Log(q), where $q_0 = 0$ and

$$q_n = -\frac{q^n}{n \log(1-q)}, \quad n \in \mathbb{N}. \tag{2.8}$$

The probability generating function of $\{q_n\}_{n\in\mathbb{N}_0}$ is

$$\varphi(s) = \frac{\log(1-qs)}{\log(1-q)}, \quad |s| \le \frac{1}{q}. \tag{2.9}$$

Again, it is well known [11] and straightforward to verify that ExtLog(k, q) is in the Panjer$(q, -kq, k)$ class for $k \in \mathbb{N}$.

3 An example of numerical instability

Since ExtLog(k, q) and ExtNegBin(α, k, p) are Panjer class distributions, we could use Panjer's recursion (cf. Theorem 4.1 below) to calculate the distribution $\{p_n\}_{n\in\mathbb{N}_0}$

of the aggregate loss S given by (1.1). Numerical stability cannot be guaranteed, however, because the term $(a+bj/n)$ in (4.2) can change its sign as j runs from 1 to n. To show that numerical inaccuracies are a real danger, let us consider the following example.

Example 3.1 Take $k \in \mathbb{N}$ and $\varepsilon, p \in (0,1)$, define $\alpha = -k+\varepsilon$, and let $\{q_n\}_{n\in\mathbb{N}_0}$ denote the distribution of $N \sim \text{ExtNegBin}(\alpha, k, p)$. Choose $\ell \in \mathbb{N}$ with $\ell \geq 3$, and $\mathbb{P}[X_1 = 1] = \mathbb{P}[X_1 = \ell] = 1/2$ as loss distribution. Since $\mathbb{P}[N \leq k-1] = 0$, we have

$$p_k = \mathbb{P}[N = k,\, X_1 = \cdots = X_k = 1] = \frac{q_k}{2^k}$$

and

$$\begin{aligned} p_{k+\ell-1} &= \sum_{j=1}^{k} \mathbb{P}\big[N = k,\, X_j = \ell,\, X_i = 1 \text{ for all } i \in \{1,\dots,k\} \setminus \{j\}\big] \\ &\quad + \mathbb{P}[N = k+\ell-1,\, X_1 = \cdots = X_{k+\ell-1} = 1] = \frac{kq_k}{2^k} + \frac{q_{k+\ell-1}}{2^{k+\ell-1}}. \end{aligned}$$

We now apply the Panjer recursion formula (4.2) for a frequency distribution in the $\text{Panjer}(q, q(\alpha-1), k)$ class, where $q = 1-p$. Since the sum S_k takes values in the set $\{k + j(\ell-1) \mid j = 0,\dots,k\}$, which does not contain $k+\ell$, the recursion formula for $p_{k+\ell}$ reduces to

$$\begin{aligned} p_{k+\ell} &= q\left(1 + \frac{\alpha-1}{k+\ell}\right)\frac{p_{k+\ell-1}}{2} + q\left(1 + \frac{\alpha-1}{k+\ell}\ell\right)\frac{p_k}{2} \\ &= q\,\frac{k(\ell-1)+\varepsilon k}{k+\ell}\left(\frac{q_k}{2^{k+1}} + \frac{q_{k+\ell-1}}{k2^{k+\ell}}\right) - q\,\frac{k(\ell-1)-\varepsilon\ell}{k+\ell}\,\frac{q_k}{2^{k+1}}. \end{aligned}$$

Hence, a severe cancellation occurs for $p_{k+\ell}$ when ε is small and $q_{k+\ell-1} \ll 2^{\ell-1}kq_k$. For example, the values $\varepsilon = 10^{-4}$, $k = 1$, $\ell = 5$ and $p = 1/10$ give

$$p_6 \approx 0.14999261827 - 0.14997008919 = 0.00002252908;$$

hence we lose four significant digits in this case.

Table 1 compares the results of Panjer's recursion and our alternative algorithm (see Theorem 4.5(a) and Corollary 5.1 below) applied to the above setting.

4 A generalization of the Panjer recursion

The famous Panjer recursion [20, 29] is contained in the following theorem.

Theorem 4.1 (Extended Panjer recursion) *Assume that the probability distribution* $\{q_n\}_{n\in\mathbb{N}_0}$ *of* N *belongs to the* $\text{Panjer}(a,b,k)$ *class and* $a\,\mathbb{P}[X_1 = 0] \neq 1$. *Then the*

Table 1 Creation and propagation of cancellation errors by Panjer's recursion for Example 3.1. The probabilities p_k in the middle column are calculated with double precision by our numerically stable algorithm. Relative errors when using Panjer's recursion with five significant digits are given in the last column. Using our numerically stable algorithm with five significant digits, the relative errors stay below 0.01% for the given probabilities

Loss k	Modified Panjer recursion with double precision	Panjer with 5 digits (relative error)
1	0.49996279266	≈0.00%
2	0.00001124916	33.33%
3	0.00000168754	33.33%
4	0.00000037971	33.33%
5	0.49996289519	≈0.00%
6	0.00002252908	77.55%
7	0.00000507252	96.47%
8	0.00000152220	104.36%
9	0.00000051380	106.99%
10	0.00001143414	34.53%

distribution $\{p_n\}_{n\in\mathbb{N}_0}$ of the random sum S given in (1.1) can be calculated by

$$p_0 = \varphi_N\big(\mathbb{P}[X_1=0]\big) = \begin{cases} q_0 & \text{if } \mathbb{P}[X_1=0]=0, \\ \mathbb{E}[(\mathbb{P}[X_1=0])^N] & \text{otherwise}, \end{cases} \tag{4.1}$$

where φ_N is the probability generating function of N, and by the recursion formula

$$p_n = \frac{1}{1-a\,\mathbb{P}[X_1=0]}\Bigg(\mathbb{P}[S_k=n]q_k + \sum_{j=1}^{n}\Big(a+\frac{bj}{n}\Big)\mathbb{P}[X_1=j]p_{n-j}\Bigg) \tag{4.2}$$

for all $n \in \mathbb{N}$, where $S_k = X_1 + \cdots + X_k$.

The distribution of S_k can be computed with at most $2\lfloor\log_2 k\rfloor$ convolutions, cf. Remark 4.3 below. The only compound distribution violating the technical condition $a\,\mathbb{P}[X_1=0] \neq 1$ arises from $a=1$ and $\mathbb{P}[X_1=0]=1$. Obviously, $p_0=1$ and $p_n=0$ for all $n\in\mathbb{N}$ in this trivial case.

Remark 4.2 (Calculation of the initial value) To apply the classical Panjer recursion (4.2), the probability p_0 of a loss of zero is needed as starting value, see (4.1). If $N \sim \text{Poisson}(\lambda)$ with $\lambda > 0$, then $\varphi_N(s) = e^{-\lambda(1-s)}$ for all $s\in\mathbb{R}$. If $N \sim \text{NegBin}(\alpha, p)$ with $\alpha > 0$ and $p \in (0,1)$, then $\varphi_N(s) = (p/(1-qs))^\alpha$ for all $|s| < 1/q$ by (2.2), where $q := 1-p$. These distributions are in the Panjer$(0,\lambda,0)$ and the Panjer$(q,(\alpha-1)q,0)$ class, respectively. When modeling large portfolios with the collective risk model (1.1) using one of these claim number distributions, it can happen for large λ or α, respectively, that p_0 is so small that it can only be represented as zero on a computer (numerical underflow). The recursion (4.2) then produces $p_n = 0$ for all $n \in \mathbb{N}$, which is clearly wrong. The standard solution, cf. [16, Sect. 6.6.2], is to perform Panjer's recursion with the reduced parameter $\lambda' := \lambda/2^n$ (resp. $\alpha' := \alpha/2^n$) instead, where n is chosen such that the new p_0 is properly representable on the computer. Afterwards, n iterative and numerically stable convolutions are needed to calculate the original probability distribution. This approach works because for independent $N_1, \ldots, N_{2^n} \sim \text{Poisson}(\lambda/2^n)$, we have that $N = N_1 + \cdots + N_{2^n} \sim \text{Poisson}(\lambda)$, similarly for the negative binomial distribution; see Remark 5.11 below for more details. In general, this works for claim number distributions closed under convolutions.

Remark 4.3 (Binomial distribution) The binomial distribution $\text{Bin}(m, p)$ with $m \in \mathbb{N}$ trials and success probability $p \in (0, 1)$ is in the $\text{Panjer}(-p/q, (m+1)p/q, 0)$ class, where $q := 1 - p$. If $N \sim \text{Bin}(m, p)$, then $\varphi_N(s) = (q + ps)^m$ for all $s \in \mathbb{R}$. If $n > m + 1$, then the term $a + bj/n$ in (4.2) changes sign while j runs from 1 to n, hence Panjer's recursion for the binomial distribution is numerically unstable due to cancellations. The problem with numerical underflow during the calculation of the initial value p_0 can also occur for large m, cf. Remark 4.2. Since

$$\varphi_S(s) = \varphi_N\big(\varphi_{X_1}(s)\big) = \big(q + p\varphi_{X_1}(s)\big)^m = \prod_{\substack{k=0\\ a_k=1}}^{\ell} \big(q + p\varphi_{X_1}(s)\big)^{2^k}, \quad s \in \mathbb{R},$$

where $m = \sum_{k=0}^{\ell} a_k 2^k$ with $a_0, \ldots, a_{\ell-1} \in \{0, 1\}$, $a_\ell = 1$ and $\ell = \lfloor \log_2 m \rfloor$ denotes the dyadic representation of m, we see that the distribution $\{p_n\}_{n\in\mathbb{N}_0}$ of S can be computed in a numerically stable way with $a_0 + \cdots + a_{\ell-1} + \ell \le 2\ell$ convolutions.

Remark 4.4 As a historical comment, we mention that Panjer's recursion for binomial, negative binomial, and extended negative binomial claim number distributions is contained in a much older result: For $\alpha \in \mathbb{R}$ and a power series $f(s) = \sum_{k=0}^{\infty} a_k s^k$ with $a_0 \neq 0$, the coefficients $\{b_n\}_{n\in\mathbb{N}_0}$ of the power series $f^{-\alpha}(s)$ satisfy the recursion

$$b_n = \frac{1}{na_0} \sum_{k=1}^{n} \big((1-\alpha)k - n\big) a_k b_{n-k}, \quad n \in \mathbb{N}.$$

Gould [8] has traced this remarkable, often rediscovered recurrence back to Euler [4, Sect. 76]. Using the probability generating functions of the above distributions and $\varphi_S = \varphi_N \circ \varphi_{X_1}$, this formula applied to $f(s) = 1 - q\,\varphi_{X_1}(s)$ yields recursions which indeed agree with the respective Panjer recursions.

Now we state and prove our main result. Instead of considering a single distribution that satisfies the Panjer recursion, we work with several claim number distributions that are linked by relation (4.4) below. We show that the corresponding compound distributions satisfy the weighted convolution relation (4.5). In this way, the calculation for a claim number distribution whose Panjer recursion is unstable can sometimes be reduced to one for which Panjer's recursion is stable. Part (b) of Theorem 4.5 aims at another relation between claim number distributions, a special case of which is given below in Corollary 4.7 by truncated distributions modified up to a certain claim number.

Theorem 4.5 *Fix $\ell \in \mathbb{N}$. Let $\{q_n\}_{n\in\mathbb{N}_0}$ and $\{\tilde{q}_{i,n}\}_{n\in\mathbb{N}_0}$ denote the probability distributions of the $\mathbb{N}_0$-valued random variables N and $\tilde{N}_i$ for $i \in \{1,\dots,\ell\}$, respectively, which are independent of the $\mathbb{N}_0$-valued i.i.d. sequence $\{X_n\}_{n\in\mathbb{N}}$. Let $\{p_n\}_{n\in\mathbb{N}_0}$ and $\{\tilde{p}_{i,n}\}_{n\in\mathbb{N}_0}$ denote the probability distributions of the random sums $S = X_1 + \cdots + X_N$ and $\tilde{S}_{(i)} = X_1 + \cdots + X_{\tilde{N}_i}$ for $i \in \{1,\dots,\ell\}$, respectively.*

(a) *Assume that there exist $k \in \mathbb{N}_0$ and $a_1,\dots,a_\ell, b_1,\dots,b_\ell \in \mathbb{R}$ such that*

$$\tilde{q}_{i,0} = \cdots = \tilde{q}_{i,k+\ell-i-1} = 0 \quad \textit{for all } i \in \big\{1,\dots,\min(\ell, k+\ell-1)\big\} \tag{4.3}$$

and

$$q_n = \sum_{i=1}^{\ell}\Big(a_i + \frac{b_i}{n}\Big)\tilde{q}_{i,n-i} \quad \textit{for all } n \in \mathbb{N} \textit{ with } n \ge k+\ell. \tag{4.4}$$

Then

$$p_n = \sum_{j=1}^{k+\ell-1} \mathbb{P}[S_j = n]q_j + \sum_{i=1}^{\ell}\sum_{j=0}^{n}\Big(a_i + \frac{b_i j}{in}\Big)\mathbb{P}[S_i = j]\tilde{p}_{i,n-j} \tag{4.5}$$

for all $n \in \mathbb{N}$, where $S_j = X_1 + \cdots + X_j$, and p_0 is given by (4.1).

(b) *Assume that there exist $\nu_1,\dots,\nu_\ell \in [0,1]$ with $\nu_1 + \cdots + \nu_\ell \le 1$ such that $q_n = \sum_{i=1}^{\ell} \nu_i \tilde{q}_{i,n}$ for all $n \in \mathbb{N}$. Then $p_n = \sum_{i=1}^{\ell} \nu_i \tilde{p}_{i,n}$ for all $n \in \mathbb{N}$.*

Note that Theorem 4.1 is a special case of Theorem 4.5(a). Indeed, if $\{q_n\}_{n\in\mathbb{N}_0}$ belongs to the Panjer(a,b,k) class, then Theorem 4.5(a) is applicable by choosing $\ell = 1$ and $\tilde{q}_{1,n} = q_n$ for all $n \in \mathbb{N}_0$, which implies $p_n = \tilde{p}_{1,n}$ for all $n \in \mathbb{N}_0$. Using $q_1 = \cdots = q_{k-1} = 0$, which implies (4.3), and solving (4.5) for p_n yields (4.2).

Remark 4.6 Algorithm 5.3 below for ExtNegBin(α,k,p) as well as Algorithm 5.6 for ExtLog(k,q), which are both based on Theorem 4.5, increases the computational effort relative to Panjer's recursion (4.2) to gain numerical stability. As Remarks 4.2 and 4.3 show, this is a classical trade-off between speed and numerical accuracy.

Proof of Theorem 4.5 (a) We extend the standard proof of Panjer's recursion (cf. Mikosch [19, Theorem 3.3.10] for the case $k = 0$ and $\ell = 1$) to our setting.

To prove the representation for the initial value given in (4.1), note that we have $\{S = 0,\ N = 0\} = \{N = 0\}$ and $\{S = 0, N \ge 1\} = \{X_1 = 0,\dots,X_N = 0,\ N \ge 1\}$. Hence

$$\begin{aligned} p_0 = \mathbb{P}[S=0] &= \mathbb{P}[S=0,\ N=0] + \mathbb{P}[S=0, N \ge 1] \\ &= q_0 + \mathbb{P}[X_1 = 0,\dots,X_N = 0,\ N \ge 1]. \end{aligned}$$

If $\mathbb{P}[X_1 = 0] = 0$, the second term is zero. Otherwise use the independence of N and $\{X_n\}_{n\in\mathbb{N}}$ as well as the i.i.d. assumption for this sequence to obtain

$$p_0 = q_0 + \sum_{n\in\mathbb{N}} \big(\mathbb{P}[X_1 = 0]\big)^n \, \mathbb{P}[N = n] = \mathbb{E}\big[\big(\mathbb{P}[X_1 = 0]\big)^N\big].$$

We now prove (4.5) for fixed $n \in \mathbb{N}$. For this we need a short preparation. Fix $i \in \{1, \dots, \ell\}$. For every $m \in \mathbb{N}$ with $m \geq i$, we use the representations $S_m = S_{m-i} + S_{i,m}$ with $S_{i,m} = X_{m-i+1} + \cdots + X_m$ and independent and identically distributed $X_1, \dots, X_m$. If $\mathbb{P}[S_m = n] > 0$, we obtain

$$\begin{aligned} 1 = \mathbb{E}\left[\frac{S_m}{n}\,\middle|\, S_m = n\right] &= \sum_{j=1}^{m} \mathbb{E}\left[\frac{X_j}{n}\,\middle|\, S_m = n\right] \\ &= m\,\mathbb{E}\left[\frac{X_m}{n}\,\middle|\, S_m = n\right] = m\,\mathbb{E}\left[\frac{S_{i,m}}{in}\,\middle|\, S_m = n\right], \end{aligned}$$

hence

$$a_i + \frac{b_i}{m} = \mathbb{E}\left[a_i + \frac{b_i S_{i,m}}{in}\,\middle|\, S_m = n\right] = \sum_{j=0}^{n} \left(a_i + \frac{b_i j}{in}\right) \mathbb{P}[S_{i,m} = j \,|\, S_m = n]. \quad (4.6)$$

For every $m \geq i$ we know that S_{m-i} and $S_{i,m}$ are independent and that $S_{i,m}$ has the same distribution as S_i, hence

$$\begin{aligned} \mathbb{P}[S_{i,m} = j,\, S_m = n] &= \mathbb{P}[S_{i,m} = j,\, S_{m-i} = n - j] \\ &= \mathbb{P}[S_i = j]\,\mathbb{P}[S_{m-i} = n - j]. \end{aligned} \quad (4.7)$$

Using the independence of S_m and N for every $m \in \mathbb{N}$,

$$p_n = \mathbb{P}[S = n] = \sum_{m\in\mathbb{N}} \mathbb{P}[S_m = n,\, N = m] = \sum_{m=1}^{k+\ell-1} \mathbb{P}[S_m = n] q_m + A_n, \quad (4.8)$$

and rewriting the abbreviation A_n using the representation (4.4), it follows that

$$A_n := \sum_{m=k+\ell}^{\infty} \mathbb{P}[S_m = n] q_m = \sum_{m=k+\ell}^{\infty} \sum_{i=1}^{\ell} \left(a_i + \frac{b_i}{m}\right) \mathbb{P}[S_m = n] \tilde{q}_{i,m-i}.$$

Inserting (4.6) and (4.7) yields

$$\begin{aligned} A_n &= \sum_{m=k+\ell}^{\infty} \sum_{i=1}^{\ell} \sum_{j=0}^{n} \left(a_i + \frac{b_i j}{in}\right) \mathbb{P}[S_i = j]\,\mathbb{P}[S_{m-i} = n - j] \tilde{q}_{i,m-i} \\ &= \sum_{i=1}^{\ell} \sum_{j=0}^{n} \left(a_i + \frac{b_i j}{in}\right) \mathbb{P}[S_i = j] \sum_{m=k+\ell}^{\infty} \mathbb{P}[S_{m-i} = n - j] \tilde{q}_{i,m-i}, \end{aligned}$$

where the rearrangement from the first to the second line is admissible, because we shall show that the series in the second line converge for every $i \in \{1, \dots, \ell\}$ and $j \in \{0, \dots, n\}$. Indeed, using (4.3), an index shift, and arguments as for (4.8),

$$\sum_{m=k+\ell}^{\infty} \mathbb{P}[S_{m-i} = n - j]\tilde{q}_{i,m-i} = \sum_{m=i}^{\infty} \mathbb{P}[S_{m-i} = n - j]\tilde{q}_{i,m-i}$$

$$= \sum_{m=0}^{\infty} \mathbb{P}[S_m = n - j, \tilde{N}_i = m] = \mathbb{P}[\tilde{S}_{(i)} = n - j] = \tilde{p}_{i,n-j}.$$

Substituting this result into A_n and then A_n into (4.8) gives (4.5).

(b) Modifying the calculation in (4.8) using $\mathbb{P}[N = m] = \sum_{i=1}^{\ell} \nu_i \, \mathbb{P}[\tilde{N}_i = m]$ for $m \in \mathbb{N}$, we obtain for every $n \in \mathbb{N}$

$$p_n = \sum_{m=1}^{\infty} \mathbb{P}[S_m = n]\,\mathbb{P}[N = m] = \sum_{i=1}^{\ell} \nu_i \sum_{m=1}^{\infty} \mathbb{P}[S_m = n]\mathbb{P}[\tilde{N}_i = m] = \sum_{i=1}^{\ell} \nu_i \, \tilde{p}_{i,n}. \qquad \square$$

The following corollary of Theorem 4.5(b) is useful when only a k-truncation of a probability distribution is in a Panjer(a, b, k) class as, e.g., in the case of a distribution of the Panjer(a, b, k) class modified at $0, \dots, k - 1$.

Corollary 4.7 *Assume that $\{q_n\}_{n\in\mathbb{N}_0}$ has mass at or above $k \in \mathbb{N}$. Let $\{\tilde{q}_n\}_{n\in\mathbb{N}_0}$ denote its k-truncated probability distribution, i.e. $\tilde{q}_0 = \cdots = \tilde{q}_{k-1} = 0$ and*

$$\tilde{q}_n := \frac{q_n}{1 - \sum_{j=0}^{k-1} q_j}, \qquad n \geq k. \tag{4.9}$$

Assume that N and $\tilde{N}$, respectively, have these distributions. Let $S = X_1 + \cdots + S_N$ and $\tilde{S} = X_1 + \cdots + N_{\tilde{N}}$ denote the corresponding random sums with distributions $\{p_n\}_{n\in\mathbb{N}_0}$ and $\{\tilde{p}_n\}_{n\in\mathbb{N}_0}$. Then p_0 is given by (4.1) and

$$p_n = \sum_{i=1}^{k-1} q_i \, \mathbb{P}[S_i = n] + \left(1 - \sum_{j=0}^{k-1} q_j\right) \tilde{p}_n, \qquad n \in \mathbb{N}. \tag{4.10}$$

Proof Apply Theorem 4.5(b) with $\ell = k$, $\nu_i = q_i$ and $\tilde{q}_{i,i} = 1$ for $i \in \{1, \dots, k - 1\}$, $\nu_k = 1 - (q_0 + \cdots + q_{k-1})$, $\tilde{q}_{k,n} = \tilde{q}_n$ for all $n \geq k$, and all other $\tilde{q}_{i,n} = 0$. $\square$

5 Application to numerical stability

5.1 Extended negative binomial distribution

As noted in Sect. 3, numerical stability of Panjer's recursion for the extended negative binomial distribution cannot be guaranteed. In this section we develop a remedy to this problem, see Algorithm 5.3 below.

Corollary 5.1 *For the parameters* $k \in \mathbb{N}_0$, $\alpha \in (-k, -k+1)$ *and* $p \in [0,1)$, *with* $p \neq 0$ *for* $k = 0$, *let* $\{q_n\}_{n\in\mathbb{N}_0}$ *denote the* $\mathrm{ExtNegBin}(\alpha-1, k+1, p)$ *distribution and* $\{\tilde{q}_n\}_{n\in\mathbb{N}_0}$ *the* $\mathrm{ExtNegBin}(\alpha, k, p)$ *distribution, where* $\mathrm{ExtNegBin}(\alpha, 0, p)$ *stands for the negative binomial distribution* $\mathrm{NegBin}(\alpha, p)$. *Then* (4.4) *holds with* $\ell = 1$ *and* $\tilde{q}_{1,n} = \tilde{q}_n$ *for* $n \geq k+1$. *The constants are given by* $a_1 = 0$ *and*

$$b_1 = (\alpha - 1)q \frac{p^{-\alpha} - \sum_{j=0}^{k-1} \binom{\alpha+j-1}{j} q^j}{p^{1-\alpha} - \sum_{j=0}^{k} \binom{\alpha+j-2}{j} q^j}, \tag{5.1}$$

hence (4.5) *simplifies to the numerically stable weighted convolution*

$$p_n = \frac{b_1}{n} \sum_{j=1}^{n} j\, \mathbb{P}[X_1 = j] \tilde{p}_{n-j}, \quad n \in \mathbb{N}, \tag{5.2}$$

and p_0 *is given by* (4.1) *with pgf* φ_N *from* (2.2).

Proof Using (2.1), we see that, for every $n \geq k+1$,

$$\binom{(\alpha-1)+n-1}{n} q^n = \frac{(\alpha-1)q}{n} \binom{\alpha+(n-1)-1}{n-1} q^{n-1},$$

hence $q_n = b_1 \tilde{q}_{n-1}/n$ and Theorem 4.5(a) is applicable. □

The case $k = 0$, $p = 0$ is excluded in the preceding corollary. We cannot reduce the calculation for a claim number $N \sim \mathrm{ExtNegBin}(\alpha-1, k+1, p)$ to the one for $N \sim \mathrm{ExtNegBin}(\alpha, k, p)$ in this case, because the negative binomial distribution is not defined for $p = 0$. However, a suitable limit $p \searrow 0$ gives the following numerically stable procedure.

Lemma 5.2 (Stable recursion for $\mathrm{ExtNegBin}(\alpha-1, 1, 0)$) *For* $\alpha \in (0,1)$ *consider a claim number* $N \sim \mathrm{ExtNegBin}(\alpha-1, 1, 0)$. *Then the distribution* $\{p_n\}_{n\in\mathbb{N}_0}$ *of the random sum* S *in* (1.1) *can be calculated by* $p_0 = 1 - (\mathbb{P}[X_1 \geq 1])^{1-\alpha}$ *and*

$$p_n = \begin{cases} \frac{1-\alpha}{n} \sum_{j=1}^{n} j\, \mathbb{P}[X_1 = j] r_{n-j} & \text{if } \mathbb{P}[X_1 \geq 1] > 0, \\ 0 & \text{if } \mathbb{P}[X_1 \geq 1] = 0, \end{cases} \quad n \in \mathbb{N},$$

where for the case $\mathbb{P}[X_1 \geq 1] > 0$ *the non-negative sequence* $\{r_n\}_{n\in\mathbb{N}_0}$ *is defined by* $r_0 = (\mathbb{P}[X_1 \geq 1])^{-\alpha}$ *and recursively in a numerically stable way by*

$$r_n = \frac{1}{\mathbb{P}[X_1 \geq 1]} \sum_{j=1}^{n} \frac{n - j + \alpha j}{n} \mathbb{P}[X_1 = j] r_{n-j}, \quad n \in \mathbb{N}.$$

Proof It is enough to consider the non-trivial case $\mathbb{P}[X_1 \geq 1] > 0$. We begin with $p \in (0,1)$ and let $\{\tilde{p}_n(p)\}_{n\in\mathbb{N}_0}$ denote the distribution of $\tilde{S} = X_1 + \cdots + X_{\tilde{N}}$, where $\tilde{N} \sim \mathrm{NegBin}(\alpha, p)$, and $\{p_n(p)\}_{n\in\mathbb{N}_0}$ the distribution of $S = X_1 + \cdots + X_N$, where

$N \sim \text{ExtNegBin}(\alpha - 1, 1, p)$. Since $\text{NegBin}(\alpha, p)$ is in the $\text{Panjer}(q, (\alpha - 1)q, 0)$ class, a recursion for the auxiliary sequence

$$r_n(p) := p^{-\alpha} \tilde{p}_n(p), \quad n \in \mathbb{N}_0, \tag{5.3}$$

follows from the Panjer recursion (4.2) for $\{\tilde{p}_n(p)\}_{n\in\mathbb{N}_0}$, namely

$$r_n(p) = \frac{1}{1 - q\,\mathbb{P}[X_1 = 0]} \sum_{j=1}^{n} q\left(1 + \frac{\alpha - 1}{n} j\right) \mathbb{P}[X_1 = j] r_{n-j}(p) \tag{5.4}$$

with starting value

$$r_0(p) = \big(1 - q\,\mathbb{P}[X_1 = 0]\big)^{-\alpha} \tag{5.5}$$

given by (4.1) with the pgf from (2.2). The weighted convolution (5.2) becomes

$$p_n(p) = \frac{p^\alpha b_1}{n} \sum_{j=1}^{n} j\,\mathbb{P}[X_1 = j] r_{n-j}(p), \quad n \in \mathbb{N}, \tag{5.6}$$

with $b_1 = (1 - \alpha) q p^{-\alpha} / (1 - p^{1-\alpha})$ from (5.1) and starting value

$$p_0(p) = \frac{1 - (1 - q\,\mathbb{P}[X_1 = 0])^{1-\alpha}}{1 - p^{1-\alpha}} \tag{5.7}$$

given by (4.1) with pgf from (2.3). The normalization in (5.3) is chosen so that we can take the limit $p \searrow 0$ (i.e., $q \nearrow 1$) in (5.4)–(5.7); in particular $p^\alpha b_1$ tends to $1 - \alpha$. With $r_n := \lim_{p\searrow 0} r_n(p)$ and $p_n := \lim_{p\searrow 0} p_n(p)$, the lemma follows. □

Algorithm 5.3 Corollary 5.1 and Lemma 5.2 lead to the following numerically stable algorithm for the calculation of the distribution of the aggregate loss in the collective risk model (1.1), where $N \sim \text{ExtNegBin}(\alpha, k, p)$ with $k \in \mathbb{N}$, $\alpha \in (-k, -k + 1)$ and $p \in [0, 1)$:

- If $p > 0$, perform a stable Panjer recursion according to Theorem 4.1 for $N \sim \text{NegBin}(\alpha + k, p)$, followed by a stable weighted convolution according to Corollary 5.1 to pass to $N \sim \text{ExtNegBin}(\alpha + k - 1, 1, p)$.
- If $p = 0$, use Lemma 5.2 to calculate the distribution of the compound sum S for $N \sim \text{ExtNegBin}(\alpha + k - 1, 1, p)$.

Calculate $k - 1$ weighted convolutions according to (5.2) to pass iteratively to $N \sim \text{ExtNegBin}(\alpha + k - 2, 2, p), \ldots,$ and finally to $N \sim \text{ExtNegBin}(\alpha, k, p)$.

Of course, compared to the ordinary (but possibly unstable) Panjer recursion of Theorem 4.1, Algorithm 5.3 increases the numerical effort by a factor of $k + 1$. Note that the weighted convolution in (5.2) is not a recurrence, hence unavoidable rounding errors do not propagate as in a recursive calculation.

5.2 Extended logarithmic distribution

Similar results as in the previous subsection can be obtained for the extended logarithmic distribution.

Corollary 5.4 *For the parameters $k \in \mathbb{N}$ and $q \in (0,1]$ with $q < 1$ in case $k = 1$, let $\{q_n\}_{n\in\mathbb{N}_0}$ denote the* $\mathrm{ExtLog}(k+1,q)$ *distribution and $\{\tilde{q}_n\}_{n\in\mathbb{N}_0}$ the* $\mathrm{ExtLog}(k,q)$ *distribution, where* $\mathrm{ExtLog}(1,q)$ *stands for* $\mathrm{Log}(q)$. *Then* (4.4) *holds with $\ell = 1$ and $\tilde{q}_{1,n} = \tilde{q}_n$ for $n \ge k+1$. The constants are given by $a_1 = 0$ and*

$$b_1 = (k+1)q\frac{\sum_{\ell=k}^{\infty}\binom{\ell}{k}^{-1}q^{\ell}}{\sum_{\ell=k+1}^{\infty}\binom{\ell}{k+1}^{-1}q^{\ell}} = \begin{cases} -\frac{q\log(1-q)}{q+(1-q)\log(1-q)} & \text{if } k=1,\\ -\frac{kq\chi_k(q)}{\chi_{k+1}(q)} & \text{if } k \ge 2, \end{cases} \tag{5.8}$$

where χ_k is given in Lemma 2.1. *Hence* (4.5) *again simplifies to the numerically stable weighted convolution* (5.2) *and p_0 is given by* (4.1) *with pgf φ_N from* (2.7).

Proof Using (2.4), we see that, for every $n \ge k+1$,

$$\frac{q^n}{\binom{n}{k+1}} = \frac{(k+1)q}{n}\frac{q^{n-1}}{\binom{n-1}{k}},$$

hence $q_n = b_1\tilde{q}_{n-1}/n$ and Theorem 4.5(a) is applicable. □

In the excluded case $(k,q) = (1,1)$, we cannot reduce the calculation for $N \sim \mathrm{ExtLog}(2,q)$ to that for $N \sim \mathrm{ExtLog}(1,q) = \mathrm{Log}(q)$, because the logarithmic distribution in (2.8) is not defined for $q = 1$. Fortunately, a similar limit consideration as for the extended negative binomial distribution works.

Lemma 5.5 (Stable recursion for ExtLog(2, 1)) *Assume that $N \sim \mathrm{ExtLog}(2,1)$. Then the distribution $\{p_n\}_{n\in\mathbb{N}_0}$ of the random sum S in* (1.1) *can be calculated by*

$$p_0 = \mathbb{P}[X_1 = 0] + \mathbb{P}[X_1 \ge 1]\log\mathbb{P}[X_1 \ge 1]$$

with the convention $0\log 0 = 0$, and

$$p_n = \begin{cases} \frac{1}{n}\sum_{j=1}^{n} j\,\mathbb{P}[X_1 = j]r_{n-j} & \text{if } \mathbb{P}[X_1 \ge 1] > 0,\\ 0 & \text{if } \mathbb{P}[X_1 \ge 1] = 0, \end{cases} \qquad n \in \mathbb{N},$$

where for the case $\mathbb{P}[X_1 \ge 1] > 0$ the non-negative sequence $\{r_n\}_{n\in\mathbb{N}_0}$ is defined by $r_0 = -\log\mathbb{P}[X_1 \ge 1]$ and recursively in a numerically stable way by

$$r_n = \frac{1}{\mathbb{P}[X_1 \ge 1]}\left(\mathbb{P}[X_1 = n] + \frac{1}{n}\sum_{j=1}^{n} j\,\mathbb{P}[X_1 = n-j]r_j\right), \quad n \in \mathbb{N}.$$

Proof Again, it suffices to consider the non-trivial case $\mathbb{P}[X_1 \geq 1] > 0$. We start with $q \in (0, 1)$ and let $\{\tilde{p}_n(q)\}_{n\in\mathbb{N}_0}$ denote the distribution of $\tilde{S} = X_1 + \cdots + X_{\tilde{N}}$, where $\tilde{N} \sim \text{Log}(q)$, and $\{p_n(q)\}_{n\in\mathbb{N}_0}$ the distribution of $S = X_1 + \cdots + X_N$, where $N \sim \text{ExtLog}(2, q)$. This time we define the auxiliary sequence $r_n(q)$ by

$$r_n(q) := -\tilde{p}_n(q) \log(1-q), \quad n \in \mathbb{N}_0. \tag{5.9}$$

Since $\text{Log}(q)$ is in the Panjer$(q, -q, 1)$ class, it satisfies

$$r_n(q) = \frac{q}{1 - q\,\mathbb{P}[X_1 = 0]} \left(\mathbb{P}[X_1 = n] + \frac{1}{n} \sum_{j=0}^{n-1} j\,\mathbb{P}[X_1 = n - j] r_j(q) \right), \quad n \in \mathbb{N}, \tag{5.10}$$

with starting value

$$r_0(q) = -\log\bigl(1 - q\,\mathbb{P}[X_1 = 0]\bigr) \tag{5.11}$$

given by (4.1) with the pgf from (2.9). Using (5.8) for $k = 1$, the weighted convolution (5.2) turns into

$$p_n(q) = \frac{q}{q + (1-q)\log(1-q)} \frac{1}{n} \sum_{j=1}^{n} j\,\mathbb{P}[X_1 = j] r_{n-j}(q), \quad n \in \mathbb{N}, \tag{5.12}$$

with starting value

$$p_0(q) = \frac{q\,\mathbb{P}[X_1 = 0] + (1 - q\,\mathbb{P}[X_1 = 0]) \log(1 - q\,\mathbb{P}[X_1 = 0])}{q + (1-q)\log(1-q)}. \tag{5.13}$$

Due to the normalization in (5.9), we can take the limit $q \nearrow 1$ in (5.10)–(5.13). Defining $r_n = \lim_{q\nearrow 1} r_n(q)$ and $p_n = \lim_{q\nearrow 1} p_n(q)$ finishes the proof. □

Algorithm 5.6 Corollary 5.4 and Lemma 5.5 lead to the following numerically stable algorithm for the calculation of the distribution of the aggregate loss S in the collective risk model (1.1), where $N \sim \text{ExtLog}(k, q)$ with $k \in \mathbb{N} \setminus \{1\}$ and $q \in (0, 1]$:

- If $q < 1$, perform a stable Panjer recursion according to Theorem 4.1 for $N \sim \text{Log}(q)$, followed by stable weighted convolution according to Corollary 5.4 to pass to $N \sim \text{ExtLog}(2, q)$.
- If $q = 1$, use Lemma 5.5 to calculate the distribution of the compound sum S for $N \sim \text{ExtLog}(2, 1)$.

If $k \geq 3$, calculate $k-2$ weighted convolutions according to (5.2) to pass iteratively to $N \sim \text{ExtLog}(3, q)$, $N \sim \text{ExtLog}(4, q)$, …, and finally to $N \sim \text{ExtLog}(k, q)$.

Of course, compared to a (possibly unstable) ordinary Panjer recursion according to Theorem 4.1 applied directly to $N \sim \text{ExtLog}(k, q)$, Algorithm 5.6 increases the numerical effort by a factor of k.

5.3 Poisson mixed over generalized tempered stable distributions

In this subsection and the following one, we show how the extended negative binomial distribution arises naturally in the collective risk model with claim number distributions mixed over a tempered stable distribution (e.g. the Lévy or the inverse Gaussian distribution) respectively mixed over a generalized tempered stable distribution, and how our weighted convolution (5.2) can improve numerical stability.

For a given parameter $\lambda > 0$ and a probability distribution function F with support contained in $[0, \infty)$, called the mixing distribution, we can define the corresponding Poisson mixture distribution $\{q_n\}_{n\in\mathbb{N}_0}$ by

$$q_n = \int_0^\infty \frac{(\lambda x)^n}{n!} e^{-\lambda x} F(dx), \quad n \in \mathbb{N}_0. \tag{5.14}$$

Mixed Poisson distributions serve as a rich class of claim number distributions, as they can exhibit effects such as heavy tails and over-dispersion.

It is well known that a mixed Poisson distribution is infinitely divisible if and only if the mixing distribution is infinitely divisible [17]. Furthermore, a theorem in Feller [5, Chap. 12, p. 290] says that any infinitely divisible distribution on the nonnegative integers can be represented as a compound Poisson distribution. Considering compound sums where the claim number follows a mixed Poisson distribution, Willmot [32] has noted that this implies that the distribution of the aggregate claims can be calculated by applying the Panjer recursion twice.

We start with a general observation on mixed Poisson distributions, which we illustrate by two examples, and give the application to the generalized tempered stable distribution afterwards.

Lemma 5.7 (Mixed claim numbers distributions) *Let F denote a probability distribution function with support contained in $[0, \infty)$. Assume that the expectation $c := \int_0^\infty x F(dx)$ is in $(0, \infty)$. Then*

$$\tilde{F}(x) = \frac{1}{c} \int_{[0,x]} y F(dy), \quad x \in [0, \infty), \tag{5.15}$$

is a probability distribution function.

(a) *For $\lambda > 0$ and F, define the Poisson mixture distribution $\{q_n\}_{n\in\mathbb{N}_0}$ by (5.14) and define $\{\tilde{q}_n\}_{n\in\mathbb{N}_0}$ similarly using $\tilde{F}$. Then*

$$q_n = \frac{c\lambda}{n} \tilde{q}_{n-1}, \quad n \in \mathbb{N}. \tag{5.16}$$

(b) *The negative binomial distribution* NegBin(r, p) *with parameters $p \in (0, 1)$ and $r > 0$ can be written as*

$$\binom{n+r-1}{n} p^r (1-p)^n = \binom{n+r-1}{n} \frac{Q^n}{(1+Q)^{n+r}}, \quad n \in \mathbb{N}_0,$$

with $Q := (1-p)/p \in (0,\infty)$. Define the mixture distribution

$$q_n(r) = \binom{n+r-1}{n} \int_0^\infty \frac{(Qx)^n}{(1+Qx)^{n+r}} F(dx), \quad n \in \mathbb{N}_0,$$

and define $\{\tilde{q}_n(r)\}_{n\in\mathbb{N}_0}$ similarly using $\tilde{F}$. Then

$$q_n(r) = \frac{cQr}{n} \tilde{q}_{n-1}(r+1), \quad n \in \mathbb{N}.$$

Proof Just note that

$$q_n = \frac{c\lambda}{n} \int_0^\infty \frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x} \frac{x}{c} F(dx), \quad n \in \mathbb{N}_0.$$

Note that

$$q_n(r) = \frac{cQr}{n} \binom{n+r-1}{n-1} \int_0^\infty \frac{(Qx)^{n-1}}{(1+Qx)^{n+r}} \frac{x}{c} F(dx), \quad n \in \mathbb{N}_0. \qquad \square$$

To illustrate the relationship between F and $\tilde{F}$, let us consider two examples.

Example 5.8 (Mixing with beta distribution) Let

$$f_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad x \in (0,1), \tag{5.17}$$

denote the density of the beta distribution with parameters $\alpha, \beta > 0$. If F has density $f_{\alpha,\beta}$, then $c = \frac{\alpha}{\alpha+\beta}$ and $f_{\alpha+1,\beta}$ is a density of $\tilde{F}$ in (5.15).

Example 5.9 (Mixing with gamma distribution) For parameters $\alpha, \beta > 0$,

$$g_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0, \tag{5.18}$$

is a density of the gamma distribution $\text{Gamma}(\alpha,\beta)$. For later use note that the Laplace transform of $\Lambda \sim \text{Gamma}(\alpha,\beta)$ is given by

$$\mathbb{E}\big[e^{-\Lambda s}\big] = \left(\frac{\beta}{\beta+s}\right)^\alpha \quad \text{for } s > -\beta, \tag{5.19}$$

as can be seen by rewriting the integrand in terms of $g_{\alpha,\beta+s}$. If F has density $g_{\alpha,\beta}$, then $c = \alpha/\beta$ and $g_{\alpha+1,\beta}$ is a density of $\tilde{F}$ in (5.15).

The family of stable distributions [25, 26] is a very flexible family of infinitely divisible distributions denoted by $\mathrm{S}_\alpha(\sigma,\beta,\mu)$ with $\alpha \in (0,2]$, $\sigma > 0$, $\beta \in [-1,1]$, and $\mu \in \mathbb{R}$. The support of $\mathrm{S}_\alpha(\sigma,\beta,\mu)$ is $[0,\infty)$ if $\alpha \in (0,1)$, $\beta = 1$ and $\mu = 0$ (see [25, p. 15]). It can thus serve as a mixing distribution F in (5.14). Note that the densities of this distribution family in general do not have a closed form, and therefore this family is usually characterized by its Laplace transform or its characteristic function. For $Y \sim S_\alpha(\sigma, 1, 0)$ the Laplace transform is given by

$$\mathbb{E}\big[\exp(-sY)\big] = \exp\big(-\gamma_{\alpha,\sigma} s^\alpha\big) \quad \text{for } s \geq 0 \text{ with } \gamma_{\alpha,\sigma} := \frac{\sigma^\alpha}{\cos(\alpha\pi/2)}, \tag{5.20}$$

cf. [25, Proposition 1.2.12]. We now generalize this distribution by introducing the additional parameters $\tau \geq 0$ and $m \in \mathbb{N}_0$ to define the distribution function

$$F_{\alpha,\sigma,\tau,m}(y) = \frac{\mathbb{E}[Y^{-m} e^{-\tau Y} 1_{\{Y \leq y\}}]}{\mathbb{E}[Y^{-m} e^{-\tau Y}]}, \quad y \in \mathbb{R}. \tag{5.21}$$

To see that this is well defined, it suffices to show that the moment $\mathbb{E}[Y^{-m}]$ exists. For $a > 0$, integration by parts yields

$$\int_a^\infty y^{-m} F_{\alpha,\sigma,0,0}(dy) = -a^{-m} F_{\alpha,\sigma,0,0}(a) + m \int_a^\infty y^{-m-1} F_{\alpha,\sigma,0,0}(y)\,dy. \tag{5.22}$$

It is a consequence of the Hardy–Littlewood–Karamata Tauberian theorem that

$$F_{\alpha,\sigma,0,0}(y) = o\big(\exp\big(-\gamma_{\alpha,\sigma} y^{-\alpha}\big)\big) \quad \text{as } y \searrow 0.$$

See Feller [6, Sect. XIII.6, Theorem 1] for a proof of this fact in the case $\gamma_{\alpha,\sigma} = 1$, which trivially extends to $\gamma_{\alpha,\sigma} > 0$. This implies that the right-hand side of (5.22) converges as $a \searrow 0$, hence $\mathbb{E}[Y^{-m}] < \infty$.

Now suppose that $\Lambda_{\tau,m} \sim F_{\alpha,\sigma,\tau,m}$, and consider a mixture model according to (5.14), where

$$\mathcal{L}\big(N^{(m)}\big|\Lambda_{\tau,m}\big) \stackrel{\text{a.s.}}{=} \text{Poisson}(\lambda\Lambda_{\tau,m}) \quad \text{with } \lambda > 0. \tag{5.23}$$

The probability generating function of $N^{(m)}$ is given by

$$\begin{aligned}\varphi_{N^{(m)}}(s) &= \mathbb{E}\big[s^{N^{(m)}}\big] = \mathbb{E}\big[\mathbb{E}\big[s^{N^{(m)}}\,\big|\,\Lambda_{\tau,m}\big]\big] = \mathbb{E}\big[e^{-\lambda(1-s)\Lambda_{\tau,m}}\big] \\ &= \frac{\mathbb{E}[Y^{-m} e^{-(\tau+\lambda(1-s))Y}]}{\mathbb{E}[Y^{-m} e^{-\tau Y}]} \quad \text{for } |s| \leq \frac{\tau+\lambda}{\lambda}.\end{aligned} \tag{5.24}$$

In the special case $m = 0$, the distribution $F_{\alpha,\sigma,\tau,0}$ is known as a τ-tempered α-stable distribution [24]. Using (5.20) and (5.21), we see that the Laplace transform of $\Lambda_{\tau,0} \sim F_{\alpha,\sigma,\tau,0}$ is given by

$$\mathcal{L}_{\Lambda_{\tau,0}}(s) = \mathbb{E}\big[\exp(-s\Lambda_{\tau,0})\big] = \exp\big(-\gamma_{\alpha,\sigma}\big((s+\tau)^\alpha - \tau^\alpha\big)\big), \qquad s \geq -\tau. \tag{5.25}$$

Since $\mathcal{L}'_{\Lambda_{\tau,0}}(s) = -\alpha\gamma_{\alpha,\sigma}(s+\tau)^{\alpha-1}\mathcal{L}_{\Lambda_{\tau,0}}(s)$ for $s > -\tau$, we obtain for $\tau > 0$

$$\mathbb{E}[\Lambda_{\tau,0}] = -\mathcal{L}'_{\Lambda_{\tau,0}}(0) = \alpha\gamma_{\alpha,\sigma}\tau^{\alpha-1} \tag{5.26}$$

and

$$\text{Var}(\Lambda_{\tau,0}) = \mathcal{L}''_{\Lambda_{\tau,0}}(0) - \big(\mathcal{L}'_{\Lambda_{\tau,0}}(0)\big)^2 = \alpha(1-\alpha)\gamma_{\alpha,\sigma}\tau^{\alpha-2} = \frac{1-\alpha}{\tau}\mathbb{E}[\Lambda_{\tau,0}]. \tag{5.27}$$

We have the following representation of the distribution of $N^{(0)}$.

Lemma 5.10 *Fix* $\alpha \in (0,1)$, $\lambda, \sigma > 0$ *and* $\tau \geq 0$. *Define* $\delta = \gamma_{\alpha,\sigma}((\lambda+\tau)^\alpha - \tau^\alpha)$ *and* $p = \frac{\tau}{\lambda+\tau}$. *Let* $M \sim \text{Poisson}(\delta)$ *and* $N_i \sim \text{ExtNegBin}(-\alpha, 1, p)$ *for* $i \in \mathbb{N}$ *be independent random variables. Then* $N^{(0)}$ *equals* $N_1 + \cdots + N_M$ *in distribution.*

Proof Define $q = 1 - p$. Using (5.20), the probability generating function (5.24) for $n = 0$ can be rewritten as

$$\begin{aligned}\varphi_{N^{(0)}}(s) &= \exp\big(-\gamma_{\alpha,\sigma}\big(\big(\tau+\lambda(1-s)\big)^\alpha - \tau^\alpha\big)\big)\\ &= \exp\big(-\gamma_{\alpha,\sigma}(\lambda+\tau)^\alpha\big(\big(p+q(1-s)\big)^\alpha - p^\alpha\big)\big), \quad |s| \leq \frac{1}{q}.\end{aligned}$$

Furthermore, $\varphi_M(s) = \exp(-\delta(1-s))$ for $s \in \mathbb{R}$ and, according to (2.3),

$$\varphi_{N_1}(s) = \frac{1-(p+q(1-s))^\alpha}{1-p^\alpha} \quad \text{for } |s| \leq \frac{1}{q}.$$

Therefore,

$$\mathbb{E}\big[s^{N_1+\cdots+N_M}\big] = \varphi_M\big(\varphi_{N_1}(s)\big) = \exp\bigg(-\delta\frac{(p+q(1-s))^\alpha - p^\alpha}{1-p^\alpha}\bigg), \quad |s| \leq \frac{1}{q}.$$

Note that $\delta/(1-p^\alpha) = \gamma_{\alpha,\sigma}(\lambda+\tau)^\alpha$, hence $\varphi_{N^{(0)}}$ and $\varphi_M \circ \varphi_{N_1}$ agree. □

Before applying this result to derive numerically stable recursions, we also need the following fact.

Remark 5.11 Let M and $\{N_i\}_{i\in\mathbb{N}}$ denote (possibly dependent) $\mathbb{N}_0$-valued random variables and define $N = N_1 + \cdots + N_M$. Here M can be interpreted as the random number of batches containing $N_1, N_2, \ldots, N_M$ claims. Let $\{X_j\}_{j\in\mathbb{N}}$ denote an i.i.d. sequence of $\mathbb{N}_0$-valued claim sizes, independent of random variables relating to claim numbers. Define the sum of all claims in batch $i \in \mathbb{N}$ by

$$S_{(i)} = \sum_{j=N_1+\cdots+N_{i-1}+1}^{N_1+\cdots+N_i} X_j \tag{5.28}$$

and note that the probability generating functions satisfy

$$\varphi_{S_{(i)}}(s) = \mathbb{E}\big[\mathbb{E}\big[s^{S_{(i)}}\,\big|\,N_1,\ldots,N_i\big]\big] = \mathbb{E}\big[\big(\varphi_{X_1}(s)\big)^{N_i}\big] = \varphi_{N_i}\big(\varphi_{X_1}(s)\big)$$

for $|s| \leq 1$, hence $S_{(i)}$ equals $X_1 + \cdots + X_{N_i}$ in distribution. According to definition (5.28), $S := X_1 + \cdots + X_N = S_{(1)} + \cdots + S_{(M)}$. Observe that, for every $i \in \mathbb{N}$,

$$\mathbb{E}\big[s^{S_{(1)}}\ldots s^{S_{(i)}}\,\big|\,N_1,\ldots,N_i\big] \overset{\text{a.s.}}{=} \big(\varphi_{X_1}(s)\big)^{N_1+\cdots+N_i}, \quad |s| \leq 1.$$

Hence, if $\{N_i\}_{i\in\mathbb{N}}$ are independent, it follows that $\{S_{(i)}\}_{i\in\mathbb{N}}$ are independent.

Algorithm 5.12 We can use Lemma 5.10 and Remark 5.11 to calculate the distribution of $S = X_1 + \cdots + X_{N^{(0)}}$ with $\mathcal{L}(N^{(0)})$ given by (5.23) in a numerically stable way. First, we calculate the distribution of $S_{(1)} := X_1 + \cdots + X_{N_1}$ with $N_1 \sim \text{ExtNegBin}(-\alpha, 1, p)$ using Algorithm 5.3. For $p > 0$, corresponding to $\tau > 0$, this means to use Panjer's recursion from Theorem 4.1 for $\text{NegBin}(1-\alpha, p)$ followed by our stable weighted convolution (5.2) from Corollary 5.1 with

$$b_1 = \alpha q \frac{p^{\alpha-1}}{1-p^\alpha}. \tag{5.29}$$

In the case $p = 0$, arising from $\tau = 0$, we apply the recursion in Lemma 5.2 for $\text{ExtNegBin}(-\alpha, 1, 0)$ to obtain $\mathcal{L}(S_{(1)})$. Second, letting $S_{(2)}, S_{(3)}, \ldots$ denote independent copies of $S_{(1)}$, we calculate the distribution of $S_{(1)} + \cdots + S_{(M)}$, which is equal to S in distribution by Remark 5.11. For this purpose, we use the previously calculated $\mathcal{L}(S_{(1)})$ as severity distribution and apply the numerically stable Panjer recursion from Theorem 4.1 for $M \sim \text{Poisson}(\delta)$ with δ given in Lemma 5.10.

To extend Algorithm 5.12 to calculate the distribution of $X_1 + \cdots + X_{N^{(m)}}$, where the distribution of the claim number $N^{(m)}$ is obtained by a Poisson mixture over $\Lambda_{\tau,m} \sim F_{\alpha,\sigma,\tau,m}$, we need the expectation and the Laplace transform of $\Lambda_{\tau,m}$. The special case $\alpha = 1/2$ is needed in Examples 5.20 and 5.22 below. The following lemma and its corollary generalize (5.25) and (5.26) and give the higher moments, too, which can be useful for calibration purposes. We start with the α-stable case.

Lemma 5.13 *Given $\alpha \in (0, 1)$ and $\sigma > 0$, let $Y \sim \mathrm{S}_\alpha(\sigma, 1, 0)$. Then*

$$\mathbb{E}\big[Y^{-m} e^{-sY}\big] = I_{\alpha,\sigma}(m, s), \quad s > 0,\ m \in \mathbb{Z}, \tag{5.30}$$

and also for $s = 0$ if $m \in \mathbb{N}_0$, where

$$I_{\alpha,\sigma}(m, s) := \int_s^\infty \frac{(t-s)^{m-1}}{(m-1)!} \exp\big(-\gamma_{\alpha,\sigma} t^\alpha\big)\, dt, \quad m \in \mathbb{N},\ s \ge 0, \tag{5.31}$$

$I_{\alpha,\sigma}(0, s) := \exp(-\gamma_{\alpha,\sigma} s^\alpha)$ for $s \ge 0$, and

$$I_{\alpha,\sigma}(-m, s) := (-1)^m \frac{d^m}{ds^m} \exp\big(-\gamma_{\alpha,\sigma} s^\alpha\big), \quad m \in \mathbb{N},\ s > 0. \tag{5.32}$$

In particular, if $s = 0$, then (5.31) simplifies to

$$I_{\alpha,\sigma}(m, 0) = \frac{\Gamma(m/\alpha)}{\alpha\, \gamma_{\alpha,\sigma}^{m/\alpha}\, (m-1)!}, \quad m \in \mathbb{N}. \tag{5.33}$$

For the special case $1/\alpha \in \mathbb{N}$, we have

$$I_{\alpha,\sigma}(m, s) = \frac{\exp(-\gamma_{\alpha,\sigma} s^\alpha)}{\gamma_{\alpha,\sigma}^{m/\alpha}}\, p_{m,1/\alpha}\big(\gamma_{\alpha,\sigma} s^\alpha\big), \quad m \in \mathbb{N}_0,\ s \ge 0, \tag{5.34}$$

where the polynomial $p_{m,n}$ of degree $m(n-1)$ is defined by

$$p_{m,n}(x) = \sum_{j=m}^{mn} \frac{c_j(m,n)\, j!}{m!} x^{mn-j}, \quad m \in \mathbb{N}_0,\ n \in \mathbb{N},\ x \in \mathbb{R}, \tag{5.35}$$

where in turn $c_j(m,n)$ denotes the natural number which appears as the coefficient of x^j in the polynomial representation $((x+1)^n - 1)^m = \sum_{j=m}^{mn} c_j(m,n)x^j$.

Proof If $m = 0$, then (5.30) is just the definition (5.20) of $Y \sim \mathrm{S}_\alpha(\sigma, 1, 0)$. For negative $m \in \mathbb{Z}$, (5.30) follows by iterated differentiation from the case $m = 0$. For $m \in \mathbb{N}$, the dominated convergence theorem implies that the left- and the right-hand sides of (5.30) converge to 0 as $s \to \infty$. Hence we can finish the proof of (5.30) by induction on $m \in \mathbb{N}$ by noting that the derivatives of both sides of (5.30) are equal due to the induction hypothesis. To prove (5.33), perform the integral substitution $u = \gamma_{\alpha,\sigma} t^\alpha$ in (5.31) and use the definition of the gamma function.

If $m = 0$, then (5.34) follows from the definitions. To prove (5.34) for general $m \in \mathbb{N}$, use integration by parts in (5.31) and afterwards the integral substitution $u = \gamma_{\alpha,\sigma}(t^\alpha - s^\alpha)$ to obtain

$$\begin{aligned} I_{\alpha,\sigma}(m,s) &= \int_s^\infty \frac{(t-s)^m}{m!} \alpha\, \gamma_{\alpha,\sigma}\, t^{\alpha-1} \exp\bigl(-\gamma_{\alpha,\sigma} t^\alpha\bigr)\, dt \\ &= \frac{\exp(-\gamma_{\alpha,\sigma} s^\alpha)}{m!} \int_0^\infty \left(\left(\frac{u}{\gamma_{\alpha,\sigma}} + s^\alpha \right)^{1/\alpha} - s \right)^m e^{-u}\, du. \end{aligned} \tag{5.36}$$

When $1/\alpha \in \mathbb{N}$, we can rewrite

$$\left(\left(\frac{u}{\gamma_{\alpha,\sigma}} + s^\alpha \right)^{1/\alpha} - s \right)^m = s^m \sum_{j=m}^{m/\alpha} c_j(m, 1/\alpha) \left(\frac{u}{\gamma_{\alpha,\sigma} s^\alpha} \right)^j.$$

Substitute this into (5.36) and use the definition of the gamma function to obtain

$$I_{\alpha,\sigma}(m,s) = \frac{s^m \exp(-\gamma_{\alpha,\sigma} s^\alpha)}{m!} \sum_{j=m}^{m/\alpha} \frac{c_j(m, 1/\alpha)\, j!}{(\gamma_{\alpha,\sigma} s^\alpha)^j}, \quad m \in \mathbb{N},\ s \geq 0,$$

which can be rearranged to give (5.34). □

Remark 5.14 By the binomial formula, $c_j(m,2) = 2^{2m-j}\binom{m}{2m-j}$ for all $m \in \mathbb{N}_0$ and $j \in \{m, \ldots, 2m\}$. Using also the multinomial formula, one can show that

$$c_j(m,n) = m! \sum_{\substack{k_1,\ldots,k_n=0 \\ k_1+\cdots+k_n=m \\ 1k_1+\cdots+nk_n=j}}^{m} \prod_{\ell=1}^{n} \frac{1}{k_\ell!} \binom{n}{\ell}^{k_\ell}, \quad m \in \mathbb{N}_0,\ n \in \mathbb{N},\ j \in \{m, \ldots, mn\}.$$

For an alternative representation, note that $c_j(m,n) = \frac{1}{j!}\frac{d^j}{dx^j} f_m(g_n(x))|_{x=0}$ with $f_m(x) := (x-1)^m$ and $g_n(x) := (x+1)^n$. With Faà di Bruno's formula we obtain

$$c_j(m,n) = \frac{m!}{j!} B_{j,m}\big(g_n'(0), \ldots, g_n^{(j-m+1)}(0)\big) \quad \text{with } g_n^{(\ell)}(0) = \prod_{i=0}^{\ell-1}(n-i)$$

for all $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $j \in \{m, \ldots, mn\}$, and $\ell \in \{1, \ldots, j-m+1\}$, where the Bell polynomial $B_{j,m}$ in the variables $x_1, \ldots, x_{j-m+1}$ is defined by

$$B_{j,m}(x_1, \ldots, x_{j-m+1}) = m! \sum_{\substack{k_1,\ldots,k_{j-m+1}=0 \\ k_1+\cdots+k_{j-m+1}=m \\ 1k_1+\cdots+(j-m+1)k_{j-m+1}=j}}^{m} \prod_{\ell=1}^{j-m+1} \frac{1}{k_\ell!}\left(\frac{x_\ell}{\ell!}\right)^{k_\ell}.$$

Remark 5.15 Using Faà di Bruno's formula, we can express (5.32), for all $m \in \mathbb{N}$ and $s > 0$, by

$$I_{\alpha,\sigma}(-m,s) = (-1)^m \frac{m! \exp(-\gamma_{\alpha,\sigma} s^\alpha)}{s^m} \sum_{\substack{k_1,\ldots,k_m=0 \\ 1k_1+\cdots+mk_m=m}}^{m} \prod_{\ell=1}^{m} \frac{1}{k_\ell!}\left(-\binom{\alpha}{\ell}\gamma_{\alpha,\sigma} s^\alpha\right)^{k_\ell}.$$

Corollary 5.16 *Given $\alpha \in (0,1)$, $\sigma > 0$, $\tau \geq 0$, and $m \in \mathbb{N}_0$, let $\Lambda_{\tau,m} \sim F_{\alpha,\sigma,\tau,m}$ as defined in* (5.21). *Then, using the notation of Lemma* 5.13,

$$\mathbb{E}\left[\Lambda_{\tau,m}^\ell \exp(-s\Lambda_{\tau,m})\right] = \frac{I_{\alpha,\sigma}(m-\ell, s+\tau)}{I_{\alpha,\sigma}(m,\tau)}, \quad \ell \in \mathbb{Z},\ s > -\tau, \tag{5.37}$$

which also holds for $s = -\tau$ if $\ell \leq m$. For the special case $\tau = 0$,

$$\mathbb{E}\big[\Lambda_{0,m}^\ell\big] = \begin{cases} \alpha\, \gamma_{\alpha,\sigma}^{m/\alpha}(m-1)!\big/\Gamma(\frac{m}{\alpha}) & \text{if } \ell = m \in \mathbb{N}, \\ \gamma_{\alpha,\sigma}^{(m-\ell)/\alpha}\frac{(m-1)!}{(\ell-1)!}\frac{\Gamma(\frac{m-\ell}{\alpha})}{\Gamma(\frac{m}{\alpha})} & \text{if } \ell \in \mathbb{Z},\ m \in \mathbb{N},\ \ell \leq m-1. \end{cases} \tag{5.38}$$

For the special case $1/\alpha \in \mathbb{N}$, we have, for all $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}_0$ with $\ell \leq m$ and for all real $s \geq -\tau$,

$$\mathbb{E}\big[\Lambda_{\tau,m}^\ell \exp(-s\Lambda_{\tau,m})\big] = \gamma_{\alpha,\sigma}^{\ell/\alpha}\frac{\exp(\gamma_{\alpha,\sigma}\tau^\alpha)}{\exp(\gamma_{\alpha,\sigma}(s+\tau)^\alpha)}\frac{p_{m-\ell,1/\alpha}(\gamma_{\alpha,\sigma}(s+\tau)^\alpha)}{p_{m,1/\alpha}(\gamma_{\alpha,\sigma}\tau^\alpha)}. \tag{5.39}$$

Proof The expectation in (5.37) follows from (5.21) and Lemma 5.13. The special case (5.38) follows from (5.37) for $s = \tau = 0$, (5.33), and $I_{\alpha,\sigma}(0,0) = 1$. The second special case (5.39) for $1/\alpha \in \mathbb{N}$ follows from (5.37) and (5.34). □

Note that the expressions given in Lemma 5.13 and its corollary can be evaluated in a numerically stable way in the sense that no cancellations occur. In particular the integrand in (5.31) and the coefficients of the polynomial in (5.35) are positive.

Remark 5.17 In general, $I_{\alpha,\sigma}(m,s)$ given by (5.31) or equivalently (5.36) can be evaluated by numerical integration. Alternatively, using the integral substitution $v = u + \gamma_{\alpha,\sigma} s^\alpha$ in (5.36) and then the binomial formula, we obtain

$$I_{\alpha,\sigma}(m,s) = \frac{1}{m!} \sum_{j=0}^{m} \binom{m}{j} \frac{(-s)^{m-j}}{\gamma_{\alpha,\sigma}^{j/\alpha}} \Gamma\big(1 + j/\alpha, \gamma_{\alpha,\sigma} s^\alpha\big), \quad s \geq 0,$$

with the upper incomplete gamma function, which is implemented in software packages. However, due to the alternating sign, this representation might not evaluate in a numerically stable way.

We can now present our algorithm for generalized τ-tempered α-stable Poisson mixture distributions.

Algorithm 5.18 For $m \in \mathbb{N}$, the distribution $\{p_n^{(m)}\}_{n\in\mathbb{N}_0}$ of $X_1 + \cdots + X_{N^{(m)}}$ can be calculated iteratively in a numerically stable way by m applications of the weighted convolution (4.5) from the distribution of $X_1 + \cdots + X_{N^{(0)}}$, which can be calculated by Algorithm 5.12. More precisely, for $\alpha \in (0,1)$, $\lambda, \sigma > 0$ and tempering parameter $\tau \geq 0$, let $\mathcal{L}(N^{(m)})$ be given by (5.23) with $\Lambda_{\tau,m} \sim F_{\alpha,\sigma,\tau,m}$ according to (5.21). Then $F_{\alpha,\sigma,\tau,m-1}$ is the $\tilde{F}$ arising from $F_{\alpha,\sigma,\tau,m}$ via (5.15), hence

$$q_n^{(m)} = \frac{\lambda\,\mathbb{E}[\Lambda_{\tau,m}]}{n} q_{n-1}^{(m-1)}, \quad m, n \in \mathbb{N},$$

by Lemma 5.7(a), where $\{q_n^{(m)}\}_{n\in\mathbb{N}_0}$ denotes the distribution of $N^{(m)}$ for $m \in \mathbb{N}_0$. Therefore, (4.4) (with $k = 0$, $\ell = 1$, $a_1 = 0$ and $b_1 = \lambda\mathbb{E}[\Lambda_{\tau,m}]$) holds and the weighted convolution (4.5) from Theorem 4.5(a) implies that

$$p_n^{(m)} = \lambda\,\mathbb{E}[\Lambda_{\tau,m}] \sum_{j=1}^{n} \frac{j}{n} \mathbb{P}[X_1 = j] p_{n-j}^{(m-1)}, \quad m, n \in \mathbb{N},$$

where $\mathbb{E}[\Lambda_{\tau,m}]$ is given by Corollary 5.16 (with $\ell = 1$). Furthermore,

$$p_0^{(m)} = \mathbb{E}\big[\exp\big(-\lambda\,\mathbb{P}[X_1 \geq 1]\Lambda_{\tau,m}\big)\big], \quad m \in \mathbb{N}_0,$$

by (4.1) and (5.24). This is the Laplace transform at $s = \lambda\,\mathbb{P}[X_1 \geq 1]$, which is given in Corollary 5.16 (with $\ell = 0$).

The class of generalized τ-tempered α-stable distributions, for which the Algorithms 5.12 and 5.18 are applicable, contains several families of well-known distributions as special cases. They all originate from the Lévy distribution corresponding to $\alpha = 1/2$, for which a density is available in closed form.

Example 5.19 Lévy distribution The Lévy distribution is the special case when $\alpha = 1/2$ and $m = \tau = 0$. A density of $\Lambda_{0,0} = Y \sim F_{1/2,\sigma,0,0} = \mathrm{S}_{1/2}(\sigma, 1, 0)$ is given by

$$f_{\mathrm{L}}(y) = \left(\frac{\sigma}{2\pi y^3}\right)^{1/2} \exp\left(-\frac{\sigma}{2y}\right), \quad y > 0, \tag{5.40}$$

cf. [25, Eq. (1.1.15)] and [6, Chap. XIII.3, Ex. (b)]. According to Lemma 5.10, the mixing distribution given by (5.14) using $F_{1/2,\sigma,0,0}$ can be represented as $N_1 + \cdots + N_M$ with independent

$$M \sim \mathrm{Poisson}(\sqrt{\lambda\sigma/2})$$

and $N_i \sim \mathrm{ExtNegBin}(-1/2, 1, 0)$ for $i \in \mathbb{N}$. Algorithm 5.12 using the stable recursion of Lemma 5.2 is applicable.

Example 5.20 (Inverse gamma distribution with a half-integer shape parameter) As in Example 5.19, let $Y \sim \mathrm{S}_{1/2}(\sigma, 1, 0)$ with density f_{L} given by (5.40). Using (5.30) and (5.33) as well as $\gamma_{1/2,\sigma} = \sqrt{2\sigma}$ from (5.20), we see that

$$\mathbb{E}\left[Y^{-m}\right] = I_{1/2,\sigma}(m, 0) = \frac{2\Gamma(2m)}{(2\sigma)^m (m-1)!} = \frac{(2m)!}{(2\sigma)^m m!} = \frac{2^m}{\sqrt{\pi}\,\sigma^m}\Gamma\left(m + \frac{1}{2}\right)$$

for every $m \in \mathbb{N}$; the resulting equation even holds for $m = 0$. By (5.21) a density of $\Lambda_{0,m} \sim F_{1/2,\sigma,0,m}$ is given by

$$f_{0,m}(y) = \frac{y^{-m} f_{\mathrm{L}}(y)}{\mathbb{E}[Y^{-m}]} = \frac{1}{y^2} g_{\alpha_m,\beta}\left(\frac{1}{y}\right), \quad y > 0,$$

for every $m \in \mathbb{N}_0$, where $g_{\alpha_m,\beta}$ denotes the gamma density given in (5.18) with parameters $\alpha_m := m + \frac{1}{2}$ and $\beta := \sigma/2$. Therefore, $1/\Lambda_{0,m} \sim \mathrm{Gamma}(\alpha_m, \beta)$ and $\Lambda_{0,m}$ has an inverse gamma distribution.

Example 5.21 (Inverse Gaussian distribution) Take $\mu, \tilde{\sigma} > 0$ and define $\sigma = \mu^2/\tilde{\sigma}^2$ and the tempering parameter $\tau = 1/(2\tilde{\sigma}^2)$. Then the Laplace transform (5.25) of $\Lambda_{\tau,0} \sim F_{1/2,\sigma,\tau,0}$ is

$$\mathbb{E}\left[\exp(-s\Lambda_{\tau,0})\right] = \exp\left(\frac{\mu}{\tilde{\sigma}^2}\left(1 - \sqrt{1 + 2\tilde{\sigma}^2 s}\right)\right) \quad \text{for } s \geq -\frac{1}{2\tilde{\sigma}^2}.$$

Using (5.21) and the Laplace transform (5.20) for $Y \sim \mathrm{S}_{1/2}(\sigma, 1, 0)$ as well as the Lévy density given in (5.40), we see that a density of $F_{1/2,\sigma,\tau,0}$ is given by

$$\begin{aligned} f_{\mathrm{IG}}(y) &= \frac{e^{-\tau y} f_{\mathrm{L}}(y)}{\mathbb{E}\left[e^{-\tau Y}\right]} = \exp\left(-\tau y + \sqrt{\sigma\tau/2}\right) f_{\mathrm{L}}(y) \\ &= \frac{\mu}{\sqrt{2\pi\tilde{\sigma}^2 y^3}} \exp\left(-\frac{(y-\mu)^2}{2\tilde{\sigma}^2 y}\right), \quad y > 0, \end{aligned} \tag{5.41}$$

which coincides with the density given in [32]. By Lemma 5.10 the mixing distribution (5.14) with $F = F_{1/2,\sigma,\tau,0}$ has the representation $N_1 + \cdots + N_M$ with independent $M \sim \mathrm{Poisson}(\sqrt{\sigma/2}(\sqrt{\lambda + \tau} - \sqrt{\tau}))$ and $N_i \sim \mathrm{ExtNegBin}(-1/2, 1, p)$ for

$i \in \mathbb{N}$, where $p = \frac{\tau}{\lambda+\tau}$. Algorithm 5.12 is applicable and the constant b_1 in (5.29) reduces to

$$b_1 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{p}}\right) = \frac{1 + \sqrt{1 + 2\lambda\tilde{\sigma}^2}}{2}.$$

Example 5.22 (Generalized inverse Gaussian distribution) For parameters $\sigma, \tau > 0$ and $m \in \mathbb{N}$, consider a Lévy distributed $Y \sim F_{1/2,\sigma,0,0}$ as in Example 5.19 and let $\Lambda_{\tau,m} \sim F_{1/2,\sigma,\tau,m}$. Using (5.21), the Lévy density f_{L} from (5.40), as well as (5.30) and (5.34), and in addition $\gamma_{1/2,\sigma} = \sqrt{2\sigma}$ from (5.20), it follows that a density of $F_{1/2,\sigma,\tau,m}$ on the half-line $y > 0$ is given by

$$f_{\tau,m}(y) = \frac{y^{-m}e^{-\tau y} f_{\mathrm{L}}(y)}{\mathbb{E}[Y^{-m}e^{-\tau Y}]} = \sqrt{\frac{\sigma}{2\pi}} \frac{(2\sigma)^m}{p_{m,2}(\sqrt{2\sigma\tau})\, y^{m+3/2}} \exp\left(-\frac{(\sqrt{2\tau}\, y - \sqrt{\sigma})^2}{2y}\right).$$

Substituting $m = 0$, $\sigma = \mu^2/\tilde{\sigma}^2$ and $\tau = 1/(2\tilde{\sigma}^2)$ into this formula, we obtain f_{IG} given by (5.41), hence the term *generalized* inverse Gaussian distribution (with half-integer parameter $m + 1/2$) for $f_{\tau,m}$ is justified. This distribution is discussed more generally (with $m + 1/2$ replaced by a real number) in [14] and [23, Example 4.14.7].

5.4 Convolutions and reciprocal generalized inverse Gaussian distribution

Fix $r \in \mathbb{N}$ and let us start with a simple remark.

Remark 5.23 For every $i \in \{1, \dots, r\}$, let N_i denote a random claim number and $\{X_{i,j}\}_{j\in\mathbb{N}}$ an i.i.d. sequence of $\mathbb{N}_0$-valued claim sizes. We assume that the collection of all these random variables is independent. If the distribution of $S_{(i)} := X_{i,1} + \cdots + X_{i,N_i}$ can be computed in a numerically stable way for every $i \in \{1, \dots, r\}$ (for example, by Panjer's recursion or one of our numerically stable versions), then the distribution of $S := S_{(1)} + \cdots + S_{(r)}$ can be computed iteratively by $r - 1$ numerically stable convolutions.

We now specialize Remark 5.11 to extend the mixture model (5.14). Suppose that $\mathbb{P}[M = r] = 1$ and that $(\Lambda_1, \dots, \Lambda_r)$ is a vector of $[0, \infty)$-valued (possibly dependent) random variables. Assume that the random claim numbers $N_1, \dots, N_r$ are conditionally independent given $\Lambda_1, \dots, \Lambda_r$. Furthermore, assume that there are parameters $\lambda_1, \dots, \lambda_r > 0$ such that, for every $i \in \{1, \dots, r\}$,

$$\mathcal{L}(N_i | \Lambda_1, \dots, \Lambda_r) \overset{\text{a.s.}}{=} \mathcal{L}(N_i | \Lambda_i) \overset{\text{a.s.}}{=} \text{Poisson}(\lambda_i \Lambda_i)\,. \tag{5.42}$$

The conditional probability generating function of the total claim number $N = N_1 + \cdots + N_r$ is given by

$$\mathbb{E}\left[s^N \,\middle|\, \Lambda_1, \dots, \Lambda_r\right] \overset{\text{a.s.}}{=} \exp\bigl(-(1-s)(\lambda_1\Lambda_1 + \cdots + \lambda_r\Lambda_r)\bigr), \quad s \in \mathbb{R}, \tag{5.43}$$

hence

$$\mathcal{L}(N | \Lambda_1, \dots, \Lambda_r) \overset{\text{a.s.}}{=} \text{Poisson}(\lambda_1\Lambda_1 + \cdots + \lambda_r\Lambda_r)\,.$$

This corresponds to the mixture model (5.14) with F denoting the distribution function of the convex combination $\Lambda := (\lambda_1\Lambda_1 + \cdots + \lambda_r\Lambda_r)/\lambda$ with $\lambda = \lambda_1 + \cdots + \lambda_r$.

Suppose now that $\Lambda_1, \ldots, \Lambda_r$ are independent. It then follows by using (5.43) that $N_1, \ldots, N_r$ are independent, hence $S_{(1)}, \ldots, S_{(r)}$ are independent by Remark 5.11 and the distribution of S can be calculated by convolutions according to Remark 5.23. Note that the distribution function of Λ can be computed iteratively from the distribution functions of $\Lambda_1, \ldots, \Lambda_r$ by $r-1$ convolutions.

The following example shows that the reciprocal generalized inverse Gaussian distribution, when used as mixing distribution, fits into the above framework.

Example 5.24 (Reciprocal generalized inverse Gaussian distribution) Fix $\sigma, \tau > 0$, $m \in \mathbb{N}_0$ and $r = 2$. Assume that Λ_1 follows the generalized inverse Gaussian distribution $F_{1/2,\sigma,\tau,m}$ from Example 5.21 with density $f_{\tau,m}$. Furthermore, consider an independent $\Lambda_2 \sim \text{Gamma}(m+1/2, \tau)$ with density given in (5.18). Then a density f_m of $\Lambda := \Lambda_1 + \Lambda_2$ is given by the convolution

$$f_m(x) = \int_0^x f_{\tau,m}(y) \frac{\tau^{m+1/2}(x-y)^{m-1/2}e^{-\tau(x-y)}}{\Gamma(m+1/2)}\,dy, \qquad x > 0.$$

Using $\Gamma(m+1/2) = \sqrt{\pi}\,4^{-m}(2m)!/m!$, substituting for $f_{\tau,m}$ and rearranging the argument of the exponential function, we see that, for every $x > 0$,

$$\begin{aligned} f_m(x) = {} & \sqrt{\frac{\tau}{\pi}} \frac{(4\tau)^m}{p_{m,2}(\sqrt{2\sigma\tau})} x^{m-1/2} \exp\biggl(-\frac{(\sqrt{2\tau}x - \sqrt{\sigma})^2}{2x}\biggr) \\ & \times \frac{2^m m!}{(2m)!} \int_0^x \frac{2}{\sqrt{2\pi}} \biggl(\sigma\frac{x-y}{xy}\biggr)^m \exp\biggl(-\frac{\sigma}{2}\frac{x-y}{xy}\biggr) \frac{\sqrt{\sigma x}\,dy}{2\sqrt{(x-y)y^3}}. \end{aligned}$$

Using the substitution $z = \sqrt{\sigma(x-y)/(xy)}$, the integral reduces to $\mathbb{E}[Z^{2m}]$, where Z has a standard normal distribution. Since $\mathbb{E}[Z^{2m}] = (2m)!/(2^m m!)$, the density f_m is given by the first line. Note that $1/\Lambda \sim F_{1/2,2\tau,\sigma/2,m}$, hence Λ has a reciprocal generalized inverse Gaussian distribution, cf. Example 5.22. In particular, f_0 is a density of the usual reciprocal inverse Gaussian distribution, cf. [32, Example 6.2].

Finally, fix $\lambda > 0$ and consider independent claim numbers N_1 and N_2 satisfying (5.42). Then N_1 follows a Poisson mixture distribution given by (5.14), where $F = F_{1/2,\sigma,\tau,m}$ is the generalized inverse Gaussian distribution from Example 5.22. Thus we can calculate the distribution of $S_{(1)}$ by the numerically stable Algorithm 5.18. It follows from (5.43) and (5.19) that, for $|s| < 1/q$,

$$\mathbb{E}\bigl[s^{N_2}\bigr] = \mathbb{E}\bigl[\exp\bigl(-(1-s)\lambda\Lambda_2\bigr)\bigr] = \biggl(\frac{\tau}{\tau + (1-s)\lambda}\biggr)^{m+1/2} = \biggl(\frac{p}{1-qs}\biggr)^{m+1/2},$$

where $p := \frac{\tau}{\lambda+\tau}$ and $q := 1-p$. Hence $N_2 \sim \text{NegBin}(m+1/2, p)$ by comparison with (2.2), which belongs to the Panjer$(q, (m-1/2)q, 0)$ class, and Panjer's recursion in Theorem 4.1 for computing the distribution of $S_{(2)}$ is numerically stable. Finally, one convolution delivers the distribution of $S = S_{(1)} + S_{(2)}$.

5.5 Application and extension of CreditRisk$^+$

We briefly recall the mathematical basis of CreditRisk$^+$, cf. [2, 9], including a slight extension to stochastic exposures/recoveries.

Let $r \in \mathbb{N}_0$ denote the number of non-idiosyncratic risk factors. For $k \in \{0, \dots, r\}$ let $\{X_{i,j}\}_{j\in\mathbb{N}}$ be independent, $\mathbb{N}_0$-valued i.i.d. sequences of credit losses due to risk factor i given default, and let $\varphi_{X_{i,1}}$ denote the corresponding probability generating function. Usually, $\mathcal{L}(X_{i,1})$ is a mixture distribution arising from individual stochastic credit losses caused by obligors exposed to risk factor i. Independently of the size of the credit losses, the $\mathbb{N}_0$-valued random variables $N_0, \dots, N_K$ are modeled, describing the number of losses caused by risk $i \in \{0, \dots, r\}$. In the standard version of CreditRisk$^+$, N_0 is assumed to be Poisson(λ_0), independent of $N_1, \dots, N_r$. Furthermore, it is assumed that the risk factors $\Lambda_1, \dots, \Lambda_r$ are independent with $\Lambda_i \sim \text{Gamma}(1/\sigma_i^2, 1/\sigma_i^2)$ and $\sigma_i^2 > 0$, meaning that $\mathbb{E}[\Lambda_i] = 1$ and $\text{Var}(\Lambda_i) = \sigma_i^2$. The numbers $N_1, \dots, N_r$ of defaults are assumed to be conditionally independent given $(\Lambda_1, \dots, \Lambda_r)$ and to satisfy (5.42) for every $i \in \{1, \dots, r\}$, meaning that Λ_i models the stochastic variability of the Poisson parameter for N_i. The task is then to calculate the distribution of the credit portfolio loss

$$S = \sum_{i=0}^{r} S_{(i)} \quad \text{with} \quad S_{(i)} = \sum_{j=1}^{N_i} X_{i,j}. \tag{5.44}$$

By (5.42) and (5.19), for every $i \in \{1, \dots, r\}$,

$$\begin{aligned} \varphi_{N_i}(s) &= \mathbb{E}\left[\mathbb{E}\left[s^{N_i} \middle| \Lambda_i\right]\right] = \mathbb{E}\big[\exp\big(-\lambda_i \Lambda_i (1-s)\big)\big] \\ &= \left(\frac{1/\sigma_i^2}{1/\sigma_i^2 + \lambda_i(1-s)}\right)^{1/\sigma_i^2} = \left(\frac{p_i}{1-q_i s}\right)^{1/\sigma_i^2} \end{aligned} \tag{5.45}$$

with $p_i = 1/(1+\lambda_i\sigma_i^2)$, $q_i = 1 - p_i$ and $|s| < 1/q_i$, hence $N_i \sim \text{NegBin}(1/\sigma_i^2, p_i)$ by comparison with (2.2). Since $N_0 \sim \text{Poisson}(\lambda_0)$, Panjer's recursion in Theorem 4.1 can be used to calculate the distribution of $S_{(i)}$ for every $i \in \{0, \dots, r\}$, and Remark 5.23 suggests to calculate the distribution of $S = S_{(0)} + \dots + S_{(r)}$ by r convolutions. This algorithm would be numerically stable.

The main advantage of CreditRisk$^+$ is the observation that these r convolutions can be replaced by the numerically much faster convex combination of probabilities in (5.46) below. The probabilistic essence is the fact that the negative binomial distribution is a compound Poisson sum, cf. [5, Example in Sect. XII.2]. To explain the details, let $M_1, \dots, M_r$ and $\{N_{1,j}\}_{j\in\mathbb{N}}, \dots, \{N_{r,j}\}_{j\in\mathbb{N}}$ be a collection of independent random variables, where the sequences are i.i.d., $M_i \sim \text{Poisson}(\lambda_i')$ with $\lambda_i' := -(\log p_i)/\sigma_i^2$ and $N_{i,1} \sim \text{Log}(q_i)$ for $i \in \{1, \dots, r\}$. Then, using (2.9), for every $i \in \{1, \dots, r\}$,

$$\mathbb{E}\big[s^{N_{i,1}+\cdots+N_{i,M_i}}\big] = \mathbb{E}\big[\mathbb{E}\big(\big[s^{N_{i,1}}\big]\big)^{M_i}\big] = \mathbb{E}\bigg[\bigg(\frac{\log(1-q_i s)}{\log p_i}\bigg)^{M_i}\bigg]$$

$$= \exp\bigg(\frac{\log p_i}{\sigma_i^2}\bigg(1 - \frac{\log(1-q_i s)}{\log p_i}\bigg)\bigg) = \bigg(\frac{p_i}{1-q_i s}\bigg)^{1/\sigma_i^2}$$

for $|s| < 1/q_i$, which agrees with (5.45); hence N_i equals $N_{i,1} + \cdots + N_{i,M_i}$ in distribution. Similarly to (5.28), we define

$$S_{(i,j)} = \sum_{k=N_{i,1}+\cdots+N_{i,j-1}+1}^{N_{i,1}+\cdots+N_{i,j}} X_{i,k}, \quad i \in \{1,\ldots,r\},\ j \in \mathbb{N}.$$

Hence, instead of replacing the Poisson parameter λ_i by the stochastic version $\lambda_i \Lambda_i$ to model the number of defaults caused by risk factor $i \in \{1,\ldots,r\}$, we may equivalently think of a Poisson(λ_i')-distributed number of events M_i due to risk i, each one causing a cluster of $N_{i,j}$ credit defaults with cluster credit loss $S_{(i,j)}$ for $j \in \{1,\ldots,M_i\}$. By Remark 5.11, for every $i \in \{1,\ldots,r\}$, the sequence $\{S_{(i,j)}\}_{j\in\mathbb{N}}$ is i.i.d. and, since Log(q_i) is in the Panjer($q_i, -q_i, 1$) class, the distribution of $S_{(i,1)}$ and therefore the coefficients of its pgf $\varphi_{S_{(i,1)}}$ can be calculated in a numerically stable way using Panjer's recursion from Theorem 4.1. The total portfolio loss (5.44) can be represented as

$$S \stackrel{\mathrm{d}}{=} \sum_{k=1}^{N_0} X_{0,k} + \sum_{i=1}^{r}\sum_{j=1}^{M_i} S_{(i,j)},$$

hence its pgf is given by

$$\mathbb{E}\big[s^S\big] = \exp\big(\lambda_0\big(\varphi_{X_{0,1}}(s) - 1\big)\big)\prod_{i=1}^{r}\exp\big(\lambda_i'\big(\varphi_{S_{(i,1)}}(s) - 1\big)\big) = \exp\big(\lambda\big(\varphi(s) - 1\big)\big)$$

for $|s| < s_0 := \min\{1/q_1,\ldots,1/q_r\}$, where $\lambda := \lambda_0 + \lambda_1' + \cdots + \lambda_r'$ and

$$\varphi(s) := \frac{\lambda_0}{\lambda}\varphi_{X_{0,1}}(s) + \sum_{i=1}^{r}\frac{\lambda_i'}{\lambda}\varphi_{S_{(i,1)}}(s), \quad |s| < s_0, \tag{5.46}$$

is the pgf of a mixture distribution of $\mathcal{L}(X_{0,1})$ and $\mathcal{L}(S_{(1,1)}),\ldots,\mathcal{L}(S_{(r,1)})$. The coefficients of φ are convex combinations of probabilities, hence cancellations are not possible. A final, numerically stable Panjer recursion for Poisson(λ) with loss size distribution given by φ produces the distribution of S.

This algorithm is essentially the same as the one for which Haaf et al. [10] proved numerical stability by directly treating the pgf, without referring to Panjer's recursion and the representation of the negative binomial distribution as compound Poisson sum.

Our results allow us to generalize CreditRisk$^+$ and the above algorithm in a numerically stable way as follows. For a risk factor $i \in \{1,\ldots,r\}$, instead of modeling the stochastic variation of λ_i by $\Lambda_i \sim \text{Gamma}(1/\sigma_i^2, 1/\sigma_i^2)$, we can take a tempered

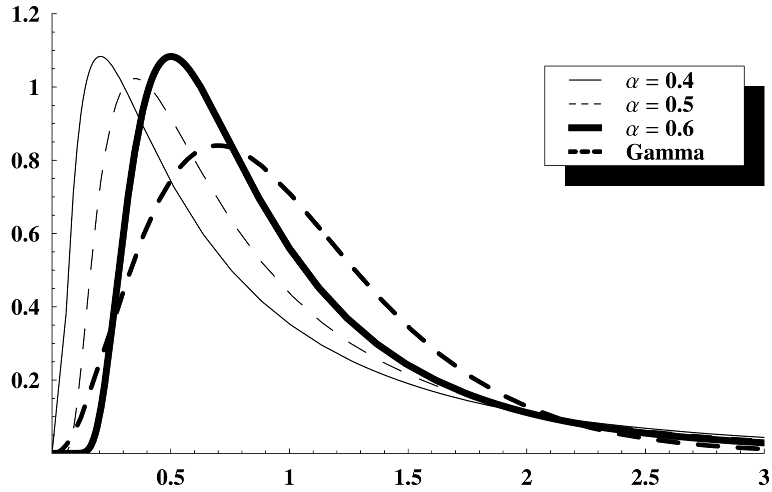

Fig. 1 Density plots of $\Lambda \sim F_{\alpha,\sigma,\tau,0}$ with varying α in comparison to $\Lambda \sim \text{Gamma}(1/\tilde{\sigma}^2, 1/\tilde{\sigma}^2)$, where $\sigma > 0$ and $\tau \geq 0$ as well as $\tilde{\sigma}^2$ are chosen such that $\mathbb{E}[\Lambda] = 1$ and $\text{Var}(\Lambda) = 0.3$

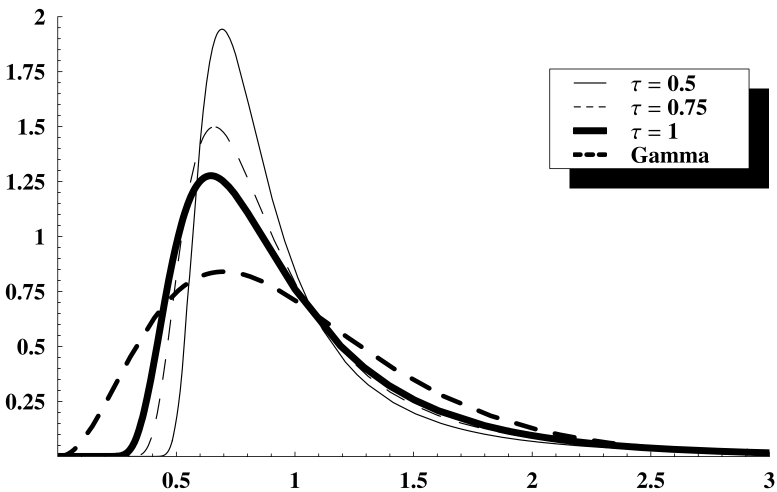

Fig. 2 Density plots of $\Lambda \sim F_{\alpha,\sigma,\tau,0}$ with varying τ in comparison to $\Lambda \sim \text{Gamma}(1/\tilde{\sigma}^2, 1/\tilde{\sigma}^2)$ where $\alpha \in (0, 1)$ and $\sigma > 0$ (resp. $\tilde{\sigma}^2$) are chosen such that $\mathbb{E}[\Lambda] = 1$ and $\text{Var}(\Lambda) = 0.3$. Note that the gamma distribution gives substantially more weight to the small values

stable distribution with $\alpha_i \in (0, 1)$, $n = 0$ and $\tau_i > 0$ as defined in (5.21), where the parameter $\tilde{\sigma}_i > 0$ is determined by $\alpha_i \tilde{\sigma}_i^{\alpha_i} \tau_i^{\alpha_i - 1} = \cos(\alpha_i \pi/2)$. Then $\mathbb{E}[\Lambda_i] = 1$ and $\text{Var}(\Lambda_i) = (1 - \alpha_i)/\tau_i$ by (5.20), (5.26) and (5.27). Therefore, for fixed variance, there remains some freedom to vary the shape of the mixing distribution, see Figs. 1 and 2. The Poisson distribution, mixed with this tempered α_i-stable distribution according to (5.14), can then be represented according to Lemma 5.10. Therefore, we can think of a Poisson(λ_i')-distributed number of credit loss clusters due to risk factor i, where we redefine

$$\lambda_i' = \frac{((\lambda_i + \tau_i)^{\alpha_i} - \tau_i^{\alpha_i})}{\alpha_i \tau_i^{\alpha_i - 1}} = \frac{1 - p_i}{\alpha_i\, p_i} \tau_i \quad \text{with } p_i = \frac{\tau_i}{\lambda_i + \tau_i},$$

and of ExtNegBin$(-\alpha_i, 1, p_i)$ distributed cluster sizes. Panjer's recursion for this extended negative binomial distribution can be replaced by our stable Algorithm 5.3, which however doubles the numerical effort in this case.

6 Further distributions for the generalized Panjer recursion

Theorem 4.5 obviously covers all distributions in Sundt's class [28] by choosing $\ell \in \mathbb{N}$ and $\tilde{q}_{n,i} = q_n$ for $n \in \mathbb{N}$. It generalizes the algorithm given by Sundt [28, Theorem 11]. The representation in (7.1) allows us to identify claim number distributions that satisfy the condition (4.4). Further claim number distributions satisfying (4.4) can be constructed in the following way.

Fix $k \in \mathbb{N}_0$, $\ell \in \mathbb{N}$ and consider claim number distributions $\{\tilde{q}_{i,n}\}_{n=k+\ell-i}^{\infty}$ for $i \in \{1, \ldots, \ell\}$ such that there are numerically stable algorithms to calculate the distributions of the corresponding compound sums. These algorithms can, for example, be Panjer's recursion in its usual form for distributions of the Panjer(a, b, k) class where it is stable. Furthermore, the algorithms given by Sundt [28] or by Wang and Sobrero [31] for the corresponding class can be applied, whenever they can be shown to be numerically stable. Further examples are, of course, algorithms using our results, like Algorithm 5.3 for ExtNegBin(α, k, p), Algorithm 5.6 for ExtLog(k, q), Algorithms 5.12 and 5.18 for the extended tempered stable distributions given by (5.21), and extensions by convolution outlined in Sect. 5.4, in particular Example 5.24. The following Theorem 6.1 helps to construct claim number distributions satisfying the requirements of Theorem 4.5(a), therefore yielding distribution for which the compound sum can be calculated in a numerically stable way. The first two moments of the constructed distribution can be calculated using (7.2) and (7.3).

Alternatively we can take subconvex combinations of claim number distributions in order to construct distributions in the context of Theorem 4.5(b). Obviously the above constructions can be iterated and combined.

Theorem 6.1 (Combination of truncated distributions) *Fix $k \in \mathbb{N}_0$, $\ell \in \mathbb{N}$ and weights $\alpha_i \geq 0$ and $\beta_i \geq -i\alpha_i$ for all $i \in \{1, \ldots, \ell\}$ such that at least one of the 2ℓ inequalities is strict. For every $i \in \{1, \ldots, \ell\}$ assume that the $\mathbb{N}_0$-valued random variable $\tilde{N}_i$ satisfies $\mathbb{P}[\tilde{N}_i < k + \ell - i] = 0$. Let $q_0, \ldots, q_{k+\ell-1} \geq 0$ with $q_0 + \cdots + q_{k+\ell-1} \leq 1$ be given and define*

$$q_n = c \sum_{i=1}^{\ell} \left(\alpha_i + \frac{\beta_i}{n} \right) \mathbb{P}[\tilde{N}_i = n - i] \quad \textit{for all } n \in \mathbb{N} \textit{ with } n \geq k + \ell, \tag{6.1}$$

where

$$c = \left(1 - \sum_{n=0}^{k+\ell-1} q_n \right) \Big/ \sum_{i=1}^{\ell} \left(\alpha_i + \beta_i\, \mathbb{E}\left[\frac{1}{i + \tilde{N}_i} \right] \right). \tag{6.2}$$

Then $\{q_n\}_{n\in\mathbb{N}_0}$ *is a probability distribution satisfying* (4.4) *in Theorem* 4.5(*a*) *with* $a_i = c\alpha_i$ *and* $b_i = c\beta_i$ *for all* $i \in \{1,\dots,\ell\}$, *and only non-negative terms are added up in* (4.5); *hence we have numerical stability.*

Remark 6.2 If numerical stability of (4.5) is not an issue, the above requirements on the weights $\alpha_i, \beta_i \in \mathbb{R}$ can be relaxed. We only need that the denominator of (6.2) differs from zero and that $q_n \ge 0$ for all $n \in \mathbb{N}$ with $n \ge k+\ell$.

Proof of Theorem 6.1 It only remains to check that c is the correct constant. Using (6.1) for $n \ge k+\ell$ we obtain that

$$\sum_{n\in\mathbb{N}_0} q_n = \sum_{n=0}^{k+\ell-1} q_n + c\sum_{i=1}^{\ell}\left(\alpha_i\,\mathbb{P}[\tilde{N}_i \ge k+\ell-i] + \beta_i \sum_{n=k+\ell}^{\infty} \frac{\mathbb{P}[\tilde{N}_i = n-i]}{n}\right).$$

Note that $\mathbb{P}[\tilde{N}_i \ge k+\ell-i] = 1$ for every $i \in \{1,\dots\ell\}$. An index shift shows that the last series is equal to $\mathbb{E}[1/(i+\tilde{N}_i)]$, hence (6.2) is the right condition to turn $\{q_n\}_{n\in\mathbb{N}_0}$ into a probability distribution. □

To apply Theorem 6.1 in practice, we need to compute

$$\mathbb{E}\left[\frac{1}{i+\tilde{N}_i}\right] = \sum_{n=k+\ell-i}^{\infty} \frac{1}{i+n}\,\mathbb{P}[\tilde{N}_i = n]$$

whenever $\beta_i \ne 0$. For the distributions in the extended Panjer class with unbounded support, closed-form expressions for this series are available and given in the following lemmas, where we use the notation of Theorem 6.1. Note that truncated distributions were defined in (4.9).

Lemma 6.3 (Truncated Poisson distribution) *Assume that the random variable* $\tilde{N}_i$ *follows a* $(k+\ell-i)$*-truncated Poisson distribution with parameter* $\lambda > 0$. *Then*

$$\mathbb{E}\left[\frac{1}{i+\tilde{N}_i}\right] = (-1)^i \frac{(i-1)!}{c_i\lambda^i}\left(e^{-\lambda} - \sum_{n=0}^{i-1} \frac{(-\lambda)^n}{n!}\right) - \frac{e^{-\lambda}}{c_i} \sum_{n=0}^{k+\ell-i-1} \frac{\lambda^n}{(i+n)n!}, \quad (6.3)$$

where

$$c_i = 1 - e^{-\lambda} \sum_{n=0}^{k+\ell-i-1} \frac{\lambda^n}{n!}$$

denotes the normalizing constant of the truncated Poisson distribution.

Proof Note that

$$\mathbb{E}\left[\frac{1}{i+\tilde{N}_i}\right] = \frac{e^{-\lambda}}{c_i\lambda^i} \sum_{n=k+\ell-i}^{\infty} \frac{\lambda^{i+n}}{(i+n)\,n!} \quad (6.4)$$

and

$$\frac{d}{dx}\sum_{n=k+\ell-i}^{\infty}\frac{x^{i+n}}{(i+n)n!}=x^{i-1}\sum_{n=k+\ell-i}^{\infty}\frac{x^n}{n!}=x^{i-1}e^x-\sum_{n=0}^{k+\ell-i-1}\frac{x^{i+n-1}}{n!}. \tag{6.5}$$

Integrating (6.5) from 0 to λ, using integration by parts and induction over i for the term $x^{i-1}e^x$, we get

$$(-1)^i(i-1)!\left(1-e^{\lambda}\sum_{n=0}^{i-1}\frac{(-\lambda)^n}{n!}\right)-\sum_{n=0}^{k+\ell-i-1}\frac{\lambda^{i+n}}{(i+n)n!}.$$

Substituting this result into (6.4) yields (6.3). □

Lemma 6.4 (Truncated extended negative binomial distribution) *Assume that the random variable $\tilde{N}_i$ follows a $(k+\ell-i)$-truncated*

(a) $\mathrm{ExtNegBin}(\alpha, m, p)$ *given by* (2.1) *with parameters* $m \in \{1, \dots, k+\ell-i\}$, $p \in (0,1)$ *and* $\alpha \in (-m, -m+1)$, *or*
(b) $\mathrm{NegBin}(\alpha, p)$ *with parameters* $\alpha > 0$ *and* $p \in (0,1)$.

Then, using the abbreviation $q = 1-p$,

$$\mathbb{E}\left[\frac{1}{i+\tilde{N}_i}\right]=\frac{1}{c_i q^i}\sum_{n=0}^{i-1}(-1)^n\binom{i-1}{n}d_n-\frac{1}{c_i}\sum_{n=0}^{k+\ell-i-1}\binom{\alpha+n-1}{n}\frac{q^n}{i+n},$$

where

$$c_i=p^{-\alpha}-\sum_{n=0}^{k+\ell-i-1}\binom{\alpha+n-1}{n}q^n$$

denotes the normalizing constant of the truncated distribution, and

$$d_n=\begin{cases}-\log(1-q) & \text{if } n=\alpha-1,\\ \frac{1-(1-q)^{n-\alpha+1}}{n-\alpha+1} & \text{otherwise.}\end{cases}$$

Proof Note that

$$\mathbb{E}\left[\frac{1}{i+\tilde{N}_i}\right]=\frac{1}{c_i q^i}\sum_{n=0}^{\infty}\binom{\alpha+n-1}{n}\frac{q^{i+n}}{i+n}-\frac{1}{c_i}\sum_{n=0}^{k+\ell-i-1}\binom{\alpha+n-1}{n}\frac{q^n}{i+n} \tag{6.6}$$

and that, using the binomial series,

$$\frac{d}{dx}\sum_{n=0}^{\infty}\binom{\alpha+n-1}{n}\frac{x^{i+n}}{i+n} = x^{i-1}\sum_{n=0}^{\infty}\binom{\alpha+n-1}{n}x^{n}$$
$$= \big(1-(1-x)\big)^{i-1}(1-x)^{-\alpha}$$
$$= \sum_{n=0}^{i-1}(-1)^{n}\binom{i-1}{n}(1-x)^{n-\alpha}. \tag{6.7}$$

Finally, integrate (6.7) from 0 to q and plug the result into (6.6). □

Lemma 6.5 (Truncated logarithmic distribution) *Assume that $k+\ell-i\geq 1$ and that $\tilde{N}_i$ follows the $(k+\ell-i)$-truncation of a logarithmic distribution with parameter $q\in(0,1)$ given by* (2.8). *Then*

$$\mathbb{E}\left[\frac{1}{i+\tilde{N}_i}\right] = \frac{1}{i\,c_i}\left(\sum_{n=k+\ell-i}^{k+\ell-1}\frac{q^n}{n} - \frac{1-q^i}{q^i}c_0\right), \tag{6.8}$$

where the normalizing constants for the truncated distributions are defined by

$$c_j = \sum_{n=k+\ell-j}^{\infty}\frac{q^n}{n} = -\log(1-q) - \sum_{n=1}^{k+\ell-j-1}\frac{q^n}{n} \quad \text{for } j\in\{0,i\}.$$

Proof Note that

$$\mathbb{E}\left[\frac{1}{i+\tilde{N}_i}\right] = \frac{1}{c_i}\sum_{n=k+\ell-i}^{\infty}\frac{q^n}{(i+n)n}. \tag{6.9}$$

Using a partial fraction decomposition and an index shift, we obtain

$$\sum_{n=k+\ell-i}^{\infty}\frac{q^n}{(i+n)n} = \frac{1}{i}\sum_{n=k+\ell-i}^{\infty}\left(\frac{q^n}{n} - \frac{1}{q^i}\frac{q^{i+n}}{i+n}\right)$$
$$= \frac{1}{i}\sum_{n=k+\ell-i}^{k+\ell-1}\frac{q^n}{n} + \frac{1}{i}\left(1-\frac{1}{q^i}\right)\sum_{n=k+\ell}^{\infty}\frac{q^n}{n},$$

where the last series equals c_0. Substitute this into (6.9) to get (6.8). □

Lemma 6.6 (Truncated extended logarithmic distribution) *Assume that $k+\ell-i\geq 1$ and that $\tilde{N}_i$ follows the $(k+\ell-i)$-truncation of an* ExtLog(m,q) *given by* (2.4) *with parameters $m\in\{1,\ldots,k+\ell-i\}$ and $q\in(0,1)$ for $m=1$ or $q\in(0,1]$ for $m\geq 2$, respectively. Then*

$$\mathbb{E}\left[\frac{1}{i+\tilde{N}_i}\right] = \frac{1}{c_0(q)}\sum_{j=1}^{i}\frac{c_j(q)}{(j+m)q^j}\prod_{n=1}^{j-1}\frac{n-i}{n+m}, \tag{6.10}$$

where $c_j(q)$ *for* $j \in \{0, 1, \dots, i\}$ *is the normalizing constant of the* $(j+k+\ell-i)$-*truncated* $\mathrm{ExtLog}(j+m, q)$ *given by*

$$c_j(q) = \sum_{n=j+k+\ell-i}^{\infty} \frac{q^n}{\binom{n}{j+m}} = (-1)^{j+m}(j+m)\chi_{j+m}(q) - \sum_{n=j+m}^{j+k+\ell-i-1} \frac{q^n}{\binom{n}{j+m}} \tag{6.11}$$

with χ_{j+m} *defined in Lemma* 2.1.

Proof Note that

$$\mathbb{E}\left[\frac{1}{i+\tilde{N}_i}\right] = \frac{q^{-i}}{c_0(q)} \sum_{n=k+\ell-i}^{\infty} \frac{q^{i+n}}{(i+n)\binom{n}{m}} = \frac{q^{-i}}{c_0(q)} \int_0^q x^{i-1} c_0(x)\, dx \tag{6.12}$$

because $x^{i-1}c_0(x)$ with $c_0(x)$ defined by (6.11) is the derivative of the series. Since $c_j'(x) = (j+m)c_{j-1}(x)$ for $j \in \{1, \dots, i\}$, integration by parts shows that

$$\int_0^q x^{i-j} c_{j-1}(x)\, dx = q^{i-j} \frac{c_j(q)}{j+m} + \frac{j-i}{j+m} \int_0^q x^{i-(j+1)} c_j(x)\, dx,$$

hence by induction

$$\int_0^q x^{i-1} c_0(x)\, dx = \sum_{j=1}^{i} q^{i-j} \frac{c_j(q)}{j+m} \prod_{n=1}^{j-1} \frac{n-i}{n+m}.$$

Substituting this result into (6.12) proves (6.10). □

7 Study of the recurrence relation

7.1 Characterization and moments of distributions

The following proposition aims at characterizing distributions satisfying the relation given in (4.4). The formulas for the first and the second moment are needed in order to fit a distribution constructed from (4.4) by its moments as described in the paragraph below.

Lemma 7.1 *Consider the assumptions of Theorem* 4.5(a) *and furthermore let* φ_N *and* $\varphi_{\tilde{N}_i}$ *for* $i \in \{1, \dots, \ell\}$ *denote the corresponding probability generating functions of the claim numbers. Then*

$$\varphi_N'(s) = \sum_{i=1}^{\ell} \big(a_i s^i \varphi_{\tilde{N}_i}'(s) + (ia_i + b_i)s^{i-1}\varphi_{\tilde{N}_i}(s)\big) + \sum_{n=1}^{k+\ell-1} n q_n s^{n-1} \tag{7.1}$$

holds at least for all $s \in \mathbb{C}$ *with* $|s| < 1$. *If* $\mathbb{E}[\tilde{N}_i] < \infty$ *for* $i = 1, \ldots, \ell$, *then the expectation of* N *is given by*

$$\mathbb{E}[N] = \sum_{i=1}^{\ell} \big(a_i \, \mathbb{E}[\tilde{N}_i] + i a_i + b_i\big) + \sum_{n=1}^{k+\ell-1} n q_n, \tag{7.2}$$

and if $\mathbb{E}[\tilde{N}_i^2] < \infty$ *for* $i = 1, \ldots, \ell$, *then the second moment of* N *is given by*

$$\begin{aligned} \mathbb{E}\big[N^2\big] = \sum_{i=1}^{\ell} \Big(a_i \, \mathbb{E}\big[\tilde{N}_i^2\big] + (2ia_i - a_i + b_i)\, \mathbb{E}[\tilde{N}_i] + (ia_i + b_i)(i-1)\Big) \\ + \sum_{n=2}^{k+\ell-1} n(n-1) q_n + \mathbb{E}[N]. \end{aligned} \tag{7.3}$$

Proof To prove the relation (7.1) between the derivatives of the probability generating functions, note that by (4.4)

$$\begin{aligned} \varphi_N'(s) &= \sum_{n=1}^{\infty} n q_n s^{n-1} = \sum_{n=1}^{k+\ell-1} n q_n s^{n-1} + \sum_{n=k+\ell}^{\infty} n \sum_{i=1}^{\ell} \Big(a_i + \frac{b_i}{n}\Big) \tilde{q}_{i,n-i} s^{n-1} \\ &= \sum_{n=1}^{k+\ell-1} n q_n s^{n-1} + \sum_{i=1}^{\ell} \sum_{n=k+\ell-i}^{\infty} (n a_i + i a_i + b_i) \tilde{q}_{i,n} s^{n+i-1}. \end{aligned}$$

Using (4.3), this yields (7.1). For $s \nearrow 1$ we get the expectation in (7.2), using that $\varphi_{\tilde{N}_i}'(1-) = \mathbb{E}[\tilde{N}_i]$ and $\varphi_{\tilde{N}_i}(1) = 1$ for $i = 1, \ldots, \ell$. Differentiating (7.1) gives

$$\begin{aligned} \varphi_N''(s) = \sum_{i=1}^{\ell} \Big(a_i s^i \varphi_{\tilde{N}_i}''(s) + (2ia_i + b_i) s^{i-1} \varphi_{\tilde{N}_i}'(s) + (ia_i + b_i)(i-1) s^{i-2} \varphi_{\tilde{N}_i}(s)\Big) \\ + \sum_{n=2}^{k+\ell-1} n(n-1) q_n s^{n-2} \end{aligned}$$

at least for $s \in \mathbb{C}$ with $|s| < 1$. Taking $s \nearrow 1$ and using $\varphi_X''(1-) = \mathbb{E}[X^2] - \mathbb{E}[X]$, valid for $\mathbb{N}_0$-valued random variables X with finite second moment, we get (7.3). □

7.2 Further distributions satisfying the recurrence relation

We want to present some distributions satisfying relation (4.4) for $k = 0$, $\ell = 1$ and $a_1 = 0$.

For $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$ define the Pochhammer symbol (also known as rising factorial) by

$$(x)_n = \prod_{k=0}^{n-1} (x + k),$$

where $(x)_0 = 1$ is the usual definition for the empty product.

Lemma 7.2 *Let $i, j \in \mathbb{N}_0$. For $i \geq 1$ let $\gamma \in \mathbb{R}^i$ and for $j \geq 1$ let $\delta \in \mathbb{R}^j$. Take $\lambda, c_{\gamma,\delta} \in \mathbb{R}$ and let $\{\mu_k\}_{k\in\mathbb{N}_0}$ be a sequence of real numbers. Assume that*

$$q_n(\gamma,\delta) := \frac{c_{\gamma,\delta}\lambda^n}{n!}\sum_{k=0}^{\infty}\mu_k \frac{(\gamma_1)_{k+n}\cdots(\gamma_i)_{k+n}}{(\delta_1)_{k+n}\cdots(\delta_j)_{k+n}}, \quad n \in \mathbb{N}, \tag{7.4}$$

together with $q_0(\gamma,\delta) \in [0,1)$, forms a well-defined probability distribution. Furthermore, assume that $q_n(\gamma+1,\delta+1)$, with the 1 added to every component of γ and δ, and defined by the analogue of (7.4) for all $n \in \mathbb{N}_0$, is a well-defined probability distribution, too. Then

$$q_n(\gamma,\delta) = \frac{\lambda c_{\gamma,\delta}}{n c_{\gamma+1,\delta+1}} \frac{\gamma_1\cdots\gamma_i}{\delta_1\cdots\delta_j} q_{n-1}(\gamma+1,\delta+1), \quad n \in \mathbb{N}.$$

Proof Just note that $(x)_{k+n} = x(x+1)_{k+n-1}$ for all $k+n \in \mathbb{N}$ and $x \in \mathbb{R}$. □

Example 7.3 (Poisson mixed over beta distribution) Consider a Poisson distribution with parameter $\lambda > 0$, which is reduced according to a beta distribution with parameters $\alpha, \beta > 0$, i.e.,

$$q_n(\alpha) = \int_0^1 \frac{(\lambda x)^n}{n!} e^{-\lambda x} f_{\alpha,\beta}(x)\,dx, \quad n \in \mathbb{N}_0,$$

with $f_{\alpha,\beta}$ from (5.17). Inserting the series of $e^{-\lambda x} = \sum_{k=0}^{\infty}\frac{(-\lambda x)^k}{k!}$, interchanging the series with the integral using dominated convergence, and rewriting the integral in terms of $f_{\alpha+k+n,\beta}$ leads to

$$q_n(\alpha) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \frac{\lambda^n}{n!} \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \frac{\Gamma(\alpha+k+n)\,\Gamma(\beta)}{\Gamma(\alpha+\beta+k+n)}, \quad n \in \mathbb{N}_0.$$

By the functional equation of the gamma function, $(x)_\ell = \Gamma(x+\ell)/\Gamma(x)$ for all $x > 0$ and $\ell \in \mathbb{N}_0$. Hence $q_n(\alpha)$ simplifies to

$$q_n(\alpha) = \frac{\lambda^n}{n!}\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}\frac{(\alpha)_{k+n}}{(\alpha+\beta)_{k+n}}, \quad n \in \mathbb{N}_0,$$

which is the representation of (7.4), hence

$$q_n(\alpha) = \frac{\alpha\lambda}{(\alpha+\beta)n} q_{n-1}(\alpha+1), \quad n \in \mathbb{N},$$

which coincides with (5.16) in the setting of Example 5.8.

Example 7.4 (Negative hypergeometric) Consider the negative hypergeometric distribution [13] (also called Pólya–Eggenberger distribution), which arises as a binomial distribution on $\{0,\ldots,m\}$ with $m \in \mathbb{N}$ mixed over a beta distribution, i.e.,

$$q_n(\alpha,m) = \int_0^1 \binom{m}{n} p^n (1-p)^{m-n} f_{\alpha,\beta}(p)\,dp, \quad n \in \{0,\ldots,m\},$$

with $f_{\alpha,\beta}$ from (5.17). Rewriting the integral using the density $f_{\alpha+n,\beta+m-n}$ yields

$$
\begin{aligned}
q_n(\alpha, m) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \frac{\Gamma(\alpha+n)\,\Gamma(\beta+m-n)}{\Gamma(\alpha+\beta+m)} \binom{m}{n} \\
&= \frac{(\alpha)_n(\beta)_{m-n}}{(\alpha+\beta)_m} \frac{m!}{n!(m-n)!} = \frac{\binom{\alpha+n-1}{n}\binom{\beta+m-n-1}{m-n}}{\binom{\alpha+\beta+m-1}{m}}, \quad n \in \{0, \ldots, m\},
\end{aligned} \tag{7.5}
$$

which justifies the name of the distribution. Note that $(x-n+1)_n = (-1)^n(-x)_n$ for all $x \in \mathbb{R}$. Applying this for $x \in \{m, \beta+m-1\}$, we obtain

$$
\frac{(\beta)_{m-n}}{(m-n)!} = \frac{(\beta)_m}{m!} \frac{(m-n+1)_n}{(\beta+m-n)_n} = \frac{(\beta)_m}{m!} \frac{(-m)_n}{(-\beta-m+1)_n};
$$

hence (7.5) can be written as

$$
q_n(\alpha, m) = \frac{(\beta)_m}{(\alpha+\beta)_m} \frac{1}{n!} \frac{(\alpha)_n(-m)_n}{(1-\beta-m)_n}, \quad n \in \mathbb{N}_0, \tag{7.6}
$$

where we consider the last quotient as zero for $n > m$ because $(-m)_n = 0$ in this case. The representation (7.6) is of the form (7.4). Hence, if $m \geq 2$, then

$$
q_n(\alpha, m) = \frac{\alpha m}{(\alpha+\beta)n} q_{n-1}(\alpha+1, m-1), \quad n \in \mathbb{N},
$$

by Lemma 7.2. Of course, this can also be seen directly using (7.5).

Let $\{p_n(\alpha, m)\}_{n\in\mathbb{N}_0}$ denote the distribution of the random sum $S = X_1 + \cdots + X_N$ with $N \sim \{q_n(\alpha, m)\}_{n\in\{0,\ldots,m\}}$ and let $\varphi_{\alpha,m}$ denote the pgf of N. Then

$$
p_0(\alpha, 1) = \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta} \mathbb{P}[X_1 = 0]
$$

and

$$
p_n(\alpha, 1) = \frac{\alpha}{\alpha+\beta} \mathbb{P}[X_1 = n], \quad n \in \mathbb{N}.
$$

For $m \geq 2$, Theorem 4.5(a) gives the recursion

$$
p_n(\alpha, m) = \frac{\alpha m}{(\alpha+\beta)n} \sum_{j=0}^{n} j\, \mathbb{P}[X_1 = j] p_{n-j}(\alpha+1, m-1), \quad n \in \mathbb{N}, \tag{7.7}
$$

with $p_0(\alpha, m) = \varphi_{\alpha,m}(\mathbb{P}[X_1 = 0])$. The weighted convolution (7.7) is numerically stable. If $\mathbb{P}[X_1 = 0] = 0$, then

$$
p_0(\alpha, m) = q_0(\alpha, m) = \frac{\beta+m-1}{\alpha+\beta+m-1} q_0(\alpha, m-1), \quad m \geq 2,
$$

with $q_0(\alpha, 1) = \frac{\beta}{\alpha+\beta}$. In this case our recursion scheme is slightly more efficient than the obvious formula

$$p_n(\alpha, m) = \sum_{k=0}^{m} \mathbb{P}[S_k = n] q_k(\alpha, m), \quad n \in \mathbb{N}_0,$$

with $S_0 := 0$, where the distribution of $S_k = X_1 + \cdots + X_k$ is calculated recursively by convolution, because (7.7) avoids the explicit computation of the distribution $\{q_n(\alpha, m)\}_{n\in\{0,\dots,m\}}$ given by (7.5). Note that Hesselager [12], as well as Panjer and Willmot [22], derives faster recursions for compound sums with loss numbers that follow a negative hypergeometric distribution, but the numerical stability of their algorithms cannot be guaranteed.

Example 7.5 Laplace–Haag The Laplace–Haag distribution arises in the context of the Laplace–Haag matching problem [13, Chapter 10, p. 410]. The probabilities are given by

$$q_n(m, M) = \frac{\lambda^n}{n!} \sum_{k=0}^{m-n} (-1)^k \frac{\lambda^k}{k!} \frac{m!(M-n-k)!}{(m-n-k)!M!}, \quad n \in \{0, \dots, m\}, \tag{7.8}$$

where $m, M \in \mathbb{N}$ with $m \le M$ and $\lambda > 0$ such that $q_0(M, m), \dots, q_{m-1}(M, m)$ given by (7.8) are non-negative. This is certainly the case for $\lambda \in (0, M/m]$, because then the summands in (7.8) are non-increasing in absolute value as k increases from 0 to $m - n$. We define $q_n(m, M) = 0$ for $n \in \mathbb{N}$ with $n > m$. Rewriting (7.8) using Pochhammer symbols we get

$$q_n(m, M) = \frac{\lambda^n}{n!} \sum_{k=0}^{m-n} (-1)^k \frac{\lambda^k}{k!} \frac{(-m)_{n+k}}{(-M)_{n+k}}, \quad n \in \{0, \dots, m\}. \tag{7.9}$$

Note that (7.9) is of the form (7.4), since $(-m)_{n+k}/(-M)_{n+k}$ vanishes whenever $n + k \ge m + 1$.

If $m, M \in \mathbb{N}$ with $2 \le m \le M$ and $\lambda \in (0, M/m]$, then also $\lambda \le (M-1)/(m-1)$ and Lemma 7.2 shows that

$$q_n(m, M) = \frac{\lambda m}{M n} q_{n-1}(m-1, M-1), \quad n \in \{1, \dots, m\}.$$

8 A generalization of De Pril's recursion

De Pril's recursion [3] allows for the calculation of higher moments of the aggregate loss S in the collective model (1.1) with claim number distributions in the Panjer$(a, b, 1)$ class, cf. [27]. We extend this recursion to the Panjer(a, b, k) class in Theorem 8.1 below (compare [11, Corollary 4.4] for binomial moments) and then apply Theorem 4.5 in order to generalize this once more in Theorem 8.2. The knowledge of higher moments of the compound sum is in particular useful when some approximate series expansion (such as the Edgeworth expansion) is used as alternative to the recursive calculation of the distribution. Using recursion (8.3) below, we can again guarantee numerical stability in these recursions for some of the cases where (8.1) or (8.2) might be unstable.

Theorem 8.1 (Extension I of De Pril's recursion) *Consider the collective model given by* (1.1). *Assume that the distribution of N belongs to the* Panjer(a, b, k) *class. Define $S_k = X_1 + \cdots + X_k$ and note that $\mathbb{E}[S^0] = 1$ by convention.*

(a) *If $a < 1$ and $\mathbb{E}[X_1^{\bar{n}}] < \infty$ for an $\bar{n} \in \mathbb{N}$, then $\mathbb{E}[S_k^n] < \infty$ and*

$$\mathbb{E}\big[S^n\big] = \frac{1}{1-a}\left(\mathbb{P}[N=k]\,\mathbb{E}\big[S_k^n\big] + \sum_{j=1}^{n}\binom{n}{j}\left(a+\frac{bj}{n}\right)\mathbb{E}\big[S^{n-j}\big]\,\mathbb{E}\big[X_1^j\big]\right) \tag{8.1}$$

for every $n \in \{1, \dots, \bar{n}\}$.

(b) *If $a = 1$ and $-b > 2$ as well as $\mathbb{E}[X_1^{\bar{n}}] < \infty$ for an $\bar{n} \in \mathbb{N} \setminus \{1\}$ and $\mathbb{E}[X_1] > 0$, then $\mathbb{E}[S_k^{n+1}] < \infty$ and*

$$\begin{aligned}\mathbb{E}\big[S^n\big] &= \frac{1}{(-b-1-n)\,\mathbb{E}[X_1]}\\ &\quad\times\left(\mathbb{P}[N=k]\,\mathbb{E}\big[S_k^{n+1}\big] + \sum_{j=1}^{n}\binom{n}{j}\left(\frac{n+1}{j+1}+b\right)\right.\\ &\quad\left.\times\,\mathbb{E}\big[S^{n-j}\big]\,\mathbb{E}\big[X_1^{j+1}\big]\right)\end{aligned} \tag{8.2}$$

for every $n \in \{1, \dots, \bar{n}-1\}$ with $n < -b-1$.

Theorem 8.1(a) applies in particular to ExtLog$(k, 1)$ with $k \geq 3$, which is in the Panjer$(1, -k, k)$ class, and to ExtNegBin$(\alpha, k, 0)$ with parameters $k \in \mathbb{N} \setminus \{1\}$ and $\alpha \in (-k, -k+1)$, which is in the Panjer$(1, \alpha-1, k)$ class. It also applies to truncations of these distributions, cf. (4.9). Note that for the cases $N \sim$ ExtLog$(k, 1)$ or $N \sim$ ExtNegBin$(\alpha, k, 0)$, the moment of order $n \in \mathbb{N}_0$ is finite if and only if $n \leq k-2$ or $n < -\alpha$, respectively. Theorem 8.1 will be proved using the proof of the following extension, which we state using the notation of Theorem 4.5.

Theorem 8.2 (Extension II of De Pril's recursion)

(a) *In addition to the assumptions of Theorem 4.5(a), suppose that X_1 and $\tilde{S}_{(i)}$ for $i = 1, \dots, \ell$ have finite moments up to order $\bar{n} \in \mathbb{N}$. Then*

$$\mathbb{E}\big[S^n\big] = \sum_{j=1}^{k+\ell-1} q_j\,\mathbb{E}\big[S_j^n\big] + \sum_{i=1}^{\ell}\sum_{j=0}^{n}\binom{n}{j}\left(a_i+\frac{b_i\,j}{in}\right)\mathbb{E}\big[\tilde{S}_{(i)}^{n-j}\big]\,\mathbb{E}\big[S_i^j\big] \tag{8.3}$$

for every $n \in \{1, \dots, \bar{n}\}$, where every $S_j = X_1 + \cdots + X_j$ has also finite moments up to order $\bar{n}$ and $\mathbb{E}[S_i^0] = \mathbb{E}[\tilde{S}_{(i)}^0] = 1$ by convention.

(b) *Under the assumptions of Theorem* 4.5(b),

$$\mathbb{E}\left[S^n\right] = \sum_{i=1}^{\ell} \nu_i \, \mathbb{E}\left[\tilde{S}_{(i)}^n\right]$$

for all $n \in \mathbb{N}$, *where both sides may be infinite.*

Proof (a) By Jensen's inequality, for every $j \in \mathbb{N}$,

$$S_j^n = j^n \left(\frac{X_1 + \cdots + X_j}{j} \right)^n \le j^{n-1} \left(X_1^n + \cdots + X_j^n \right), \tag{8.4}$$

hence S_j has finite moments up to order $\bar{n}$. By (4.5) from Theorem 4.5(a) we have

$$\sum_{m=1}^{M} m^n p_m = \sum_{j=1}^{k+\ell-1} q_j \sum_{m=1}^{M} m^n \, \mathbb{P}[S_j = m] + \sum_{i=1}^{\ell} E_{i,M} \tag{8.5}$$

for every $M \in \mathbb{N}$ with the expressions

$$E_{i,M} = \sum_{m=1}^{M} m^n \sum_{r=0}^{m} \left(a_i + \frac{b_i r}{im} \right) \mathbb{P}[S_i = r] \tilde{p}_{i,m-r}, \quad i \in \{1, \ldots, \ell\}.$$

Changing the order of summation in $E_{i,M}$, we see that

$$E_{i,M} = \sum_{r=0}^{M} \sum_{m=\max\{1,r\}}^{M} \left(a_i m^n + \frac{b_i r}{i} m^{n-1} \right) \mathbb{P}[S_i = r] \tilde{p}_{i,m-r}.$$

In the case $r = 0$ we may add the term for $m = 0$, because it is zero. Shifting the summation index m down by r yields

$$E_{i,M} = \sum_{r=0}^{M} \sum_{m=0}^{M-r} \left(a_i (m+r)^n + \frac{b_i r}{i} (m+r)^{n-1} \right) \mathbb{P}[S_i = r] \tilde{p}_{i,m}.$$

By using the binomial formula for $(m+r)^{n-1}$, shifting the summation index j up by 1 and adding the term for $j = 0$, which is zero, we get that

$$r(m+r)^{n-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} m^{n-1-j} r^{j+1} = \sum_{j=0}^{n} \frac{j}{n} \binom{n}{j} m^{n-j} r^j.$$

Using the binomial formula also for $(m+r)^n$ and changing the order of summation,

$$E_{i,M} = \sum_{j=0}^{n} \binom{n}{j} \left(a_i + \frac{b_i j}{in} \right) \sum_{m=0}^{M} m^{n-j} \tilde{p}_{i,m} \sum_{r=0}^{M-m} r^j \, \mathbb{P}[S_i = r]. \tag{8.6}$$

Substituting (8.6) into (8.5), sending $M \to \infty$, and using that all the expectations on the right-hand side of (8.3) are finite, we see that $\mathbb{E}[S^n]$ is finite and (8.3) holds.

(b) Using Theorem 4.5(b) and exchanging the order of summation, which is allowed because all terms are non-negative, we get that

$$\mathbb{E}\big[S^n\big] = \sum_{m=1}^{\infty} m^n p_m = \sum_{m=1}^{\infty}\sum_{i=1}^{\ell} m^n \nu_i \tilde{p}_{i,m} = \sum_{i=1}^{\ell} \nu_i \sum_{m=1}^{\infty} m^n \tilde{p}_{i,m} = \sum_{i=1}^{\ell} \nu_i \, \mathbb{E}\big[\tilde{S}^n_{(i)}\big]. \qquad \square$$

Proof of Theorem 8.1 (a) If $N \sim \{q_n\}_{n\in\mathbb{N}_0}$ belongs to the Panjer(a,b,k) class, then the proof of Theorem 8.2(b) is applicable by choosing $\ell = 1$ and $\tilde{q}_{1,m} = q_m$, hence $p_m = \tilde{p}_{1,m}$ for all $m \in \mathbb{N}_0$. Substituting (8.6) into (8.5), using $q_1 = \cdots = q_{k-1} = 0$ and moving the term for $j = 0$ to the left-hand side, we see that

$$\begin{aligned}
&\sum_{m=1}^{M}\big(1 - a\,\mathbb{P}[X_1 \le M-m]\big) m^n p_m \\
&\quad = q_k \sum_{m=1}^{M} m^n\, \mathbb{P}[S_k = m] \\
&\qquad + \sum_{j=1}^{n}\binom{n}{j}\Big(a + \frac{bj}{n}\Big)\sum_{m=0}^{M} m^{n-j} p_m \sum_{r=0}^{M-m} r^j\, \mathbb{P}[X_1 = r].
\end{aligned} \tag{8.7}$$

For $n \in \{1,\dots,\bar{n}\}$ assume as induction hypothesis that $\mathbb{E}[S^0],\dots,\mathbb{E}[S^{n-1}]$ are finite, which is true for $n = 1$. For $M \to \infty$, the right-hand side of (8.7) converges to the term in parentheses on the right-hand side of (8.1), where every expectation is finite. Since $a < 1$, all terms on the left-hand side of (8.7) are non-negative, and Fatou's lemma applied to the counting measure on $\mathbb{N}$ and the functions defined by $f_M(m) = (1 - a\,\mathbb{P}[X_1 \le M-m]) m^n p_m 1_{\{1,\dots,M\}}(m)$ for $m, M \in \mathbb{N}$ therefore implies that $(1-a)\,\mathbb{E}[S^n] < \infty$. Hence $\mathbb{N} \ni m \mapsto (1+|a|) m^n p_m$ is summable, and the dominated convergence theorem implies that the left-hand side of (8.7) converges to $(1-a)\,\mathbb{E}[S^n]$ as $M \to \infty$. This proves (8.1).

(b) Set $a = 1$ in (8.7). Moving the term for $j = 1$ to the left-hand side of (8.7), replacing n by $n+1$, shifting the summation index j down by 1, and using the identity $\binom{n+1}{j+1} = \binom{n}{j}\frac{n+1}{j+1}$, we obtain

$$\begin{aligned}
&\sum_{m=1}^{M} \mathbb{P}[X_1 > M-m] m^{n+1} p_m - (b+n+1)\sum_{m=0}^{M} m^n p_m \sum_{r=0}^{M-m} r\,\mathbb{P}[X_1 = r] \\
&\quad = q_k \sum_{m=1}^{M} m^{n+1}\,\mathbb{P}[S_k = m] \\
&\qquad + \sum_{j=1}^{n}\binom{n}{j}\Big(\frac{n+1}{j+1} + b\Big)\sum_{m=0}^{M} m^{n-j} p_m \sum_{r=0}^{M-m} r^{j+1}\,\mathbb{P}[X_1 = r].
\end{aligned} \tag{8.8}$$

For $n \in \{1, \dots, \bar{n} - 1\}$ with $n < -b - 1$ assume as induction hypothesis that $\mathbb{E}[S^0], \dots, \mathbb{E}[S^{n-1}]$ are finite, which is true for $n = 1$. It follows from Jensen's inequality that $\mathbb{E}[S_k^{n+1}] < \infty$, cf. (8.4). By assumption, $\mathbb{E}[X_1^2], \dots, \mathbb{E}[X_1^{n+1}]$ are finite. Letting $M \to \infty$ in (8.8), we see that the right-hand side converges to the term in parentheses on the right-hand side of (8.2), which is finite. Since the second term on the left-hand side of (8.8) converges to $-(b + n + 1)\,\mathbb{E}[S^n]\,\mathbb{E}[X_1] \geq 0$, it follows that $\mathbb{E}[S^n]$ is finite. To prove (8.2), it remains to show that the first term in (8.8), which is

$$I_M := \sum_{m=1}^{M} \mathbb{P}[X_0 > M - m] m^{n+1} p_m = \mathbb{E}\big[S^{n+1} 1_{\{S \leq M < S + X_0\}}\big]$$

with X_0 such that $\{X_j\}_{j \in \mathbb{N}_0}$ is i.i.d., converges to zero as $M \to \infty$.

By (8.4), $S_j^{n+1} \leq j^n (X_1^{n+1} + \dots + X_j^{n+1})$ for every $j \in \mathbb{N}$. Since (S_j, S_{j+1}) equals $(S_j, S_j + X_0)$ in distribution, the independence of N and the i.i.d. sequence $\{X_j\}_{j \in \mathbb{N}_0}$ implies

$$I_M = \sum_{j=k}^{\infty} \mathbb{E}\big[S_j^{n+1} 1_{\{S_j \leq M < S_{j+1}\}}\big] q_j \leq \sum_{j=k}^{\infty} j^{n+1} q_j\, \mathbb{E}\big[X_1^{n+1} 1_{\{S_j \leq M < S_{j+1}\}}\big].$$

Define $\tau_M = \sum_{j \in \mathbb{N}} 1_{\{S_j \leq M\}}$. Then $\tau_M = j$ on $\{S_j \leq M < S_{j+1}\}$ for every $j \in \mathbb{N}_0$ and the upper estimate for I_M can be rewritten as

$$I_M \leq \mathbb{E}\left[X_1^{n+1} \tau_M^{n+1} q_{\tau_M} 1_{\{k \leq \tau_M < \infty\}}\right]. \tag{8.9}$$

Using (1.2), we see that, for all $j \geq k + 1$,

$$j^{n+1} q_j = \left(\frac{j}{j-1}\right)^{n+1} \left(1 + \frac{b}{j}\right) (j-1)^{n+1} q_{j-1} \leq \left(1 - \frac{1}{j}\right)^{-b-n-1} (j-1)^{n+1} q_{j-1},$$

where we used Bernoulli's inequality

$$1 + bx \leq (1-x)^{-b} = (1-x)^{n+1} (1-x)^{-b-n-1} \quad \text{for } x = 1/j.$$

Since $1 - x \leq e^{-x}$ for all $x \in \mathbb{R}$ and since the harmonic series diverges, it follows that

$$\prod_{j=k+1}^{K} \left(1 - \frac{1}{j}\right) \leq \exp\left(-\sum_{j=k+1}^{K} \frac{1}{j}\right) \searrow 0 \quad \text{as } K \to \infty.$$

Since $-b - n - 1 > 0$, it follows that $j^{n+1} q_j \searrow 0$ as $j \to \infty$. Since $\tau_M \to \infty$ a.s. as $M \to \infty$ and $\mathbb{E}[X_1^{n+1}] < \infty$ by assumption, it follows from (8.9) by the dominated convergence theorem that $I_M \to 0$ as $M \to \infty$. □

For the case that only a k-truncation of the claim number distribution belongs to a Panjer(a, b, k) class, we can nevertheless calculate higher moments recursively using the following corollary of Theorem 8.2(a).

Corollary 8.3 *Under the assumptions of Corollary 4.7,*

$$\mathbb{E}\left[S^n\right]=\sum_{j=1}^{k-1} q_j\,\mathbb{E}\left[S_j^n\right]+\left(1-\sum_{j=0}^{k-1} q_j\right)\mathbb{E}\left[\tilde{S}^n\right]$$

for all $n \in \mathbb{N}$, where both sides may be infinite.

Proof Either use Theorem 8.2(b) with the right parameters or repeat the above proof of Theorem 8.2(b) using (4.10) instead of Theorem 4.5(b). □

9 Extension of Panjer's recursion to severities with mixed support

In this section we analyze the situation in the collective model $S = X_1 + \cdots + X_N$ given in (1.1) with severities X_n having mixed support. More precisely, we assume that the distribution of the non-negative i.i.d. sequence $\{X_n\}_{n\in\mathbb{N}}$ has a possible atom of $a_X \in [0, 1]$ at zero and a (substochastic) density f on $[0,\infty)$ for the absolutely continuous part, meaning that $\mathbb{P}[X_1 \in A] = a_X\,\delta_0(A) + \int_A f(x)\,dx$ for all Lebesgue-measurable subsets A of $[0,\infty)$, where δ_0 denotes the Dirac measure at 0. Obviously the distribution of S also has a mixed support with an atom of a_S at zero and a (substochastic) density f_S on $[0,\infty)$. If the claim number distribution belongs to a Panjer(a, b, k) class, an integral equation for f_S can be derived, cf. [23]. For example, in the case $N \sim \text{ExtNegBin}(\alpha, 1, p)$ the integral equation [22, (2.7)] is

$$f_S(x)=\left(\frac{\alpha p^\alpha q}{1-p^\alpha}+\alpha a_S q\right)\frac{f(x)}{1-a_X q} + \frac{q}{1-a_X q}\int_0^x\left(1-(1-\alpha)\frac{y}{x}\right)f(y)\,f_S(x-y)\,dy$$

for almost all $x > 0$. As $\alpha \in (-1, 0)$, the integrand can change its sign in the interval $[0, x]$ and cancellation effects can occur in the numerical solution of this equation for f_S. In the following we propose a remedy for this problem by providing a result analogous to Theorem 4.5. The statement is split into two parts. The first theorem deals with the relation of Laplace transforms of the aggregate loss in collective risk models if the claim numbers fulfill certain conditions.

Recall that the Laplace transform of a non-negative random variable Y is given by $\mathcal{L}_Y(s) = \mathbb{E}[\exp(-sY)]$ at least for all real $s \ge 0$ and that it uniquely determines the distribution of Y [15, p. 86]. Furthermore, $\mathcal{L}_Y$ is infinitely differentiable on the positive half-line, because $Y^n \exp(-sY) \le (n/s)^n e^{-n}$ for all $n \in \mathbb{N}_0$ and $s > 0$.

Theorem 9.1 *Fix $\ell \in \mathbb{N}$. Let $\{q_n\}_{n\in\mathbb{N}_0}$ and $\{\tilde{q}_{i,n}\}_{n\in\mathbb{N}_0}$ denote the probability distributions of the $\mathbb{N}_0$-valued random variables N and $\tilde{N}_i$ for $i \in \{1,\ldots,\ell\}$, respectively, which are independent of the non-negative i.i.d. sequence $\{X_n\}_{n\in\mathbb{N}}$. Define $S = X_1 + \cdots + X_N$ and $\tilde{S}_{(i)} = X_1 + \cdots + X_{\tilde{N}_i}$ for $i \in \{1,\ldots,\ell\}$.*

(a) *Assume that there exist* $k \in \mathbb{N}_0$ *and* $a_1, \ldots, a_\ell, b_1, \ldots, b_\ell \in \mathbb{R}$ *such that* (4.3) *and* (4.4) *hold. Then the derivative* $\mathcal{L}'_S$ *satisfies*

$$\mathcal{L}'_S(s) = \sum_{i=1}^{\ell}\big(a_i\,\mathcal{L}_{S_i}(s)\,\mathcal{L}'_{\tilde{S}_{(i)}}(s) + (ia_i + b_i)\mathcal{L}_{S_{i-1}}(s)\,\mathcal{L}_{\tilde{S}_{(i)}}(s)\,\mathcal{L}'_{X_1}(s)\big) + \sum_{j=1}^{k+\ell-1} q_j\,\mathcal{L}'_{S_j}(s), \quad s > 0, \tag{9.1}$$

where $S_j = X_1 + \cdots + X_j$ *for* $j \in \mathbb{N}$.

(b) *Assume that there exist* $\nu_1, \ldots, \nu_\ell \in [0, 1]$ *with* $\nu_1 + \cdots + \nu_\ell \le 1$ *such that* $q_n = \sum_{i=1}^{\ell} \nu_i\,\tilde{q}_{i,n}$ *for all* $n \in \mathbb{N}$. *Then*

$$\mathcal{L}'_S(s) = \sum_{i=1}^{\ell} \nu_i\,\mathcal{L}'_{\tilde{S}_{(i)}}(s), \quad s > 0. \tag{9.2}$$

Proof Condition (4.4) can be rewritten as $nq_n = \sum_{i=1}^{\ell}(a_i n\tilde{q}_{i,n-i} + b_i\tilde{q}_{i,n-i})$ for all $n \in \mathbb{N}$ with $n \ge k + \ell$. Multiplying each side by $\mathcal{L}^{n-1}_{X_1}(s)$, summing over n and, for the second equality, rearranging and shifting the index of summation leads to

$$\begin{aligned}\sum_{n=k+\ell}^{\infty} nq_n\,\mathcal{L}^{n-1}_{X_1}(s) &= \sum_{i=1}^{\ell}\left(a_i\sum_{n=k+\ell}^{\infty} n\tilde{q}_{i,n-i}\,\mathcal{L}^{n-1}_{X_1}(s) + b_i\sum_{n=k+\ell}^{\infty}\tilde{q}_{i,n-i}\,\mathcal{L}^{n-1}_{X_1}(s)\right)\\ &= \sum_{i=1}^{\ell}\left(a_i\mathcal{L}^{i}_{X_1}(s)\sum_{n=k+\ell-i}^{\infty}(n+i)\tilde{q}_{i,n}\,\mathcal{L}^{n-1}_{X_1}(s) + b_i\mathcal{L}^{i-1}_{X_1}(s)\sum_{n=k+\ell-i}^{\infty}\tilde{q}_{i,n}\,\mathcal{L}^{n}_{X_1}(s)\right).\end{aligned}$$

Using $\mathcal{L}^{j}_{X_1}(s) = \mathcal{L}_{S_j}(s)$ for $j \in \mathbb{N}_0$, multiplying both sides by $\mathcal{L}'_{X_1}(s)$ and adding the terms for $n = 1, \ldots, k + l - 1$ to both sides yields (9.1).

(b) Considering that $q_n = \sum_{i=1}^{\ell} \nu_i\tilde{q}_{i,n}$ for $n \in \mathbb{N}$, we multiply the equation by $n\mathcal{L}^{n-1}_{X_1}(s)\mathcal{L}'_{X_1}(s)$ and sum over $n \in \mathbb{N}$ to obtain

$$\sum_{n=1}^{\infty} nq_n\,\mathcal{L}^{n-1}_{X_1}(s)\,\mathcal{L}'_{X_1}(s) = \sum_{n=1}^{\infty}\sum_{i=1}^{\ell} n\nu_i\tilde{q}_{i,n}\,\mathcal{L}^{n-1}_{X_1}(s)\,\mathcal{L}'_{X_1}(s).$$

Finally, exchanging the order of summation yields (9.2). □

The distribution of severities in Theorem 9.1 is a general one with support in $[0, \infty)$. In Corollary 9.2 we specialize to severities with mixed support (with a possible atom only at zero) and find integral equations that can improve the stability of numerical solution procedures for the resulting integral equations. Note that these results usually significantly simplify if the claim size distribution is assumed to have no atom at all. For every $i \in \mathbb{N}_0$ we use the distribution of $S_i = X_1 + \cdots + X_i$, where a_{S_i} denotes its atom at zero and f_{S_i} denotes its substochastic (also called defective) density on $[0, \infty)$. In particular, $a_{S_0} = 1$ and $f_{S_0} \overset{\text{a.e.}}{=} 0$.

Corollary 9.2 *In addition to the assumptions of Theorem 9.1, assume that the distribution of X_1 is absolutely continuous on $(0, \infty)$ with substochastic density f and with a possible atom of a_X at zero. Then the following integral representation for a substochastic density f_S of the absolutely continuous part of the distribution of S holds for almost all $x > 0$:*

(a)

$$
\begin{aligned}
f_S(x) = \sum_{j=1}^{k+\ell-1} q_j f_{S_j}(x) + \sum_{i=1}^{\ell}\Bigg(a_i \bigg(\int_0^x f_{S_i}(x-y)\frac{y}{x} f_{\tilde{S}_{(i)}}(y)\,dy + a_{S_i} f_{\tilde{S}_{(i)}}(x)\bigg) \\
+ (ia_i + b_i)\bigg(\int_0^x \alpha(x-y)\frac{y}{x} f(y)\,dy \\
+ a_{S_{i-1}} \int_0^x f_{\tilde{S}_{(i)}}(x-y)\frac{y}{x} f(y)\,dy \\
+ a_{\tilde{S}_{(i)}} \int_0^x f_{S_{i-1}}(x-y)\frac{y}{x} f(y)\,dy + a_{S_{i-1}} a_{\tilde{S}_{(i)}} f(x)\bigg)\Bigg),
\end{aligned}
\tag{9.3}
$$

where the auxiliary function α is given by $\alpha(x) = \int_0^x f_{S_{i-1}}(x-y) f_{\tilde{S}_{(i)}}(y)\,dy$ for $x > 0$ and $a_{\tilde{S}} = \mathbb{P}[\tilde{S} = 0] = \varphi_{\tilde{N}}(a_X)$ with the pgf $\varphi_{\tilde{N}}$ of $\tilde{N}$;

(b)

$$
f_S(x) = \sum_{i=1}^{\ell} \nu_i f_{\tilde{S}_{(i)}}(x). \tag{9.4}
$$

In both cases the distribution of S has the atom

$$
a_S = \mathbb{P}[S = 0] = \varphi_N(a_X) \tag{9.5}
$$

at zero, where φ_N is the probability generating function of N.

Proof Let $\hat{f}$, $\hat{f}_S$, $\hat{f}_{\tilde{S}_{(i)}}$ and $\hat{f}_{S_i}$ denote the Laplace transforms of the absolutely continuous part of the distribution of X_1, S, $\tilde{S}_{(i)}$ and S_i for $i = 1, \ldots, \ell$, respectively. Then $\mathcal{L}_{X_1}(s) = \hat{f}(s) + a_X$ and analogously for the other transforms.

Note that $\hat{f}_S'(s) = -\int_0^\infty e^{-sx} x f_S(x)\,dx$ for $s > 0$. Using (9.3) and noting that Laplace transforms of convolutions are products of Laplace transforms, we get

$$\hat{f}'_S(s) = \sum_{j=1}^{k+\ell-1} q_j \hat{f}'_{S_j}(s) + \sum_{i=1}^{\ell} \Big(a_i \big(\hat{f}_{S_i}(s) + a_{S_i}\big) \hat{f}'_{\tilde{S}_{(i)}}(s) \\ + (ia_i + b_i)\Big(\big(\hat{f}_{S_{i-1}}(s) + a_{S_{i-1}}\big)\big(\hat{f}_{\tilde{S}_{(i)}}(s) + a_{\tilde{S}_{(i)}}\big) \hat{f}'(s)\Big)\Big), \quad s > 0. \quad (9.6)$$

Comparing (9.6) and (9.1) and using the uniqueness of the Laplace transform [15, p. 86], the representation of the substochastic density in (9.3) is proved.

For case (9.2) we multiply (9.4) by $-x$ and calculate the Laplace transform to obtain $\hat{f}'_S(s) = \sum_{i=1}^{\ell} \nu_i \hat{f}'_{\tilde{S}_{(i)}}(s)$ for $s > 0$. Comparing this to (9.2) we get the assertion by the uniqueness of the Laplace transform. The representation of the atom a_S is clear. □

Again we find an analogue to Corollary 4.7 for the case when the k-truncation of a probability distribution is in a Panjer(a, b, k) class.

Corollary 9.3 *Assume that* $\{q_n\}_{n\in\mathbb{N}_0}$ *has mass at or above* $k \in \mathbb{N}$. *Let* $\{\tilde{q}_n\}_{n\in\mathbb{N}_0}$ *denote its* k*-truncated probability distribution, i.e.,* $\tilde{q}_0 = \cdots = \tilde{q}_{k-1} = 0$ *and* (4.9) *holds. Assume that* N *and* $\tilde{N}$, *respectively, have these distributions, and let* $S = X_1 + \cdots + X_N$ *and* $\tilde{S} = X_1 + \cdots + X_{\tilde{N}}$ *denote the corresponding random sums with (substochastic) densities* f_S *and* $f_{\tilde{S}}$ *on* $(0, \infty)$ *and atoms* a_S *and* $a_{\tilde{S}}$ *at zero. Then* a_S *is given by* (9.5) *and* f_S *satisfies*

$$f_S(x) = \sum_{j=1}^{k-1} q_j f_{S_j}(x) + \left(1 - \sum_{j=0}^{k-1} q_j\right) f_{\tilde{S}}(x)$$

for almost all $x > 0$, *where* f_{S_j} *denotes a density of the absolutely continuous part of the distribution of* $S_j = X_1 + \cdots + X_j$ *for* $j = 1, \ldots, k-1$.

Proof Apply Corollary 9.2(a) with $\ell = k$, $\nu_i = q_i$ and $\tilde{q}_{i,i} = 1$ for $i \in \{1, \ldots, k-1\}$, $\nu_k = 1 - (q_0 + \cdots + q_{k-1})$, $\tilde{q}_{k,n} = \tilde{q}_n$ for all $n \geq k$, and all other $\tilde{q}_{i,n} = 0$. □

The above results can directly help to improve the numerical stability in the case of the extended negative binomial and the extended logarithmic distribution.

Corollary 9.4 *For the parameters* $k \in \mathbb{N}_0$, $\alpha \in (-k, -k+1)$ *and* $p \in [0, 1)$, *with* $p \neq 0$ *for* $k = 0$, *let* $\{q_n\}_{n\in\mathbb{N}_0}$ *denote the* ExtNegBin$(\alpha - 1, k+1, p)$ *distribution and* $\{\tilde{q}_n\}_{n\in\mathbb{N}_0}$ *the* ExtNegBin(α, k, p) *distribution, where* ExtNegBin$(\alpha, 0, p)$ *stands for the negative binomial distribution* NegBin(α, p). *Then* (4.4) *holds with* $\ell = 1$ *and* $\tilde{q}_{1,n} = \tilde{q}_n$ *for* $n \geq k+1$. *The constants are given by* $a_1 = 0$ *and* (5.1), *hence* (9.3) *simplifies to*

$$f_S(x) = b_1 a_{\tilde{S}} f(x) + \frac{b_1}{x} \int_0^x y\, f(y)\, f_{\tilde{S}}(x-y)\, dy \quad \text{for almost all } x > 0. \quad (9.7)$$

Proof Analogously to Corollary 5.1, Corollary 9.2(b) is applicable. □

Corollary 9.5 *For the parameters $k \in \mathbb{N}$ and $q \in (0,1]$ with $q < 1$ if $k = 1$, let $\{q_n\}_{n\in\mathbb{N}_0}$ denote the* $\mathrm{ExtLog}(k+1, q)$ *distribution and $\{\tilde{q}_n\}_{n\in\mathbb{N}_0}$ the* $\mathrm{ExtLog}(k, q)$ *distribution, where* $\mathrm{ExtLog}(1, q)$ *stands for* $\mathrm{Log}(q)$. *Then* (4.4) *holds with $\ell = 1$ and $\tilde{q}_{1,n} = \tilde{q}_n$ for $n \geq k+1$. The constants are given by $a_1 = 0$ and* (5.8), *hence* (9.3) *again simplifies to the numerically stable weighted convolution* (9.7).

Proof Analogously to Corollary 5.4, Corollary 9.2(b) is applicable. □

Remark 9.6 In Lemmas 5.2 and 5.5 we gave, for discrete claim size distributions, limit arguments to deal with special cases $\mathrm{ExtNegBin}(\alpha - 1, 1, 0)$ for $\alpha \in (0, 1)$ and $\mathrm{ExtLog}(2, 1)$, respectively. If the claim size distribution is absolutely continuous (without an atom at zero), then a completely analogous reasoning applies. First, consider N and $\tilde{N}$ as in Lemma 5.2. Then an integral equation for the auxiliary function $r(p, x) = p^{-\alpha} f_{\tilde{S}}(x)$, defined for $p \in (0, 1)$ and $x \geq 0$, follows from the Panjer-style integral equation of $f_{\tilde{S}}$, cf. [22, (2.7)], and the limit $q \nearrow 1$ can be taken in this equation and the integral representation of f_S by r, which is obtained from (9.7). For the continuous variant of Lemma 5.5, one can proceed in the same way, using $r(q, x) = -\log(1-q) f_{\tilde{S}}(x)$. This approach does not work with a claim size atom at zero, though: The normalization can keep the factor b_1 at the integral in (9.7) from exploding in the limit, but not the b_1 that sits at the summand before it. The details are left to the reader.

References

1. Basel Committee of Banking Supervision: Basel II: international convergence of capital measurement and capital standards: a revised framework (2005). Available at www.bis.org/publ/bcbs118.htm
2. Credit Suisse Financial Products: CreditRisk$^+$: a credit risk management framework (1997). Available at www.csfb.com/creditrisk
3. De Pril, N.: Moments of a class of compound distributions. Scand. Actuar. J. **2**, 117–120 (1986)
4. Euler, L.: Introductio in Analysin Infinitorum, vol. 1. Lausanne (1748). Available at www.math.dartmouth.edu/~euler/
5. Feller, W.: An Introduction to Probability Theory and Its Applications, vol. 1, 3rd edn. Wiley, New York (1968)
6. Feller, W.: An Introduction to Probability Theory and Its Applications, vol. 2, 2nd edn. Wiley, New York (1971)
7. Gordy, M.: Saddlepoint approximation of CreditRisk$^+$. J. Bank. Finance **26**, 1335–1353 (2002)
8. Gould, H.: Coefficient identities for powers of Taylor and Dirichlet series. Am. Math. Mont. **81**, 3–14 (1974)
9. Gundlach, M., Lehrbass, F. (eds.): CreditRisk$^+$ in the Banking Industry. Springer, Berlin (2004)
10. Haaf, H., Reiß, O., Schoenmakers, J.: Numerically stable computation of CreditRisk$^+$. In: Gundlach, M., Lehrbass, F. (eds.), CreditRisk$^+$ in the Banking Industry, pp. 69–77. Springer, Berlin (2004)
11. Hess, K., Liewald, A., Schmidt, K.: An extension of Panjer's recursion. Astin Bull. **32**, 283–297 (2002)
12. Hesselager, O.: A recursive procedure for calculation of some compound distributions. Astin Bull. **24**, 19–32 (1994)
13. Johnson, N., Kotz, S., Kemp, A.: Univariate Discrete Distributions, 3th edn. Wiley, Hoboken (2005)
14. Jørgensen, B.: Statistical Properties of the Generalized Inverse Gaussian Distribution. Lecture Notes in Statistics, vol. 9. Springer, New York (1982)
15. Kallenberg, O.: Foundations of Modern Probability, 2nd edn. Probability and Its Applications. Springer, New York (2002)

16. Klugman, S., Panjer, H., Willmot, G.: Loss Models: From Data to Decisions, 2nd edn. Wiley, New York (2004)
17. Maceda, E.: On the compound and generalized Poisson distributions. Ann. Math. Stat. **19**, 414–416 (1948)
18. McNeil, A., Frey, R., Embrechts, P.: Quantative Risk Management: Concepts, Techniques, and Tools. Princeton University Press, Princeton (2005)
19. Mikosch, T.: Non-Life Insurance Mathematics, An Introduction with Stochastic Processes. Springer, Berlin (2004)
20. Panjer, H.: Recursive evaluation of a family of compound distributions. Astin Bull. **12**, 22–26 (1981)
21. Panjer, H., Wang, S.: On the stability of recursive formulas. Astin Bull. **23**, 227–258 (1993)
22. Panjer, H., Willmot, G.: Difference equation approaches in evaluation of compound distributions. Insur. Math. Econ. **6**, 43–56 (1987)
23. Panjer, H., Willmot, G.: Insurance Risk Models. Society of Actuaries (1992)
24. Rosiński, J.: Tempering stable processes. Stoch. Process. Appl. **117**, 677–707 (2007)
25. Samorodnitsky, G., Taqqu, M.: Stable Non-Gaussian Random Processes. Stochastic Modeling. Chapman & Hall, New York (1994)
26. Sato, K.I.: Lévy Processes and Infinitely Divisible Distributions. Cambridge Studies in Advanced Mathematics, vol. 68. Cambridge University Press, Cambridge (1999)
27. Schmidt, K.: Lectures on Risk Theory. Teubner, Leipzig (1995)
28. Sundt, B.: On some extensions of Panjer's class of counting distributions. Astin Bull. **22**, 61–80 (1992)
29. Sundt, B., Jewell, W.: Further results on recursive evaluation of compound distributions. Astin Bull. **12**, 27–39 (1981)
30. Temnov, G., Warnung, R.: Comparison of methods of loss aggregation for operational risk. J. Oper. Risk **3**(1), 3–23 (2008)
31. Wang, S., Sobrero, M.: Further results on Hesselager's recursive procedure for calculation of some compound distributions. Astin Bull. **24**, 161–166 (1994)
32. Willmot, G., Mixed compound Poisson distributions. Astin. Bull. **16**(S), 59–76 (1986)
33. Willmot, G.: Sundt and Jewell's family of discrete distributions. Astin Bull. **18**, 17–29 (1988)