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Asymptotic Analysis of Some Discrete Distributions by the Saddle Point Method

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We present asymptotic formulas for the probability mass functions of three discrete distributions: the Neyman type A, the compound Poisson–Katz, and the convolution of negative binomial and Pólya–Aeppli. An approximation of the moments of the Neyman type A distribution is also given. All of these results are found by Hayman's encapsulation of the saddle point method.

Keywords Asymptotics; Discrete distribution; Probability generating function; Saddle point method.

Mathematics Subject Classification Primary 62E20; Secondary 05A16.

1. Introduction

For many discrete distributions, the probability mass function (pmf) is not available in closed form. Then asymptotic formulas for the pmf are useful for several purposes. For instance, they shed some light on the tail behavior of the distribution, a crucial point in model choice. On the numerical side, they allow for very fast approximate computation of the pmf, and they can be instrumental in parameter estimation. For instance, the asymptotic formula (4) for the pmf of the Neyman type A distribution, which yields a good approximation throughout the support of the distribution, could be used to set up simple approximate maximum likelihood equations. Another important potential application consists of assessing the numerical stability of recursive formulas for the pmf (Wimp, 1984). Namely, for recursions of finite order, stability can only be guaranteed if we are computing an asymptotically dominant solution of the recurrence relation.

For the distributions that we consider there is a closed form expression for the probability generating function (pgf), but not for the pmf $g_n = \mathbf{P}[X = n]$, where X is a random variable with the desired distribution. To obtain asymptotic results on

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the pmf, a fruitful line of attack is to subject the probability generating function

$$G(z) = \sum_{n=0}^{\infty} \mathbf{P}[X = n] z^n = \sum_{n=0}^{\infty} g_n z^n$$

to methods from complex analysis, via Cauchy's formula

$$g_n = \frac{1}{2\pi \mathrm{i}} \oint \frac{G(z)}{z^{n+1}} \mathrm{d}z. \tag{1}$$

In this article, we use the saddle point method (also known as method of steepest descent), which applies to functions $G(z)$ that are of rapid growth near their smallest singularity (as $\exp(z)$ is near infinity, for instance). If we move the integration contour in (1) towards a saddle point of the integrand, then it is often the case that a small neighborhood of the saddle point captures most of the integral, and that this concentration increases with n. Performing a second-order Taylor approximation on the integrand and integrating termwise then yields an asymptotic formula for g_n. See de Bruijn (1958) or Flajolet and Sedgewick (2008) for excellent introductions to this procedure, including Hayman's method (1956). The gist of that approach are sufficient conditions that ensure applicability of the saddle point method. To formulate them, we define

$$h(z) = \log G(z), \quad A(r) = rh'(r), \quad \text{and} \quad B(r) = r^2 h''(r) + rh'(r).$$

Definition 1.1. A function $G(z)$ that has radius of convergence ρ and is positive on some subinterval (r_0, ρ) of $(0, \rho)$ is called *Hayman-admissible*, if it satisfies the following three conditions:

H_1: $\lim_{r \to \rho^-} B(r) = +\infty$.
H_2: For some function $\theta_0(r)$ defined on (r_0, ρ) and satisfying $0 < \theta_0 < \pi$, we have

$$G(re^{\mathrm{i}\theta}) \sim G(r) e^{\mathrm{i}\theta A(r) - \theta^2 B(r)/2}, \quad r \to \rho^-,$$

uniformly for $|\theta| \le \theta_0(r)$.
H_3: Uniformly in $\theta_0(r) \le |\theta| < \pi$

$$G(re^{\mathrm{i}\theta}) = \mathrm{o}(B(r)^{-1/2} G(r)), \quad r \to \rho^-.$$

Then, Hayman (1956) has shown that the Taylor coefficients g_n of $G(z)$ satisfy

$$g_n \sim \frac{G(\zeta)}{\zeta^n \sqrt{2\pi B(\zeta)}}, \quad n \to \infty, \tag{2}$$

where $\zeta = \zeta(n)$ is the unique solution of the saddle point equation

$$\zeta \frac{G'(\zeta)}{G(\zeta)} = n \tag{3}$$

in the interval (r_0, ρ). Admissible functions enjoy closure properties which often allow to establish admissibility without getting one's hands dirty. If $G(z)$ and

$H(z)$ are admissible functions and $P(z)$ is a polynomial, then $G(z)H(z)$, $\exp G(z)$, and $G(z)+P(z)$ are admissible. If the leading coefficient of $P(z)$ is positive, then $G(z)P(z)$ and $P(G(z))$ are admissible. Finally, if $\exp P(z)$ has positive Taylor coefficients, then it is also admissible.

These properties describe a class of discrete distributions that can be subjected to Hayman's method, yielding an asymptotic estimate for the pmf. An example is provided by the pmf of the Neyman type A distribution; see Sec. 2. Similarly, if the moment generating function is in this class, moment estimates can be found almost mechanically. The closure properties above do not yield all admissible functions, though. Our third example, the pmf of the compound Poisson–Katz family, applies another admissibility result of Hayman. In our last example, the convolution of negative binomial and Pólya–Aeppli distributions, we show admissibility directly by Definition 1.1.

2. Examples

2.1. *Pmf of the Neyman Type A Distribution*

The first example we consider is the Neyman type A or compound Poisson–Poisson distribution (Johnson et al., 2005, p. 403) with pgf

$$G_{\mathrm{NA}}(z)=\sum_{n=0}^{\infty} g_n^{\mathrm{NA}} z^n=\exp(\lambda(\mathrm{e}^{\phi(z-1)}-1)), \quad \lambda, \phi>0.$$

We can treat this example very briefly, since Douglas (1965) has already stated the formula

$$g_n^{\mathrm{NA}} \sim \frac{\mathrm{e}^{-\lambda}}{\sqrt{2\pi}} \times \frac{\phi^n \exp(n/W(cn))}{W(cn)^n(n(1+W(cn)))^{1/2}}, \tag{4}$$

where $c=\mathrm{e}^{\phi}/\lambda$, and $W(z)$, defined by $W(z)\exp(W(z))=z$, is the Lambert W function (Corless et al., 1996). Douglas did not justify why the saddle point method is applicable, though. (See, e.g., Knopfmacher et al., 1999, for an example of a "naturally occurring" generating function that does not satisfy the concentration property required by saddle point analysis, so that a blindfold application of the method yields a wrong result. In fact, a very simple example (Flajolet and Sedgewick, 2008) of such a function is $1/(1-z)$.) To prove (4), note that Hayman (1956, Theorem X) has shown that $\exp(\phi z)$ is admissible (which, for $\phi=1$, leads to a proof of Stirling's formula), and also that the exponential of an admissible function is admissible (Hayman, 1956, Theorem VI). Therefore, the pgf of the Neyman type A distribution is admissible, and (4) holds indeed. The formula yields good numerical results even for small n; see Table 1 for two examples. To compute the exact values of the pmf, we used the recurrence relation (Beall, 1940; Johnson et al., 2005):

$$g_n^{\mathrm{NA}}=\frac{\lambda\phi\mathrm{e}^{-\phi}}{n}\sum_{j=0}^{n-1}\frac{\phi^j}{j!}g_{n-j-1}^{\mathrm{NA}}.$$

Table 1
The pmf of the Neyman type A distribution for the parameter values $(\lambda, \phi) = (0.8, 0.2)$ and $(1.0, 2.0)$, in comparison to the approximation (4)

n	1	4	8	12	16
g_n^{NA}	0.113314	0.000318785	4.33161×10^{-8}	3.30548×10^{-12}	1.74844×10^{-16}
approx.	0.145213	0.000338659	4.48516×10^{-8}	3.38959×10^{-12}	1.78356×10^{-16}
g_n^{NA}	0.114004	0.0782724	0.0146437	0.00190362	0.000191942
approx.	0.224049	0.0837621	0.0153626	0.00196942	0.000197092

2.2. *Moments of the Neyman Type A Distribution*

The moments of the Neyman type A distribution do not have a closed form, either. But the moment generating function

$$\sum_{n=0}^{\infty} \mathbf{E}[X^n]\frac{z^n}{n!} = M_{\mathrm{NA}}(z) = G_{\mathrm{NA}}(\mathrm{e}^z) = \exp(\lambda(\mathrm{e}^{\phi(\mathrm{e}^z-1)} - 1)),$$

where X follows the Neyman type A distribution, is Hayman-admissible, again by appealing to closure under taking exponentials. To approximate the moments, we first define $\zeta(n) > 0$ by (3), i.e.,

$$n = \lambda\phi\zeta(n)\exp\big(\zeta(n) + \phi(\mathrm{e}^{\zeta(n)} - 1)\big).$$

The appearance of such implicitly defined quantities is characteristic of the saddle point method. Plugging into (2) yields

$$\begin{aligned}\mathbf{E}[X^n]/n! &\sim \frac{\exp(\lambda(\mathrm{e}^{\phi(\mathrm{e}^{\zeta(n)}-1)} - 1))}{\zeta(n)^n\sqrt{2\pi\phi n\zeta(n)\mathrm{e}^{\zeta(n)}}} \\ &= \frac{\mathrm{e}^{-\lambda}}{\sqrt{2\pi\phi n}}\zeta(n)^{-n-1/2}\exp\left(n\mathrm{e}^{-\zeta(n)}/(\phi\zeta(n)) - \frac{1}{2}\zeta(n)\right),\end{aligned}$$

hence

$$\mathbf{E}[X^n] \sim \frac{\mathrm{e}^{-\lambda}}{\sqrt{\phi}}\zeta(n)^{-n-1/2}n^n\exp\left(n\mathrm{e}^{-\zeta(n)}/(\phi\zeta(n)) - n - \frac{1}{2}\zeta(n)\right) =: a_n \tag{5}$$

by Stirling's formula. The exact values of the moments can be computed using Shenton's recursion (Johnson et al., 2005; Shenton, 1949):

$$\kappa_{n+1} = \phi\left(\sum_{j=0}^{n-1}\binom{n}{j}\kappa_{n-j} + \lambda\right)$$

for the cumulants of the Neyman type A and the general formula (Johnson et al., 2005)

$$\mathbf{E}[X^n] = \sum_{j=0}^{n-1}\binom{n-1}{j}\kappa_{n-j}\mathbf{E}[X^j].$$

Table 2
The moments of the Neyman type A distribution for parameters $(\lambda, \phi) = (0.05, 0.1)$ and $(1.0, 2.0)$, compared to the approximation a_n from (5)

n	20	40	60	80	100	120
$a_n/\mathbf{E}[X^n]$	1.24423	1.1984	1.17245	1.15751	1.14768	1.14055
$a_n/\mathbf{E}[X^n]$	1.29314	1.24185	1.21849	1.20418	1.19418	1.18664

Table 2 shows that the approximation (5), while giving the correct order of magnitude, converges rather slowly, presumably due to the presence of the very slowly increasing function $\zeta(n)$. Similar behavior is to be expected for the moments of the distributions we will meet in the following examples, so that we refrain from analyzing those.

2.3. *Pmf of the Compound Poisson–Katz Family*

The compound Poisson–Katz (Johnson et al., 2005, p. 420) is the distribution of $X = \sum_{k=1}^{N} Y_k$, where N is Poisson distributed, and the Y_k are i.i.d., independent of N, and follow the Katz distribution (Johnson et al., 2005, p. 82). The pgf

$$G_{\mathrm{P}\vee\mathrm{K}}(z) = \exp\left(\lambda\left(\left(1 - \frac{\beta}{1-\beta}(z-1)\right)^{-\alpha/\beta} - 1\right)\right)$$

with positive λ, α, and $0 < \beta < 1$ is Hayman-admissible. In fact, Hayman (1956, Theorem XII) has shown that any function of the form

$$\exp\left(\lambda(1-z)^{-\beta_0}\left(\log\frac{1}{1-z}\right)^{\beta_1}\left(\log\log\frac{1}{1-z}\right)^{\beta_2}\cdots\left(\log\ldots\log\frac{1}{1-z}\right)^{\beta_m}\right),$$

with positive λ, β_0 and real $\beta_1, \ldots, \beta_m$ is admissible. The auxiliary function $B(r)$ from Definition 1.1 equals

$$B(r) = \frac{\alpha\lambda r(1+\alpha r)}{(1-\beta r)^2} \times \left(\frac{1-\beta}{1-\beta r}\right)^{\alpha/\beta}.$$

The saddle point of $G_{\mathrm{P}\vee\mathrm{K}}(z)/z^{n+1}$ is located at $z = \zeta(n)$, defined by

$$\zeta(n)(1-\beta\zeta(n))^{-\alpha/\beta-1} = (\alpha\lambda)^{-1}(1-\beta)^{-\alpha/\beta} n.$$

(Of course, this is a different function than the $\zeta(n)$ from the preceding example.) By inversion of formal power series (Knuth, 1998), it is easy to see that $\zeta(n)$ has an asymptotic expansion of the form

$$\zeta(n) \sim \sum_{k\geq 0} c_k n^{-\frac{k\beta}{\alpha+\beta}}, \quad n \to \infty, \tag{6}$$

with real c_k. The first two coefficients are

$$c_0 = 1/\beta, \quad c_1 = -\beta^{-\frac{\alpha+2\beta}{\alpha+\beta}}(\lambda\alpha)^{\frac{\beta}{\alpha+\beta}}(1-\beta)^{\frac{\alpha}{\alpha+\beta}}. \tag{7}$$

There seems to be no simple general pattern, but the coefficients c_k can be computed recursively (Knuth, 1998). To write down an asymptotic formula for $\zeta(n)^{-n}$, we must expand $\zeta(n)$ up to order $o(1/n)$, which amounts to determining c_k for $0 \leq k \leq \lceil \alpha/\beta \rceil$ in (6). Then we find

$$\begin{aligned}\log \zeta(n) &= -\log \beta + \log\left(1 + \sum_{k=1}^{\lceil \alpha/\beta \rceil} \beta c_k n^{-\frac{k\beta}{\alpha+\beta}} + o(1/n)\right)\\ &= -\log \beta + \sum_{k=1}^{\lceil \alpha/\beta \rceil} d_k n^{-\frac{k\beta}{\alpha+\beta}} + o(1/n)\end{aligned}$$

for some real coefficients d_k, which can also be calculated recursively (Knuth, 1998). We thus obtain

$$\begin{aligned}\zeta(n)^{-n} &= \exp(-n \log \zeta(n))\\ &\sim \beta^n \exp\left(-\sum_{k=1}^{\lceil \alpha/\beta \rceil} d_k n^{\frac{\alpha+\beta-k\beta}{\alpha+\beta}}\right).\end{aligned} \tag{8}$$

As for $G(\zeta(n))$, the binomial theorem yields an expansion

$$\left(\frac{1}{\beta} - \zeta(n)\right)^{-\alpha/\beta} = n^{\frac{\alpha}{\alpha+\beta}} \sum_{k=0}^{\lceil \alpha/\beta \rceil - 1} e_k n^{-\frac{k\beta}{\alpha+\beta}} + o(1),$$

where once again recursive evaluation of the coefficients e_k is possible (Gould, 1974). Therefore, we find

$$G(\zeta(n)) \sim \exp\left(\lambda\left(\left(\frac{\beta-1}{\beta}\right)^{\alpha/\beta} n^{\frac{\alpha}{\alpha+\beta}} \sum_{k=0}^{\lceil \alpha/\beta \rceil - 1} e_k n^{-\frac{k\beta}{\alpha+\beta}} - 1\right)\right). \tag{9}$$

For $B(\zeta(n))$, we only need the expansion of $\zeta(n)$ up to $k = 1$ to get

$$B(\zeta(n))^{-1/2} \sim \beta(\lambda\alpha(\alpha+\beta))^{-1/2}(1-\beta)^{-\alpha/(2\beta)}(-c_1\beta)^{\alpha/(2\beta)+1} \times n^{-\frac{\alpha+2\beta}{2(\alpha+\beta)}}. \tag{10}$$

Plugging (8)–(10) into (2) yields the desired asymptotic formula for the Taylor coefficients $g_n^{P\vee K}$ of $G_{P\vee K}(z)$, depending on the quantities d_k and e_k. In the special case $\alpha < \beta$, the formula can be stated explicitly:

$$g_n^{P\vee K} \sim C \times n^{-\frac{\alpha+2\beta}{2(\alpha+\beta)}} \beta^n \exp\left(\left(\lambda\left(\frac{1-\beta}{-c_1\beta}\right)^{\alpha/\beta} - c_1\beta\right) n^{\frac{\alpha}{\alpha+\beta}}\right), \tag{11}$$

where c_1 is defined in (7), and C is given by

$$C = \frac{e^{-\lambda}}{\sqrt{2\pi\lambda\alpha(\alpha+\beta)}}(1-\beta)^{-\alpha/(2\beta)}(-c_1\beta)^{\alpha/(2\beta)+1}.$$

Numerical examples are presented in Table 3. Formula (11) clearly captures the order of magnitude of the pmf, even for small n, but convergence seems to be

Table 3
The pmf of the compound Poisson-Katz for the parameter values $(\lambda, \alpha, \beta) = (4.0, 0.5, 0.8)$ and $(1.0, 0.2, 0.8)$, compared to the approximation (11)

n	10	20	30	40	50
$g_n^{\mathrm{P}\vee\mathrm{K}}$	0.00417348	0.000332238	0.000030429	2.94281×10^{-6}	2.92849×10^{-7}
approx.	0.00721246	0.000550061	0.0000492799	4.69712×10^{-6}	4.6245×10^{-7}
$g_n^{\mathrm{P}\vee\mathrm{K}}$	0.0427618	0.0160611	0.00459123	0.00112699	0.000250184
approx.	0.0454144	0.0172368	0.00496519	0.00122584	0.000273376

rather sensitive to the parameter values. For instance, it is much faster in the second example of Table 3 than in the first one. Again, the exact pmf values have been computed recursively, using the recurrence relation (Johnson et al., 2005):

$$g_{n+1}^{\mathrm{P}\vee\mathrm{K}} = \frac{\lambda}{n+1} \sum_{j=0}^{n} \frac{(\alpha/\beta + j)!\beta^{j+1}(1-\beta)^{\alpha/\beta}}{(\alpha/\beta - 1)!j!} g_{n-j}^{\mathrm{P}\vee\mathrm{K}}.$$

2.4. *Pmf of the Convolution of Negative Binomial and Pólya–Aeppli Distribution*

The convolution of a negative binomial variable with a Pólya–Aeppli variable (Johnson et al., 2005, p. 242) has pgf

$$G_{\mathrm{NBPA}}(z) = \left(\frac{1-a}{1-az}\right)^{v} \exp\left(\lambda\left(\frac{1-a}{1-az} - 1\right)\right), \tag{12}$$

with positive v and λ, and $0 < a < 1$. Ong and Lee (1979) obtain the distribution as a mixture of a negative binomial variable and a shifted Poisson variable. From (2) we find the asymptotics

$$g_n^{\mathrm{NBPA}} \sim \frac{\lambda^{1/4-v/2}}{2\sqrt{\pi}}(1-a)^{v/2+1/4}\mathrm{e}^{-\lambda(1+a)/2} \times n^{v/2-3/4}a^n \mathrm{e}^{2\sqrt{\lambda(1-a)n}} \tag{13}$$

for the pmf, to be justified below. Numerical evidence (see Table 4) shows that the approximate values are usually quite satisfactory, at least if v and λ are not too

Table 4
The pmf of the convolution of a negative binomial variable with a Pólya–Aeppli variable for the parameter values $(v, \lambda, a) = (1.0, 1.0, 0.2)$ and $(3.0, 2.0, 0.6)$, in comparison to the approximation (13)

n	5	10	15	20	25
g_n^{NBPA}	0.00191599	2.54826×10^{-6}	2.55493×10^{-9}	2.18426×10^{-12}	1.67928×10^{-15}
approx.	0.00153011	2.15862×10^{-6}	2.22558×10^{-9}	1.93568×10^{-12}	1.50608×10^{-15}
g_n^{NBPA}	0.0931052	0.0566591	0.0203377	0.00558899	0.00129975
approx.	0.0683916	0.0468915	0.0176222	0.00496586	0.00117361

large. The exact values were calculated by the recursion (Ong and Lee, 1979):

$$(n+1)g_{n+1}^{\mathrm{NBPA}} = a(2n+v+\lambda(1-a))g_n^{\mathrm{NBPA}} - a^2(n+v-1)g_{n-1}^{\mathrm{NBPA}}.$$

Formula (13) has been obtained by Macintyre and Wilson (1954) by a more complicated method involving asymptotics of Bessel functions and fractional integration, and by Häusler (1930), who applied the saddle point method to a class of functions which includes (12) as a special case. For a short proof of (13), we show that the pgf (12) is Hayman-admissible. (An even shorter proof, but not in the spirit of the present note, would invoke Ong and Lee's (1979) expression of g_n^{NBPA} in terms of Laguerre polynomials and Perron's asymptotic formula (Szegö, 1975, Theorem 8.22.3) for this family of orthogonal polynomials.) We begin by putting

$$G(z) = \frac{1}{(1-z)^v}\exp\left(\frac{c}{1-z}\right) = (1-a)^{-v}\mathrm{e}^{\lambda}G_{\mathrm{NBPA}}(z/a), \tag{14}$$

where $c = \lambda(1-a) > 0$. The functions $A(r)$ and $B(r)$ from Definition 1.1 are given by

$$A(r) = \frac{r(c+v(1-r))}{(1-r)^2} \quad \text{and} \quad B(r) = \frac{r(c+cr+v(1-r))}{(1-r)^3}.$$

Condition H_1 is clearly satisfied. The function $\theta_0(r)$ from Definition 1.1 must satisfy (Flajolet and Sedgewick, 2008):

$$\alpha_2(r)^{-1/2} \ll \theta_0(r) \ll \alpha_3(r)^{-1/3},$$

where the α_k are defined by

$$\log G(r\mathrm{e}^{\mathrm{i}\theta}) = \log G(r) + \sum_{k=1}^{\infty}\alpha_k(r)\frac{(\mathrm{i}\theta)^k}{k!}.$$

This prompts us to choose $\theta_0(r) := (1-r)^{7/5}$, which lies in between of $\alpha_2(r)^{-1/2} \sim (2c)^{-1/2}(1-r)^{3/2}$ and $\alpha_3(r)^{-1/3} \sim (6c)^{-1/3}(1-r)^{4/3}$.

Using $\mathrm{e}^{\mathrm{i}\theta} = 1 + \mathrm{O}(1-r)^{7/5}$, we easily obtain $(1-r\mathrm{e}^{\mathrm{i}\theta})^{-v} \sim (1-r)^{-v}$ uniformly in the required range. Expanding the argument of exp in (14) to second-order w.r.t. θ gives

$$\frac{c}{1-r\mathrm{e}^{\mathrm{i}\theta}} = \frac{c}{1-r} + \frac{\mathrm{i}cr\theta}{(1-r)^2} - \frac{cr(1+r)\theta^2}{2(1-r)^3} + \mathrm{O}(1-r)^{21/5},$$

whence it easily follows that $G(r\mathrm{e}^{\mathrm{i}\theta})$ satisfies H_2. Finally, the decay condition H_3 calls for an estimate of

$$\left|\exp\left(\frac{c}{1-r\mathrm{e}^{\mathrm{i}\theta}}\right)\right| = \exp\left(\frac{c(1-r\cos\theta)}{(1-r\cos\theta)^2 + r^2\sin^2\theta}\right). \tag{15}$$

For $(1-r)^{9/10} \le |\theta| < \pi$, this can be bounded by

$$\exp\left(\frac{c}{1-r\cos\theta}\right) \le \exp\left(\frac{c}{1-r(1-\frac{1}{2}(1-r)^{9/5}+\mathrm{O}(1-r)^{18/5})}\right)$$
$$= \exp\left(\frac{c}{1-r} - \frac{c}{2(1-r)^{1/5}} + \mathrm{o}(1)\right), \tag{16}$$

where the first inequality follows from using $\cos\theta \le \cos((1-r)^{9/10})$ and plugging in the second-order expansion of cos. In the remaining range $(1-r)^{7/5} \le |\theta| \le (1-r)^{9/10}$, we use

$$1 - \frac{1}{2}(1-r)^{9/5} \le \cos\theta \le 1 - \frac{1}{2}(1-r)^{14/5} + \mathrm{O}(1-r)^{28/5} \tag{17}$$

in (15) and obtain the same estimate as in (16). Therefore, (16) indeed holds uniformly in $|\theta| \ge \theta_0$. Moreover, the upper bound in (17) easily yields $(1-re^{i\theta})^{-v} = \mathrm{O}(1-r)^{-v}$. Since

$$\frac{G(r)}{\sqrt{B(r)}} \sim \frac{1}{\sqrt{2c}}(1-r)^{3/2-v}\exp\left(\frac{c}{1-r}\right),$$

the proof of Hayman-admissibility of $G(z)$, and hence of $G_{\mathrm{NBPA}}(z)$, is complete, which establishes (13).

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