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On Turán's inequality for Legendre polynomials

Horst Alzer[a], Stefan Gerhold[b], Manuel Kauers[c,*], Alexandru Lupaş[d]

[a] *Morsbacher Str. 10, 51545 Waldbröl, Germany*
[b] *Christian Doppler Laboratory for Portfolio Risk Management, Vienna University of Technology, Vienna, Austria*
[c] *Research Institute for Symbolic Computation, J. Kepler University, Linz, Austria*
[d] *Department of Mathematics, University of Sibiu, 2400 Sibiu, Romania*

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Abstract

Let

$$\Delta_n(x) = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x),$$

where P_n is the Legendre polynomial of degree n. A classical result of Turán states that $\Delta_n(x) \geqslant 0$ for $x \in [-1, 1]$ and $n = 1, 2, 3, \ldots$. Recently, Constantinescu improved this result. He established

$$\frac{h_n}{n(n+1)}(1 - x^2) \leqslant \Delta_n(x) \quad (-1 \leqslant x \leqslant 1;\ n = 1, 2, 3, \ldots),$$

where h_n denotes the nth harmonic number. We present the following refinement. Let $n \geqslant 1$ be an integer. Then we have for all $x \in [-1, 1]$

$$\alpha_n(1 - x^2) \leqslant \Delta_n(x)$$

*Corresponding author. Tel.: +43 732 2468 9958; fax: +43 732 2468 9930.
E-mail addresses: alzerhorst@freenet.de (H. Alzer), sgerhold@fam.tuwien.ac.at (S. Gerhold), manuel.kauers@risc.uni-linz.ac.at (M. Kauers), alexandru.lupas@ulbsibiu.ro (A. Lupaş).

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with the best possible factor

$$\alpha_n = \mu_{[n/2]}\mu_{[(n+1)/2]}.$$

Here, $\mu_n = 2^{-2n}\binom{2n}{n}$ is the normalized binomial mid-coefficient.

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1. Introduction

The Legendre polynomial of degree n can be defined by

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n \quad (n = 0, 1, 2, \ldots)$$

which leads to the explicit representation

$$P_n(x) = \frac{1}{2^n}\sum_{v=0}^{[n/2]}(-1)^v \frac{(2n-2v)!}{v!(n-v)!(n-2v)!}x^{n-2v}.$$

(As usual, $[x]$ denotes the greatest integer not greater than x.) The most important properties of $P_n(x)$ are collected, for example, in [1,16]. Legendre polynomials belong to the class of Jacobi polynomials, which are studied in detail in [3,13]. These functions have various interesting applications. For instance, they play an important role in numerical integration; see [12].

The following beautiful inequality for Legendre polynomials is due to P. Turán [15]:

$$\Delta_n(x) = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geqslant 0 \quad \text{for } -1 \leqslant x \leqslant 1 \text{ and } n \geqslant 1.^{1} \tag{1.1}$$

This inequality has found much attention and several mathematicians provided new proofs, far-reaching generalizations, and refinements of (1.1). We refer to [8,9,11,14] and the references given therein.

In this paper we are concerned with a remarkable result published by E. Constantinescu [7] in 2005. He offered a new refinement and a converse of Turán's inequality. More precisely, he proved that the double-inequality

$$\frac{h_n}{n(n+1)}(1-x^2) \leqslant \Delta_n(x) \leqslant \frac{1}{2}(1-x^2) \tag{1.2}$$

is valid for $x \in [-1, 1]$ and $n \geqslant 1$. Here, $h_n = 1 + 1/2 + \cdots + 1/n$ denotes the nth harmonic number.

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[1] A nice anecdote about Turán reveals that he used (1.1) as his 'visiting card'; see [4].

It is natural to ask whether the bounds given in (1.2) can be improved. In the next section, we determine the largest number α_n and the smallest number β_n such that we have for all $x \in [-1, 1]$

$$\alpha_n(1-x^2) \leqslant \Delta_n(x) \leqslant \beta_n(1-x^2).$$

We show that the right-hand side of (1.2) is sharp, but the left-hand side can be improved. It turns out that the best possible factor α_n can be expressed in terms of the normalized binomial mid-coefficient

$$\mu_n = 2^{-2n}\binom{2n}{n} = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)} \quad (n = 0, 1, 2, \ldots).$$

We remark that μ_n has been the subject of recent number theoretic research; see [2,5].

In our proof we reduce the desired refinement of Turán's inequality to another inequality, which also depends polynomially on Legendre polynomials. This latter inequality is amenable to a recent computer algebra procedure [10,11]. The procedure sets up a formula that encodes the induction step of an inductive proof of the inequality and, replacing the quantities $P_n(x), P_{n+1}(x), \ldots$ by real variables $Y_1, Y_2, \ldots,$ transforms the induction step formula into a polynomial formula in finitely many variables. The recurrence relation of the Legendre polynomials translates into polynomial equations in the Y_k, which are added to the induction step formula. The truth of the resulting formula for all real $Y_1, Y_2, \ldots$ can be decided algorithmically and is a sufficient (in general not necessary!) condition for the truth of the initial inequality, if we assume that sufficiently many initial values have been checked.

2. Main result

The following refinement of (1.2) is valid.

Theorem. *Let n be a natural number. For all real numbers* $x \in [-1, 1]$ *we have*

$$\alpha_n(1-x^2) \leqslant P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \leqslant \beta_n(1-x^2) \tag{2.1}$$

with the best possible factors

$$\alpha_n = \mu_{[n/2]}\mu_{[(n+1)/2]} \quad \text{and} \quad \beta_n = \tfrac{1}{2}. \tag{2.2}$$

Proof. We define for $x \in (-1, 1)$ and $n \geqslant 1$

$$f_n(x) = \frac{\Delta_n(x)}{1-x^2}.$$

We have $f_1(x) \equiv \alpha_1 = \beta_1 = 1/2$. First, we prove that f_n is strictly increasing on $(0, 1)$ for $n \geqslant 2$. Differentiation yields

$$f_n'(x) = \frac{2x\Delta_n(x) + (1-x^2)\Delta_n'(x)}{(1-x^2)^2}.$$

Using the well-known formulas

$$P_n'(x) = \frac{n+1}{1-x^2}(xP_n(x) - P_{n+1}(x))$$

and

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

we obtain the representation

$$n(1-x^2)^2 f_n'(x) = (n-1)xP_n(x)^2 - (2nx^2 + x^2 - 1)P_n(x)P_{n+1}(x) + (n+1)xP_{n+1}(x)^2. \tag{2.3}$$

We prove the positivity of the right-hand side of (2.3) on (0, 1) by typing

In[1] :=≪ **SumCracker.m**
SumCracker Package by Manuel Kauers – © RISC Linz – V 0.3 2006-05-24
In[2] := **ProveInequality**[
$((\boldsymbol{n}-\mathbf{1})\boldsymbol{x}\textbf{LegendreP}[\boldsymbol{n},\boldsymbol{x}]^{\mathbf{2}}$
$-(\mathbf{2}\boldsymbol{n}\boldsymbol{x}^{\mathbf{2}}+\boldsymbol{x}^{\mathbf{2}}-\mathbf{1})\textbf{LegendreP}[\boldsymbol{n},\boldsymbol{x}]\textbf{LegendreP}[\boldsymbol{n}+\mathbf{1},\boldsymbol{x}]$
$+(\boldsymbol{n}+\mathbf{1})\boldsymbol{x}\textbf{LegendreP}[\boldsymbol{n}+\mathbf{1},\boldsymbol{x}]^{\mathbf{2}})>\mathbf{0},$
From→ **2**, **Using**→ {**0** < ***x*** < **1**}, **Variable**→ ***n***]

into Mathematica, obtaining, after a couple of seconds, the output

Out[2]= True.

It follows from this that f_n is strictly increasing on (0, 1) for $n \geqslant 2$. Since

$$P_n(x) = (-1)^n P_n(-x),$$

we conclude that f_n is even. Thus, we obtain

$$f_n(0) < f_n(x) < f_n(1) \quad \text{for } -1 < x < 1, \; x \neq 0. \tag{2.4}$$

We have

$$P_n(1) = 1 \quad \text{and} \quad P_n'(1) = \tfrac{1}{2}n(n+1).$$

Therefore,

$$\Delta_n(1) = 0 \quad \text{and} \quad \Delta_n'(1) = -1.$$

Applying l'Hospital's rule gives

$$f_n(1) = \lim_{x \to 1} \frac{\Delta_n(x)}{1-x^2} = -\frac{1}{2}\Delta_n'(1) = \frac{1}{2}. \tag{2.5}$$

Since

$$P_{2k-1}(0) = 0 \quad \text{and} \quad P_{2k}(0) = (-1)^k \mu_k$$

we get

$$f_{2k-1}(0) = \mu_{k-1}\mu_k \quad \text{and} \quad f_{2k}(0) = \mu_k^2. \tag{2.6}$$

Combining (2.4)–(2.6) we conclude that (2.1) holds with the best possible factors α_n and β_n given in (2.2). □

Remarks. (1) The proof of the Theorem reveals that for $n \geqslant 2$ the sign of equality holds on the left-hand side of (2.1) if and only if $x = -1, 0, 1$ and on the right-hand side if and only if $x = -1, 1$.

(2) The numbers $\mu_p\mu_q$ $(p, q = 0, 1, 2, \dots; p \leqslant q)$ are the eigenvalues of Liouville's integral operator for the case of a planar circular disc of radius 1 lying in $\mathbf{R}^3$; see [6].

(3) The automated proving procedure can be applied to (2.1) directly. However, owing to the computational complexity of the method, we did not obtain any output after a reasonable amount of computation time.

(4) The Mathematica package SumCracker used in the proof of the Theorem contains an implementation of the proving procedure described in [10]. It is available online at http://www.risc.uni-linz.ac.at/research/combinat/software

(5) The normalized Jacobi polynomial of degree n is defined for $\alpha, \beta > -1$ by

$$R_n^{(\alpha,\beta)}(x) = {}_2F_1(-n, n + \alpha + \beta + 1; \alpha + 1; (1 - x)/2).$$

The special case $\alpha = \beta$ leads to the normalized ultraspherical polynomial

$$\begin{aligned} R_n^{(\alpha,\alpha)}(x) &= {}_2F_1(-n, n + 2\alpha + 1; \alpha + 1; (1 - x)/2) \\ &= \frac{(-1)^n}{2^n(\alpha + 1)_n} \frac{1}{(1 - x^2)^\alpha} \frac{d^n}{dx^n}(1 - x^2)^{n+\alpha}, \end{aligned}$$

where $(a)_n$ denotes the Pochhammer symbol. Obviously, we have $R_n^{(0,0)}(x) = P_n(x)$. We conjecture that the following extension of our Theorem holds.

Conjecture. *Let $\alpha > -1/2$ and $n \geqslant 1$. For all $x \in [-1, 1]$ we have*

$$a_n^{(\alpha)}(1 - x^2) \leqslant R_n^{(\alpha,\alpha)}(x)^2 - R_{n-1}^{(\alpha,\alpha)}(x)R_{n+1}^{(\alpha,\alpha)}(x) \leqslant b_n^{(\alpha)}(1 - x^2)$$

with the best possible factors

$$a_n^{(\alpha)} = \mu_{[n/2]}^{(\alpha)}\mu_{[(n+1)/2]}^{(\alpha)} \quad \textit{and} \quad b_n^{(\alpha)} = \frac{1}{2(\alpha + 1)}.$$

Here, $\mu_n^{(\alpha)} = \mu_n / \binom{n+\alpha}{n}$.

(6) Gasper [9] has shown that the normalized Jacobi polynomials satisfy

$$R_n^{(\alpha,\beta)}(x)^2 - R_{n-1}^{(\alpha,\beta)}(x)R_{n+1}^{(\alpha,\beta)}(x) \geqslant 0 \quad (-1 \leqslant x \leqslant 1)$$

if and only if $\beta \geqslant \alpha > -1$. More general criteria for a family of orthogonal polynomials to satisfy a Turán-type inequality are given by Szwarc [14].

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