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Finding efficient recursions for risk aggregation by computer algebra

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Abstract

We derive recursions for the probability distribution of random sums by computer algebra. Unlike the well-known Panjer-type recursions, they are of finite order and thus allow for computation in linear time. This efficiency is bought by the assumption that the probability generating function of the claim size be algebraic. The probability generating function of the claim number is supposed to be from the rather general class of D-finite functions.

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1. Introduction

Random sums

$$L = X_1 + \cdots + X_N$$

play a prominent role in risk theory. We refer to the X_i, which are independent copies of a discrete random variable X, as claims, and to N, which is independent of the X_i, as the claim number. A lot of research has been devoted to recursive calculation of the distribution of L. The classical Panjer recursion and its numerous extensions [8,19,20] provide infinite-order linear recursions for this problem for various claim number distributions. Recently, Hipp [9] has found that finite-order recursions can be obtained for phase-type claim size distributions, with obvious advantages concerning computation time. The generalized discrete phase-type distributions are the distributions of hitting times in a finite-state discrete-time Markov chain. They serve as a very flexible class of severity distributions with several useful properties [2,13]. Simple examples are the geometric and the negative binomial distribution (with integral parameter α).

Another approach by De Pril [4] also leads to finite-order recursions for some claim size distributions, for example for piecewise constant or piecewise linear claim size distributions. However, the recursion by De Pril is not necessarily of finite order for the class of algebraic probability generating functions, which we consider in this article.

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We present an alternative method to obtain recursions of finite order satisfied by the distribution $\mathbb{P}[L = n]$ of L. Our assumption on the claim number N is that its probability generating function (pgf) be a D-finite function. This broad class of functions is characterized by linear differential equations with polynomial coefficients. The pgf of the claims is assumed to be algebraic. Then the theory of D-finite functions ensures the existence of a finite-order linear recursion with polynomial coefficients for the distribution of L. Our approach thus goes beyond the assumptions of Hipp [9], in respect of both the claim number distribution and the claim size distribution. The recursion can be determined by computer algebra, which avoids tedious hand calculations.

Section 2 collects the parts of the theory of D-finite functions that we want to use, together with their algorithmic realization in computer algebra systems. The latter is applied in Section 3 to some concrete examples. Although we focus on computational efficiency, we do care about numerical stability of the proposed recursions. Therefore, in Section 4 we apply the stability theory of finite-order recursions to our examples.

2. *D*-finite functions

We will assume throughout that the probability generating function (pgf)

$$\varphi_N(z) = \mathbb{E}[z^N]$$

of the claim number N is of the following kind.

Definition 2.1. Let $f(z)$ be a function that is analytic at zero. Then $f(z)$ is called *D-finite* if it satisfies a linear differential equation

$$Q_0(z)f(z) + Q_1(z)f'(z) + \cdots + Q_d(z)f^{(d)}(z) = 0 \tag{2.1}$$

with polynomial coefficients $Q_0(z), \ldots, Q_d(z)$, not all identically zero.

Most discrete distributions that are used in practice have D-finite pgfs. See, e.g., the comprehensive list of hypergeometric distributions in Johnson, Kemp, Kotz [11]. Many properties of D-finite functions follow from the classical theory of ordinary differential equations [10]; see Stanley [17,18] for an introduction from a combinatorial viewpoint including a proof of the following result.

Theorem 2.2. (i) *An analytic function $f(z) = \sum_{n\geq 0} a_n z^n$ is D-finite if and only if its coefficient sequence $(a_n)_{n\geq 0}$ satisfies a finite-order linear recursion with polynomial coefficients.*
(ii) *The sum and the product of two D-finite functions are D-finite. The composition $f(g(z))$ of a D-finite function $f(z)$ and an algebraic function $g(z)$ is D-finite.*

Part (i) says that a differential equation of the form Eq. (2.1) always translates into a recursion

$$R_0(n)a_n + R_1(n)a_{n+1} + \cdots + R_e(n)a_{n+e} = 0, \quad n \geq 0,$$

with polynomial coefficients $R_k(n)$ for the power series coefficients $(a_n)_{n\geq 0}$ of $f(z)$, and vice versa. Therefore, the distributions from the Panjer class are simple examples of distributions with D-finite pgf. More examples of such distributions can be found, e.g., in Panjer and Willmot [15]. Hesselager [8] has found a recursion for the distribution of L for claim number distributions with arbitrary D-finite pgf. It is valid for any claim size distribution, but is of infinite order in general. We, on the other hand, aim at finite-order recursions for the distribution of L. Part (ii) of Theorem 2.2 is central for our approach. Recall that an algebraic function $g(z)$ satisfies $P(z, g(z)) \equiv 0$ for some non-trivial bivariate polynomial P. We explicitly note the result that part (ii) of Theorem 2.2 implies in our situation.

Corollary 2.3. *If the pgf $\varphi_N(z)$ of the claim numbers is D-finite and the pgf $\varphi_X(z)$ of the severity distribution is algebraic, then*

$$\varphi_L(z) = \varphi_N(\varphi_X(z)) = \sum_{n\geq 0} a_n z^n$$

is D-finite, and its coefficients a_n satisfy a linear recurrence of finite order with polynomial coefficients.

Examples of admissible severity distributions include those whose pgfs are polynomials or rational functions, in particular, the phase-type distributions mentioned in the introduction. Furthermore, consider the negative binomial distribution NBin (α, p) with $\alpha > 0$ and $p \in (0, 1)$, i.e.,

$$\mathbb{P}[X = n] = \binom{\alpha + n - 1}{n} p^\alpha (1 - p)^n, \quad n \geq 0.$$

Its pgf

$$\varphi_X(z) = \left(\frac{p}{1 - (1 - p)z}\right)^\alpha$$

is algebraic if α is a rational number. Other examples of distributions with algebraic pgf are the binomial distribution, the discrete Mittag–Leffler distribution, and its generalization, the discrete Linnik distribution [11]. Again, parameters appearing in the exponent have to be constrained to the rational numbers. The distribution of the number of games lost by the ruined gambler in the classical gambler's ruin problem [11] has an algebraic pgf, too.

All operations described in Theorem 2.2 are constructive and have been implemented in computer algebra packages. We have done the computations below with Mallinger's Mathematica package GeneratingFunctions [12]. Another possible choice would have been Salvy and Zimmermann's Maple package gfun [16]. GeneratingFunctions and the diploma thesis [12] which Mallinger wrote about it can be downloaded from http://www.risc.uni-linz.ac.at/research/combinat/. The package offers numerous commands for the manipulation of D-finite power series and their coefficient sequences, which can be used, e.g., to prove identities involving such objects.

To realize the operations described in Corollary 2.3, the package requires an algebraic equation for the claim size pgf and a differential equation for the claim number pgf. We assume that our claim size pgf is an explicit algebraic (or even rational) function, so the first of these two equations is obvious in our examples. As for the second one, suppose we have a closed form expression for the D-finite claim number pgf. A differential equation for it can be built up step by step, by starting from obvious differential equations (for the exponential function, say) and using commands that implement the closure properties from Theorem 2.2. Finally, the package allows to convert the differential equation that we have thus found for $\varphi_L(z)$ into the desired finite-order recursion for its coefficients.

The package GeneratingFunctions can deal with undetermined parameters. For instance, the differential equation $(az+b)f'(z) - cz^2 f(z) = 0$ with parameters a, b, c would be a valid input, and we could, e.g., compute a recurrence relation for the power series coefficients of $f(z)$. When inputting algebraic relations, however, exponents must be fixed: We can specify $f(z)^2 = (az + b)^3$, e.g., but not $f(z)^\alpha = (az + b)^3$ with undetermined α.

3. Examples

We illustrate our approach by three examples. The intermediate differential equations obtained during the process are not displayed.

Example 3.1. First we consider $N \sim$ NBin (α, p) and $X \sim$ NBin (β, q) with $p, q \in (0, 1)$. Then the pgf of the aggregate loss is given by

$$\varphi_L(z) = \left(\frac{p}{1 - (1 - p)(\frac{q}{1-(1-q)z})^\beta}\right)^\alpha. \tag{3.1}$$

Our goal is to find a differential equation for $\varphi_L(z)$ and thence a recurrence relation for $a_n = \mathbb{P}[L = n]$. As a byproduct of the stability analysis in Section 4, we will obtain the asymptotics of a_n for general α and β. To compute a recurrence for a_n by computer algebra, however, these parameters have to be concrete rational numbers (see the last paragraph of Section 2). We choose $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$. Then the function $\varphi_N(z)$ satisfies the algebraic equation

$$(1 - (1 - p)z)\varphi_N(z)^2 = p,$$

from which the command AlgebraicEquationToDifferentialEquation computes a differential equation for $\varphi_N(z)$. From the latter and the algebraic equation

$$(1 - (1 - q)z)\varphi_X(z)^3 = q$$

of $\varphi_X(z)$, the command AlgebraicCompose (cf. Corollary 2.3) derives the differential equation

$$\begin{aligned}&-5(p-1)^3(q-1)^3 q f(z) + 10(q-1)^2(1-z-qz)(16-q+3pq-3p^2q+p^3q-16z+16qz)f'(z)\\&\quad +36(q-1)(1-z+qz)^2(8-3q+9pq-9p^2q+3p^3q-8z+8qz)f''(z)\\&\quad +72(1-z+qz)^3(1-q3pq-3p^2q+p^3q-z+qz)f'''(z)=0\end{aligned}\tag{3.2}$$

for $\varphi_L(z) = \varphi_N(\varphi_X(z))$. Finally, this equation is transformed into the recurrence

$$\begin{aligned}&8n(1+3n)(2+3n)(q-1)^4 a_n + (q-1)^3(320+896n+864n^2+288n^3-5q-46nq-108n^2q-72n^3q\\&\quad +15pq+138npq+324n^2pq+216n^3pq-15p^2q-138np^2q-324n^2p^2q-216n^3p^2q+5p^3q\\&\quad +46np^3q+108n^2p^3q+72n^3p^3q)a_{n+1}+(4+2n)(q-1)^2(512+648n+216n^2-113q-216n^2\\&\quad -113q-216nq-108n^2q+339pq+648npq+324n^2pq-339p^2q-648np^2q-324n^2p^2q\\&\quad +113p^3q+216np^3q+108n^2p^3q)a_{n+2}+36(2+n)(3+n)(q-1)(16+8n-9q-6nq+27pq\\&\quad +18npq-27p^2q-18np^2q+9p^3q+6np^3q)a_{n+3}\\&\ = 72(2+n)(3+n)(4+n)(-1+q-3pq+3p^2q-p^3q)a_{n+4}\end{aligned}\tag{3.3}$$

for the a_n by DifferentialEquationToRecurrenceEquation. Using this recurrence, the probability a_n can be computed with $\mathrm{O}(n)$ operations. We will show in Section 4 that the computation is numerically stable.

Example 3.2. In our second example we suppose that $N \sim$ Poisson (Λ) and $\Lambda \sim \mathrm{GIG}(\psi, \chi, \theta)$, the generalized inverse Gaussian distribution with parameters θ and $\psi, \chi > 0$. In this case we have

$$\varphi_N(z) = C \cdot (\psi + 2 - 2z)^{-\theta/2} \cdot K_\theta(\sqrt{\chi(\psi+2-2z)}),\tag{3.4}$$

where $K_\theta(z)$ is a modified Bessel function of the second kind [1, p. 374] and $C = \psi^{\theta/2}/K_\theta(\sqrt{\psi\chi})$. Provided that θ is a rational number, the second factor is an algebraic function, hence D-finite. (Below we will fix $\theta = \frac{2}{3}$.) The Bessel function $K_\theta(z)$ is D-finite for any θ, by virtue of its classical second-order differential equation. Therefore, by Theorem 2.2, our $\varphi_N(z)$ is indeed a D-finite function.

As for the severities, we take them to be shifted geometrically distributed: $X \sim \mathrm{Geo}(1, q)$ with $q \in (0, 1)$, so that

$$\varphi_X(z) = \frac{qz}{1-(1-q)z}.$$

Once again we want to find a differential equation, and thence a recurrence for the power series coefficients, for the function

$$\varphi_L(z) = \varphi_N(\varphi_X(z)) = C \cdot f(\varphi_X(z)) \cdot K_\theta(g(\varphi_X(z))),\tag{3.5}$$

where $f(z)$ and $g(z)$ are algebraic functions defined according to Eq. (3.4). The command AlgebraicEquationTo DifferentialEquation computes a differential equation for $f(\varphi_X(z))$ from the algebraic equation

$$f(\varphi_X(z))^{-2/\theta} = \psi + 2 - \frac{2qz}{1-(1-q)z}.$$

As mentioned above, this works for any rational θ; to perform the calculation step, we have to fix its value though, say $\theta = \frac{2}{3}$. A differential equation for $K_{2/3}(g(\varphi_X(z)))$ can be found with AlgebraicCompose. It takes as input the differential equation of $K_{2/3}(z)$ and the algebraic equation

$$g(\varphi_X(z))^2 = \chi\left(\psi + 2 - \frac{2qz}{1-(1-q)z}\right).$$

Now that we have a differential equation for each of the two (non-constant) factors in Eq. (3.5), the command DECauchy computes a differential equation for $\varphi_N(\varphi_X(z))$, which is transformed into a recursion for its power series coefficients a_n by DifferentialEquationToRecurrenceEquation. The recurrence we find is

$$
\begin{aligned}
&3n(1+n)(-1+q)^3(-2-\psi+\psi q)\cdot a_n \\
&\quad +2(1+n)(-1+q)^2(-18-12n-9\psi-6n\psi+q+3nq+9\psi q+6n\psi q)\cdot a_{n+1} \\
&\quad +(144+144n+36n^2+72\psi+72n\psi+18n^2\psi-188q-202nq-54n^2q-144\psi q-144n\psi q \\
&\quad -36n^2\psi q+44q^2-3\chi q^2+58nq^2+18n^2q^2+72\psi q^2+72n\psi q^2+18n^2\psi q^2)\cdot a_{n+2} \\
&\quad +2(3+n)(-30-12n-15\psi-6n\psi+19q+9nq+15\psi q+6n\psi q)\cdot a_{n+3} \\
&\quad +3(3+n)(4+n)(2+\psi)\cdot a_{n+4}=0.
\end{aligned}
\tag{3.6}
$$

Summing up, if N has a mixed Poisson distribution Poisson (Λ) with $\Lambda \sim \mathrm{GIG}(\psi, \chi, \frac{2}{3})$, and $X \sim \mathrm{Geo}(1, q)$, then we can compute the total loss probabilities with $\mathrm{O}(n)$ operations by the recurrence Eq. (3.6).

Example 3.3. The third example we consider is $N \sim$ Poisson (λ) and $X \sim \mathrm{NBin}\left(\frac{1}{2}, p\right)$ with $p \in (0, 1)$. To determine a differential equation for the pgf

$$\varphi_L(z) = \exp\left(\lambda\left(\left(\frac{p}{1-(1-p)z}\right)^{1/2} - 1\right)\right),$$

we use again the command AlgebraicCompose. Its input are the algebraic equation

$$(1-(1-p)z)\varphi_X(z)^2 = p$$

and the differential equation

$$\varphi_N'(z) = \lambda\varphi_N(z).$$

AlgebraicCompose then finds the differential equation

$$\lambda^2 p(1-p)^2\varphi_L(z) + 6(1-p)(1-(1-p)z)^2\varphi_L'(z) - 4(1-(1-p)z)^3\varphi_L''(z) = 0. \tag{3.7}$$

Using DifferentialEquationToRecurrenceEquation, we obtain the recurrence

$$
\begin{aligned}
&2n(2n+1)(1-p)^3 a_n - (1-p)^2(-\lambda^2 p + 12n^2 + 24n + 12)a_{n+1} \\
&\quad + 6(n+2)(2n+3)(1-p)a_{n+2} = 4(n+2)(n+3)a_{n+3}
\end{aligned}
\tag{3.8}
$$

for the probabilities a_n.

4. Numerical stability and asymptotics

The computation of a sequence by a linear recurrence relation of finite order is numerically stable if the sequence grows at least as fast as any other solution of the recurrence [14,21]. This is intuitively clear, since rounding errors will always add a portion of each member of a fundamental system of the recurrence to the solution we are computing. Asymptotically dominant solutions will therefore wipe out subordinate solutions in the long run.

In this section we show how to apply methods from asymptotic analysis to the examples from Section 3. The growth of the coefficients a_n depends on the location and nature of the singularity of the generating function $\varphi_L(z)$ that is closest to the origin [6]. To assess the growth of the other solutions of our recurrences, we have to analyze the dominating singularity of the differential equation for $\varphi_L(z)$. If it is regular, then a fundamental system can in principle be determined by Frobenius' method. Flajolet and Odlyzko's singularity analysis [5,6] allows to obtain the growth rate of the power series coefficients of these solutions. Then, hopefully, we can read off that the solution we are interested in dominates the other ones. This fairly general method works in the first two examples from Section 3.

In what follows we use the symbol $\sim$ not only in the sense "is distributed as", but also for asymptotic equality of sequences ($a_n \sim b_n$ if and only if $a_n/b_n \to 1$ as $n \to \infty$); no confusion should arise.

Proposition 4.1. *Let $N \sim \mathrm{NBin}(\alpha, p)$ and $X \sim \mathrm{NBin}(\beta, q)$ with $p, q \in (0, 1)$. Then the probabilities $a_n = \mathbb{P}[L = n]$ satisfy*

$$a_n \sim C z_1^{-n} n^{\alpha-1} \tag{4.1}$$

as $n \to \infty$, where

$$z_1 = \frac{1 - q(1-p)^{1/\beta}}{1-q} \quad \text{and} \quad C = \frac{(pq)^\alpha (1-p)^{\alpha/\beta}}{\Gamma(\alpha)\beta^\alpha (1 - q(1-p)^{1/\beta})^\alpha}.$$

Proof. The dominating singularity of $\varphi_L(z)$ is located at $z = z_1$. Moving the singularity to $z = 1$ and putting $c := 1 - q(1-p)^{1/\beta}$, we find

$$\varphi_L(zz_1) = \left(\frac{p}{1 - (1-p)\left(\frac{q}{1-cz}\right)^\beta} \right)^\alpha .$$

From the expansion

$$\begin{aligned}\left(\frac{q}{1-cz}\right)^\beta &= \left(\frac{q}{1-c}\right)^\beta \left(1 + \frac{c}{1-c}(1-z)\right)^{-\beta} \\ &= \frac{1}{1-p}\left(1 - \frac{\beta c}{1-c}(1-z) + \mathrm{O}((1-z)^2)\right)\end{aligned}$$

we thus obtain

$$\begin{aligned}\varphi_L(zz_1) &= p^\alpha \left(\frac{\beta c}{1-c}(1-z) + \mathrm{O}((1-z)^2)\right)^{-\alpha} \\ &= \frac{(pq)^\alpha (1-p)^{\alpha/\beta}}{\beta^\alpha (1 - q(1-p)^{1/\beta})^\alpha}(1-z)^{-\alpha} + \mathrm{O}((1-z)^{-\alpha+1}) \qquad (4.2)\end{aligned}$$

as z tends to 1. The nth power series coefficient of $(1-z)^{-\alpha}$ asymptotically equals $n^{\alpha-1}/\Gamma(\alpha)$ [6, Chapter VI]. The coefficients of the error term in Eq. (4.2) are of smaller order [6, Chapter VI], whence the desired result. We note that an asymptotic expansion to arbitrary order can be obtained in the same way. □

To be able to derive a concrete recursion we had to assign values to the parameters α and β in Example 3.1.

Proposition 4.2. *Let $N \sim \mathrm{NBin}\left(\frac{1}{2}, p\right)$ and $X \sim \mathrm{NBin}\left(\frac{1}{3}, q\right)$ with $p, q \in (0, 1)$. Then the computation of the a_n by the recursion Eq. (3.3) is numerically stable.*

Proof. We show, by Frobenius' theory of power series solutions of differential equations [10], that no solution of Eq. (3.3) grows faster than Eq. (4.1) (with $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$). Instead of working directly with Eq. (3.2), we transform Eq. (3.3) into a differential equation. This is necessary because new solutions might creep in when passing from a differential equation to a recurrence or vice versa. The new differential equation, called $\mathcal{E}$ in what follows, is solved by all generating functions of solutions of Eq. (3.3). It is of order four and has the same leading coefficient as Eq. (3.2). By equating this coefficient to zero, we locate the dominating singularity $z_1 = (1 - q(1-p)^3)/(1-q)$. The indicial polynomial of $\mathcal{E}$ at the regular singularity $z = z_1$ is $(w-2)(w-1)w(1+2w)$. The roots of the indicial polynomial are the possible values of the exponent w in the generalized power series solution. The root $w = -\frac{1}{2}$ leads, by singularity analysis, to a solution of $\mathcal{E}$ whose coefficients grow like Eq. (4.1) (with $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$). The other solutions of the fundamental system that Frobenius' method yields have either no singularity at z_1 or a logarithmic singularity at z_1. The latter type occurs because some roots of the indicial polynomial differ by integers, and the coefficients of the corresponding solutions of $\mathcal{E}$ grow like $1/n$ times a power of $\log n$ [6, Chapter VI]. □

The second example from Section 3 can be treated analogously:

Proposition 4.3. *Let $N \sim \mathrm{Poisson}(\Lambda)$ and $\Lambda \sim \mathrm{GIG}(\psi, \chi, \theta)$, where ψ, χ and θ are positive. Furthermore assume that $X \sim \mathrm{Geo}(1, q)$ with $q \in (0, 1)$. Then the probabilities $a_n = \mathbb{P}[L = n]$ satisfy*

$$a_n \sim C\chi^{-\theta/2} D^{-\theta} (2n)^{\theta-1} z_1^{-n} \qquad (4.3)$$

as $n \to \infty$, where

$$C = \frac{\psi^{\theta/2}}{K_\theta(\sqrt{\chi\psi})}, \qquad z_1 = \frac{1}{1-\psi q/(2+\psi)}, \quad \textit{and} \quad D = \frac{(2+\psi)(2+\psi(1-q))}{2q}.$$

Proof. We proceed analogously to Proposition 4.1. The dominating singularity of $\varphi_L(z)$ is located at $z = z_1$. We use the expansion

$$K_\theta(z) = 2^{\theta-1}\Gamma(\theta)z^{-\theta} + \mathrm{O}(z^{\min\{\theta,2-\theta\}}), \quad z \to 0,$$

valid for $\theta > 0$. From this we find

$$\varphi_L(z_1 z) \sim CD^{-\theta}\chi^{-\theta/2}2^{\theta-1}\Gamma(\theta)(1-z)^{-\theta}, \quad z \to 1.$$

The result now follows from singularity analysis, since the coefficients of $(1-z)^{-\theta}$ asymptotically equal $n^{\theta-1}/\Gamma(\theta)$. Once again, an asymptotic expansion to arbitrary order can be readily obtained. □

Proposition 4.4. *Assume the setup of Example 3.2, i.e., $N \sim$ Poisson (Λ) with $\Lambda \sim \mathrm{GIG}(\psi, \chi, \frac{2}{3})$, and $X \sim$ Geo$(1, q)$ with $q \in (0, 1)$. Then the computation of the a_n by the recursion Eq. (3.6) is numerically stable.*

Proof. Completely analogous to the proof of Proposition 4.2. The indicial polynomial is $w(w-1)(w-2)(3w+2)$. The root $w = -\frac{2}{3}$ leads to a solution whose coefficients grow like Eq. (4.3) (with $\theta = \frac{2}{3}$), whereas the coefficients of the other solutions grow slower. □

The approach we have just illustrated works whenever the dominating singularities of the differential equation satisfied by $\varphi_L(z)$ are regular. In Example 3.3, however, the dominating singularity is irregular. In general, it is difficult to say anything about the growth order of the power series coefficients of the solutions in this case. In our example, though, all solutions of the differential equation for $\varphi_L(z)$ can be expressed in closed form, and their coefficients can be analyzed by Cauchy's integral formula and the saddle point method (also known as method of steepest descent).

Proposition 4.5. *Let $N \sim$ Poisson (λ) and $X \sim$ NBin$\left(\frac{1}{2}, p\right)$ with $p \in (0, 1)$, as in Example 3.3. Then the computation of the a_n by the recursion Eq. (3.8) is numerically stable, and the probabilities satisfy*

$$a_n \sim \frac{\lambda^{1/3}p^{1/6}}{2^{1/3}\sqrt{3\pi}}(1-p)^n n^{-5/6}\exp\left(3p^{1/3}(\lambda/2)^{2/3}n^{1/3} - \lambda\right) \tag{4.4}$$

as $n \to \infty$.

Proof. We present the proof for $p = \frac{1}{2}$ and $\lambda = 1$. The general case yields no additional complications. The functions

$$\left\{\exp\frac{\pm 1}{\sqrt{2-z}}\right\}$$

form a fundamental system for the differential equation Eq. (3.7). Therefore, a fundamental system of the third-order recursion Eq. (3.8) is given by our a_n, the coefficients of $\exp(-1/\sqrt{2-z})$, and $(1, 0, 0, 0, \ldots)$. We have to show that the coefficients a_n of $\varphi_L(z) = \exp(1/\sqrt{2-z})$ have the announced asymptotic behavior, and that those of $\exp(-1/\sqrt{2-z})$ grow slower. (The additional solution $(1, 0, 0, 0, \ldots)$ of Eq. (3.8) cannot make the computation unstable, of course.) To do so, we appeal to Cauchy's integral formula:

$$\begin{aligned} a_n &= \frac{1}{2\mathrm{i}\pi}\int_{|z|=r}\frac{\varphi_L(z)}{z^{n+1}}\mathrm{d}z \\ &= \frac{1}{2\pi r^n}\int_{-\pi}^{\pi}\mathrm{e}^{-\mathrm{i}n\theta}\varphi_L(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta, \quad 0 < r < 2. \end{aligned}$$

We will determine the asymptotics of the integral by the saddle point method [3,6]. To find an approximate saddle point, we equate the derivative of the integrand to zero, which leads to the equation

$$4n^2(2-z)^3 = z^2.$$

Clearly, we must have $z \to 2$ as $n \to \infty$ here. By plugging $z = 2 - u$ with unknown $u = \mathrm{o}(1)$ into the equation, we obtain $u \sim n^{-2/3}$. Therefore, we choose the integration contour $|z| = r := 2 - n^{-2/3}$. The dominant part of the integral arises near the saddle point, for $\theta = \mathrm{O}(n^{-\alpha})$, where α is a fixed parameter with $\frac{7}{9} < \alpha < \frac{5}{6}$. Outside this central part we have

$$\cos\theta \le \cos(n^{-\alpha}) = 1 - \frac{1}{2}n^{-2\alpha} + \mathrm{O}(n^{-4\alpha}),$$

and using this estimate in

$$|2 - z|^{-1/2} = (4 - 4r\cos\theta + r^2)^{-1/4},$$

we obtain

$$\exp\frac{1}{|2 - z|^{1/2}} \le \exp(n^{1/3} - n^{5/3-2\alpha})(1 + \mathrm{o}(1)). \tag{4.5}$$

To calculate the central part of the integral, we compute the second-order approximation

$$(2 - r\mathrm{e}^{\mathrm{i}\theta})^{-1/2} = n^{1/3} + \mathrm{i}n\theta - \frac{3}{2}n^{5/3}\theta^2 + \mathrm{O}(n^{7/3-3\alpha}).$$

Since

$$\int_{-n^{-\alpha}}^{n^{-\alpha}} \exp(-\frac{3}{2}n^{5/3}\theta^2)\mathrm{d}\theta \sim \sqrt{\frac{2\pi}{3}}n^{-5/6}$$

and $r^{-n} \sim 2^{-n}\exp(\frac{1}{2}n^{1/3})$, we find

$$a_n \sim \frac{1}{2\pi r^n}\int_{-n^{-\alpha}}^{n^{-\alpha}} \mathrm{e}^{-\mathrm{i}n\theta}\exp\left(\frac{1}{\sqrt{2 - r\mathrm{e}^{\mathrm{i}\theta}}}\right)\mathrm{d}\theta \sim \frac{1}{\sqrt{6\pi}}\frac{\exp(\frac{3}{2}n^{1/3})}{2^n n^{5/6}}.$$

Note that we have shown above that the remaining portion of the integral, where $n^{-\alpha} < |\theta| < \pi$, grows slower, by virtue of the factor $\exp(-n^{5/3-2\alpha})$ in Eq. (4.5).

Now that we have established the asymptotics of a_n, it remains to show that the coefficients, b_n say, of $\exp(-1/\sqrt{2 - z})$ grow slower. To see this, we use Cauchy's integral formula with the same contour as above:

$$b_n = \frac{1}{2\pi r^n}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}n\theta}\exp\left(\frac{-1}{\sqrt{2 - r\mathrm{e}^{\mathrm{i}\theta}}}\right)\mathrm{d}\theta.$$

Here the integrand has no saddle point near $z = r$, but a bound good enough for our purpose can still be deduced. The tail $|\theta| > n^{-\alpha}$ satisfies the same estimate as for $\varphi_L(z)$. Near the real axis, for $\theta = \mathrm{o}(n^{-\alpha})$, we have

$$\begin{aligned}|\exp(-(2 - r\mathrm{e}^{\mathrm{i}\theta})^{-1/2})| &\sim \exp\left(-n^{1/3} + \frac{3}{2}n^{5/3}\theta^2\right)\\ &\le \exp\left(-\frac{1}{2}n^{1/3}\right)\end{aligned}$$

for large n, which shows that the integral over the central part grows slower than that for a_n (it even tends to zero), whence $b_n = \mathrm{o}(a_n)$.

Note that the proof of Eq. (4.4) can be shortened by using that the function $\varphi_L(z)$ is Hayman-admissible [6] [7, Section 17]. We gave the detailed proof above in order to recycle parts of it in the estimate of b_n. □

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References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, in: National Bureau of Standards Applied Mathematics Series, vol. 55, 1964, For sale by the Superintendent of Documents U.S., Government Printing Office, Washington, DC.

[2] S. Asmussen, O. Nerman, M. Olsson, Fitting phasetype distributions via the EM algorithm, Scand. J. Statist. 23 (1996) 419–441.

[3] N.G. de Bruijn, Asymptotic methods in analysis, in: Bibliotheca Mathematica, vol. 4, North-Holland Publishing Co., Amsterdam, 1958.

[4] N. De Pril, Improved recursions for some compound Poisson distributions, Insurance Math. Econom. 5 (2) (1986) 129–132.

[5] P. Flajolet, A. Odlyzko, Singularity analysis of generating functions, SIAM J. Discrete Math. 3 (2) (1990) 216–240.

[6] P. Flajolet, R. Sedgewick, Analytic Combinatorics, 2008, Book in preparation: Individual chapters are available as INRIA Research Reports 1888, 2026, 2376, 2956, 3162, 4103 and electronically under http://algo.inria.fr/flajolet/Publications/books.html.

[7] W.K. Hayman, A generalisation of Stirling's formula, J. Reine Angew. Math. 196 (1956) 67–95.

[8] O. Hesselager, A recursive procedure for calculation of some compound distributions, ASTIN Bulletin 24 (1) (1994) 19–32.

[9] C. Hipp, Speedy convolution algorithms and Panjer recursions for phase-type distributions, Insurance Math. Econom. 38 (1) (2006) 176–188.

[10] E.L. Ince, Ordinary Differential Equations, Dover Publications Inc., 1956.

[11] N.L. Johnson, A.W. Kemp, S. Kotz, Univariate discrete distributions, third ed., in: Wiley Series in Probability and Statistics, Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, 2005.

[12] C. Mallinger, Algorithmic Manipulations and Transformations of Univariate Holonomic Functions and Sequences, Master's Thesis, RISC, J. Kepler University, August 1996. Available online at: http://www.risc.uni-linz.ac.at/research/combinat/.

[13] M. Neuts, Matrix-Geometric Solutions in Stochastic Models, Johns Hopkins University Press, 1981.

[14] H.H. Panjer, S. Wang, On the stability of recursive formulas, ASTIN Bulletin 23 (2) (1993) 227–258.

[15] H.H. Panjer, G.E. Willmot, Difference equation approaches in evaluation of compound distributions, Insurance Math. Econom. 6 (1987) 43–56.

[16] B. Salvy, P. Zimmerman, GFUN: A Maple package for the manipulation of generating and holonomic functions in one variable, ACM Trans. Math. Softw. 20 (2) (1994) 163–177.

[17] R.P. Stanley, Differentiably finite power series, European J. Combin. 1 (2) (1980) 175–188.

[18] R.P. Stanley, Enumerative Combinatorics. Vol. 2, in: Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[19] B. Sundt, On some extensions of Panjer's class of counting distributions, ASTIN Bulletin 22 (1992) 61–80.

[20] S. Wang, M. Sobrero, Further results on Hesselager's recursive procedure for calculation of some compound distributions, ASTIN Bulletin 24 (2) (1994) 161–166.

[21] J. Wimp, Computations with Recurrence Relations, Pitman Advanced Publishing Program, 1984.